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Existence and Uniqueness Results for a Nonlinear Coupled System of Nonlinear Fractional Langevin Equations with a New Kind of Boundary Conditions

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Abstract. Nonlinear coupled system of fractional Langevin equations with new boundary conditions are considered. By using fixed point theorem and fractional calculus, existence and uniqueness solution for the considered problem are studied. Finally, an example is constructed to illustrate the obtained results.

1. Introduction

Recently, time fractional differential equations plays an important role in describing and studying several applied fields such as control engineering, physical science, economic, fluid dynamic and so on, (see [1-7]). On the other hand, Several problems in physics and engineering can be described by Langevin equation (see [8-11]. The generalization of the Langevin equations is acquired naturally by replacing the ordinary derivative with a fractional-order derivative that yields the popular fractional Langevin equations (see [12-21].

The purpose of this work is to study existence and uniqueness solution for coupled system of fractional Langevin equations in the following form:

$$\begin{cases} {}^cD^{\kappa_1}_{0+}({}^cD^{\upsilon_1}_{0+} + \chi_1)z_1(t) = \Psi_1(t, z_1(t), z_2(t)),\ t \in J := [0, T], 1 < \upsilon_1 \leq 2,\ 1 < \kappa_1 \leq 2 \\ {}^cD^{\kappa_2}_{0+}({}^cD^{\upsilon_2}_{0+} + \chi_2)z_2(t) = \Psi_2(t, z_1(t), z_2(t)),\ t \in J := [0, T], 1 < \upsilon_2 \leq 2,\ 1 < \kappa_2 \leq 2 \end{cases} \tag{1}$$

subject to the following coupled boundary conditions

$$\begin{cases} z_1(0) = 0,\ \ z_1(T) = \delta_1 z_1(\eta_1),\ \ z_1'(T) = \epsilon_1 z_1'(\xi_1) \\ z_2(0) = 0,\ \ z_2(T) = \delta_2 z_1(\eta_2),\ \ z_2'(T) = \epsilon_2 z_1'(\xi_2) \end{cases} \tag{2}$$

where ${}^cD^{\kappa_1}_{0+}, {}^cD^{\kappa_2}_{0+}, {}^cD^{\upsilon_1}_{0+}, {}^cD^{\upsilon_2}_{0+}$ denote the Caputo fractional derivative of order $\kappa_1,\ \kappa_2,\ \upsilon_1$ and υ_2 respectively, $\Psi_1,\ \Psi_2 : [0, T] \times R \times R \to R$ are continuous functions, $\chi_1,\ \chi_2 \in R$ are the dissipative parameters and $\delta_i,\ \epsilon_i \in R$ and $0 < \eta_i,\ \xi_i < 1$ for $i = 1, 2$.

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2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 2.1 (see [22]). The fractional integral of order $\kappa > 0$ with the lower limit zero for a function f can be defined as

$$I^{\kappa} f(t) = \frac{1}{\Gamma(\kappa)} \int_0^t \frac{f(s)}{(t-s)^{1-\kappa}} ds, \ \ t > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where

$$\Gamma(\kappa) = \int_0^{\infty} t^{\kappa-1} e^{-t} dt, \ \ \kappa > 0.$$

Definition 2.2 (see [22]). The Caputo derivative of order κ with the lower limit zero for a function f can be written as

$${}^{c}D^{\kappa} f(t) = \frac{1}{\Gamma(n-\kappa)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\kappa+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \ \ t > 0, \ 0 \le n-1 < \kappa < n.$$

Lemma 2.1. Let $m_1, m_2 \in C([0,T], R)$ then the solution of linear system (1.1) is equivalent to the system of integral equations

$$\begin{aligned} z_1(t) &= \left[\frac{\lambda_1}{k_1} + \frac{k_1k_2\lambda_2 - k_2k_3\lambda_1}{k_1(k_2k_3 - k_1k_4)}\right] \frac{t^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{\lambda_1 k_3 - \lambda_2 k_1}{k_2k_3 - k_1k_4} \frac{t^{(\upsilon_1+1)}}{\Gamma(\upsilon_1+2)} \\ &+ \int_0^t \frac{(t-s)^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1+\upsilon_1)} \Psi_1(s) ds - \chi_1 \int_0^t \frac{(t-s)^{\upsilon_1-1}}{\Gamma(\upsilon_1)} z_1(s) ds, \end{aligned} \tag{3}$$

and

$$\begin{aligned} z_2(t) &= \left(\frac{t^{\upsilon_2}}{T^{\upsilon_2}} - \frac{B_1 T t^{\upsilon_2}}{\Gamma(\upsilon_2+2)} + \frac{B_1 t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\right)\zeta_1 + \left(\frac{B_2 T t^{\upsilon_2}}{\Gamma(\upsilon_2+2)} - \frac{B_2 t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\right)\zeta_2 \\ &+ \left(\frac{\delta_2 \eta_2^{\upsilon_1} t^{\upsilon_2}}{T^{\upsilon_2}\Gamma(\upsilon_1+1)} - \frac{B_3 T t^{\upsilon_2}}{\Gamma(\upsilon_2+2)} + \frac{B_3 t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\right)\left(\frac{\lambda_1}{k_1} + \frac{k_1k_2\lambda_2 - k_2k_3\lambda_1}{k_1(k_2k_3 - k_1k_4)}\right) \\ &+ \left(\frac{\delta_2 \eta_2^{\upsilon_1+1} t^{\upsilon_2}}{T^{\upsilon_2}\Gamma(\upsilon_1+2)} - \frac{B_4 T t^{\upsilon_2}}{\Gamma(\upsilon_2+2)} + \frac{B_4 t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\right)\left(\frac{\lambda_1 k_3 - \lambda_2 k_1}{k_2k_3 - k_1k_4}\right) \\ &+ \int_0^t \frac{(t-s)^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2+\upsilon_2)} \Psi_2(s) ds - \chi_2 \int_0^t \frac{(t-s)^{\upsilon_2-1}}{\Gamma(\upsilon_2)} z_2(s) ds, \end{aligned} \tag{4}$$

where

$$k_1 = \frac{T^{\upsilon_1} - \delta_1 \eta_1^{\upsilon_1}}{\Gamma(\upsilon_1+1)}, \ k_2 = \frac{T^{\upsilon_1+1} - \delta_1 \eta_1^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)}, \ k_3 = \frac{T^{\upsilon_1-1} - \epsilon_1 \xi_1^{\upsilon_1-1}}{\Gamma(\upsilon_1)}, \ k_4 = \frac{T^{\upsilon_1} - \epsilon_1 \xi_1^{\upsilon_1}}{\Gamma(\upsilon_1+1)},$$

$$\begin{aligned} B_1 &= \frac{-\upsilon_2 \Gamma(\upsilon_2+1)}{T^{\upsilon_2+1}}, \ B_2 = \frac{-\Gamma(\upsilon_2+2)}{T^{\upsilon_2}}, \\ B_3 &= \frac{\Gamma(\upsilon_2+2)[\upsilon_2 \delta_2 \eta_2^{\upsilon_1} T^{-1} - \upsilon_1 \epsilon_2 \xi_2^{\upsilon_1-1}] T^{\upsilon_2}}{\Gamma(\upsilon_1+1)[\upsilon_2 T^{-1} - (\upsilon_2+1) T^{2\upsilon_2}]}, \\ B_4 &= \frac{\Gamma(\upsilon_2+2)[\upsilon_2 \delta_2 \eta_2^{\upsilon_1+1} T^{-1} - (\upsilon_1+1) \epsilon_2 \xi_2^{\upsilon_1}] T^{\upsilon_2}}{\Gamma(\upsilon_1+2)[\upsilon_2 T^{-1} - (\upsilon_2+1) T^{2\upsilon_2}]}, \end{aligned}$$

$$\begin{aligned}\lambda_1 &= \frac{1}{\Gamma(\kappa_1+\upsilon_1)}\Big[\delta_1\int_0^{\eta_1}(\eta_1-s)^{\kappa_1+\upsilon_1-1}\Psi_1(s)ds-\int_0^T(T-s)^{\kappa_1+\upsilon_1-1}\Psi_1(s)ds\Big]\\ &- \frac{\chi_1}{\Gamma(\upsilon_1)}\Big[\delta_1\int_0^{\eta_1}(\eta_1-s)^{\upsilon_1-1}z_1(s)ds-\int_0^T(T-s)^{\upsilon_1-1}z_1(s)ds\Big],\end{aligned}$$

$$\begin{aligned}\lambda_2 &= \frac{1}{\Gamma(\kappa_1+\upsilon_1-1)}\Big[\epsilon_1\int_0^{\xi_1}(\xi_1-s)^{\kappa_1+\upsilon_1-2}\Psi_1(s)ds-\int_0^T(T-s)^{\kappa_1+\upsilon_1-2}\Psi_1(s)ds\Big]\\ &- \frac{\chi_1}{\Gamma(\upsilon_1-1)}\Big[\epsilon_1\int_0^{\xi_1}(\xi_1-s)^{\upsilon_1-2}z_1(s)ds-\int_0^T(T-s)^{\upsilon_1-2}z_1(s)ds\Big],\end{aligned}$$

$$\begin{aligned}\zeta_1 &= \frac{\chi_2}{\Gamma(\upsilon_2)}\int_0^T(T-s)^{\upsilon_2-1}z_2(s)ds-\frac{\delta_2\chi_1}{\Gamma(\upsilon_1)}\int_0^{\eta_2}(\eta_2-s)^{\upsilon_1-1}z_1(s)ds\\ &+ \frac{\delta_2}{\Gamma(\kappa_1+\upsilon_1)}\int_0^{\eta_2}(\eta_2-s)^{\kappa_1+\upsilon_1-1}\Psi_1(s)ds-\frac{1}{\Gamma(\kappa_2+\upsilon_2)}\int_0^T(T-s)^{\kappa_2+\upsilon_2-1}\Psi_2(s),ds\end{aligned}$$

$$\begin{aligned}\zeta_2 &= \frac{\chi_2}{\Gamma(\upsilon_2-1)}\int_0^T(T-s)^{\upsilon_2-2}z_2(s)ds-\frac{\epsilon_2\chi_1}{\Gamma(\upsilon_1-1)}\int_0^{\xi_2}(\xi_2-s)^{\upsilon_1-2}z_1(s)ds\\ &+ \frac{\epsilon_2}{\Gamma(\kappa_1+\upsilon_1-1)}\int_0^{\xi_2}(\xi_2-s)^{\kappa_1+\upsilon_1-2}\Psi_1(s)ds-\frac{1}{\Gamma(\kappa_2+\upsilon_2-1)}\int_0^T(T-s)^{\kappa_2+\upsilon_2-2}\Psi_2(s)ds.\end{aligned}$$

Proof. We know that

$$({}^cD_{0+}^{\upsilon_1}+\chi_1)z_1(t)=a_0+a_1t+\int_0^t\frac{(t-s)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Psi_1(s)ds$$

then

$${}^cD_{0+}^{\upsilon_1}z_1(t)=a_0+a_1t+\int_0^t\frac{(t-s)^{\kappa_1-1}}{\Gamma(\kappa_1)}\Psi_1(s)ds-\chi_1z_1(t).$$

Hence,

$$z_1(t)=\frac{a_0t^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{a_1t^{(\upsilon_1+1)}}{\Gamma(\upsilon_1+2)}+\int_0^t\frac{(t-s)^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1+\upsilon_1)}\Psi_1(s)ds-\chi_1\int_0^t\frac{(t-s)^{\upsilon_1-1}}{\Gamma(\upsilon_1)}z_1(s)ds+a_2. \tag{5}$$

Also,

$$z_2(t)=\frac{b_0t^{\upsilon_2}}{\Gamma(\upsilon_2+1)}+\frac{b_1t^{(\upsilon_2+1)}}{\Gamma(\upsilon_2+2)}+\int_0^t\frac{(t-s)^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2+\upsilon_2)}\Psi_2(s)ds-\chi_2\int_0^t\frac{(t-s)^{\upsilon_2-1}}{\Gamma(\upsilon_2)}z_2(s)ds+b_2. \tag{6}$$

where $a_i, b_i,\ i=0,1,2$ are arbitrary real constant.
From boundary conditions (1.2), we have

$$\begin{aligned}a_2 &= 0,\ b_2=0,\\ a_0 &= \frac{\lambda_1}{k_1}+\frac{k_1k_2\lambda_2-k_2k_3\lambda_1}{k_1(k_2k_3-k_1k_4)},\quad a_1=\frac{\lambda_1k_3-\lambda_2k_1}{k_2k_3-k_1k_4},\end{aligned}$$

$$
\begin{aligned}
b_0 &= \frac{\Gamma(\upsilon_2+1)}{T^{\upsilon_2}}\Big[(1-\frac{B_1T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})\zeta_1+(\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)}-\frac{B_3T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{\lambda_1}{k_1}+\frac{k_1k_2\lambda_2-k_2k_3\lambda_1}{k_1(k_2k_3-k_1k_4)})\\
&+ (\frac{\delta_2\eta_2^{\upsilon_1+1}-B_4T^{\upsilon_2+1}}{\Gamma(\upsilon_1+2)})(\frac{\lambda_1k_3-\lambda_2k_1}{k_2k_3-k_1k_4})+\frac{\zeta_2B_2T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\Big],\\
b_1 &= B_1\zeta_1-B_2\zeta_2+B_3(\frac{\lambda_1}{k_1}+\frac{k_1k_2\lambda_2-k_2k_3\lambda_1}{k_1(k_2k_3-k_1k_4)})+B_4(\frac{\lambda_1k_3-\lambda_2k_1}{k_2k_3-k_1k_4}).
\end{aligned}
$$

Inserting the values of $a_i,\ b_i,\ i=0,1$ in (5) and (6), we get solutions (3) and (4).

3. Main Results

Let us introduce the space $Z=\{z(t)|z(t)\in C([0,T],R)\}$ endowed with the norm $\|z\|=\sup\{z(t),t\in[0,T]\}$. Obviously, $(Z,\|\cdot\|)$ is a Banach space. Then the product space $(Z\times Z,\|(z_1,z_2)\|)$ is also a Banach space equipped with the norm $\|(z_1,z_2)\|=\|z_1\|+\|z_2\|$.
From Lemma 2.1, we define the operator $\Phi:Z\times Z\to Z\times Z$ by

$$\Phi(z_1,z_2)(t)=\begin{pmatrix}\Phi_1(z_1,z_2)(t)\\ \Phi_2(z_1,z_2)(t)\end{pmatrix},$$

where

$$
\begin{aligned}
\Phi_1(z_1,z_2)(t) &= \Big[\frac{\lambda_{11}}{k_1}+\frac{k_1k_2\lambda_{22}-k_2k_3\lambda_{11}}{k_1(k_2k_3-k_1k_4)}\Big]\frac{t^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{\lambda_{11}k_3-\lambda_{22}k_1}{k_2k_3-k_1k_4}\frac{t^{(\upsilon_1+1)}}{\Gamma(\upsilon_1+2)}\\
&+ \int_0^t\frac{(t-s)^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1+\upsilon_1)}\Psi_1(s,z_1(s),z_2(s))ds-\chi_1\int_0^t\frac{(t-s)^{\upsilon_1-1}}{\Gamma(\upsilon_1)}z_1(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_2(z_1,z_2)(t) &= (\frac{t^{\upsilon_2}}{T^{\upsilon_2}}-\frac{B_1Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)}+\frac{B_1t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})\zeta_{11}+(\frac{B_2Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)}-\frac{B_2t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})\zeta_{22}\\
&+ (\frac{\delta_2\eta_2^{\upsilon_1}t^{\upsilon_2}}{T^{\upsilon_2}\Gamma(\upsilon_1+1)}-\frac{B_3Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)}+\frac{B_3t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{\lambda_{11}}{k_1}+\frac{k_1k_2\lambda_{22}-k_2k_3\lambda_{11}}{k_1(k_2k_3-k_1k_4)})\\
&+ (\frac{\delta_2\eta_2^{\upsilon_1+1}t^{\upsilon_2}}{T^{\upsilon_2}\Gamma(\upsilon_1+2)}-\frac{B_4Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)}+\frac{B_4t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{\lambda_{11}k_3-\lambda_{22}k_1}{k_2k_3-k_1k_4})\\
&+ \int_0^t\frac{(t-s)^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2+\upsilon_2)}\Psi_2(s,z_1(s),z_2(s))ds-\chi_2\int_0^t\frac{(t-s)^{\upsilon_2-1}}{\Gamma(\upsilon_2)}z_2(s)ds,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_{11} &= \frac{1}{\Gamma(\kappa_1+\upsilon_1)}\Big[\delta_1\int_0^{\eta_1}(\eta_1-s)^{\kappa_1+\upsilon_1-1}\Psi_1(s,z_1(s),z_2(s))ds\\
&- \int_0^T(T-s)^{\kappa_1+\upsilon_1-1}\Psi_1(s,z_1(s),z_2(s))ds\Big]\\
&- \frac{\chi_1}{\Gamma(\upsilon_1)}\Big[\delta_1\int_0^{\eta_1}(\eta_1-s)^{\upsilon_1-1}z_1(s)ds-\int_0^T(T-s)^{\upsilon_1-1}z_1(s)ds\Big],\\
\lambda_{22} &= \frac{1}{\Gamma(\kappa_1+\upsilon_1-1)}\Big[\epsilon_1\int_0^{\xi_1}(\xi_1-s)^{\kappa_1+\upsilon_1-2}\Psi_1(s,z_1(s),z_2(s))ds\\
&- \int_0^T(T-s)^{\kappa_1+\upsilon_1-2}\Psi_1(s,z_1(s),z_2(s))ds\Big]\\
&- \frac{\chi_1}{\Gamma(\upsilon_1-1)}\Big[\epsilon_1\int_0^{\xi_1}(\xi_1-s)^{\upsilon_1-2}z_1(s)ds-\int_0^T(T-s)^{\upsilon_1-2}z_1(s)ds\Big],
\end{aligned}
$$

$$\begin{aligned}
\zeta_{11} &= \frac{\chi_2}{\Gamma(\upsilon_2)}\int_0^T (T-s)^{\upsilon_2-1} z_2(s)ds - \frac{\delta_2\chi_1}{\Gamma(\upsilon_1)}\int_0^{\eta_2}(\eta_2-s)^{\upsilon_1-1}z_1(s)ds \\
&+ \frac{\delta_2}{\Gamma(\kappa_1+\upsilon_1)}\int_0^{\eta_2}(\eta_2-s)^{\kappa_1+\upsilon_1-1}\Psi_1(s,z_1(s),z_2(s))ds \\
&- \frac{1}{\Gamma(\kappa_2+\upsilon_2)}\int_0^T (T-s)^{\kappa_2+\upsilon_2-1}\Psi_2(s,z_1(s),z_2(s)),ds \\
\zeta_{22} &= \frac{\chi_2}{\Gamma(\upsilon_2-1)}\int_0^T (T-s)^{\upsilon_2-2} z_2(s)ds - \frac{\epsilon_2\chi_1}{\Gamma(\upsilon_1-1)}\int_0^{\xi_2}(\xi_2-s)^{\upsilon_1-2}z_1(s)ds \\
&+ \frac{\epsilon_2}{\Gamma(\kappa_1+\upsilon_1-1)}\int_0^{\xi_2}(\xi_2-s)^{\kappa_1+\upsilon_1-2}\Psi_1(s,z_1(s),z_2(s))ds \\
&- \frac{1}{\Gamma(\kappa_2+\upsilon_2-1)}\int_0^T (T-s)^{\kappa_2+\upsilon_2-2}\Psi_2(s,z_1(s),z_2(s))ds.
\end{aligned}$$

For convenience, we set

$$\begin{aligned}
\varrho_1 &= \frac{\delta_1\eta_1^{\kappa_1+\upsilon_1}+T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}, \quad \varrho_2 = \frac{\epsilon_1\xi_1^{\kappa_1+\upsilon_1-1}+T^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1+\upsilon_1)}, \\
\varrho_3 &= \frac{\delta_2\eta_2^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}, \quad \varrho_4 = \frac{T^{\kappa_2+\upsilon_2}}{\Gamma(\kappa_2+\upsilon_2+1)}, \\
\varrho_5 &= \frac{\epsilon_2\xi_2^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1+\upsilon_1)}, \quad \varrho_6 = \frac{T^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2+\upsilon_2)}, \\
M_1 &= C_1\varrho_1 + \frac{\chi_1(\delta_1\eta_1^{\upsilon_1}+T^{\upsilon_1})}{\Gamma(\upsilon_1+1)}, \quad M_2 = C_1\varrho_2 + \frac{\chi_1(\epsilon_1\xi_1^{\upsilon_1-1}+T^{\upsilon_1-1})}{\Gamma(\upsilon_1)}, \\
M_3 &= \frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2+1)} + \frac{\delta_2\chi_1\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + C_1\varrho_3 + C_2\varrho_4, \\
M_4 &= \frac{\chi_2T^{\upsilon_2-1}}{\Gamma(\upsilon_2)} + \frac{\epsilon_2\chi_1\xi_2^{\upsilon_1-1}}{\Gamma(\upsilon_1)} + C_1\varrho_5 + C_2\varrho_6.
\end{aligned}$$

In this paper, to prove the main results, we need the following assumption:
(H) $\Psi_1,\Psi_2:[0,T]\times R\times R\to R$ are continuous functions and there exist positive constants C_1 and C_2 such that for all $t\in[0,T]$ and $u_i,\ v_i,\ i=1,2$ we have

$$|\Psi_1(t,u_1,u_2)) - \Psi_1(t,v_1,v_2)| \le C_1(|u_1-v_1|+|u_2-v_2|),$$
$$|\Psi_2(t,u_1,u_2)) - \Psi_2(t,v_1,v_2)| \le C_2(|u_1-v_1|+|u_2-v_2|).$$

Theorem 3.1. If the assumption (H) is satisfied, then the system (1)-(2) has a unique solution provided that $\hbar_1+\hbar_2<1$ where

$$\begin{aligned}
\hbar_1 &= \frac{(M_1+\varrho_1)T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1+1)} + \frac{[|k_1k_2|(M_2+\varrho_2)+|k_2k_3|(M_1+\varrho_1)]T^{\upsilon_1}}{|k_1(k_2k_3-k_1k_4)|\Gamma(\upsilon_1+1)} \\
&+ \frac{[|k_3|(M_1+\varrho_1)+|k_1|(M_2+\varrho_2)]T^{(\upsilon_1+1)}}{(|k_2k_3-k_1k_4|)\Gamma(\upsilon_1+2)} + \frac{T^{\kappa_1+\upsilon_1}C_1}{\Gamma(\kappa_1+\upsilon_1+1)} + \frac{T^{\upsilon_1}\chi_1}{\Gamma(\upsilon_1+1)}, \\
\hbar_2 &= [1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\delta_2\chi_1\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2+1)} + C_1\varrho_3 + C_2\varrho_4] \\
&+ [\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\epsilon_2\chi_1\xi_2^{\upsilon_1-1}}{\Gamma(\upsilon_1)} + \frac{\chi_2T^{\upsilon_2-1}}{\Gamma(\upsilon_2)} + C_1\varrho_5 + C_2\varrho_6]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{M_1+\varrho_1}{|k_1|} + \frac{|k_1k_2|(M_2+\varrho_2)+|k_2k_3|(M_1+\varrho_1)}{|k_1(k_2k_3-k_1k_4)|}) \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{(M_1+\varrho_1)|k_3|+(M_2+\varrho_2)|k_1|}{|k_2k_3-k_1k_4|}) + \frac{T^{\kappa_2+\upsilon_2}C_2}{\Gamma(\kappa_2+\upsilon_2+1)}.
\end{aligned}
$$

Proof. Define $\sup_{t\in[0,T]}|\Psi_1(t,0,0)| = \aleph_1 < \infty$ and $\sup_{t\in[0,1]}|\Psi_2(t,0,0)| = \aleph_2 < \infty$ and $r > 0$ such that

$$
r > \frac{\rho_2\aleph_1+\rho_3\aleph_2}{1-\rho_1},
$$

where

$$
\begin{aligned}
\rho_1 &= \frac{M_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1+1)} + \frac{(|k_1k_2|M_2+|k_2k_3|M_1)T^{\upsilon_1}}{|k_1(k_2k_3-k_1k_4)|\Gamma(\upsilon_1+1)} + \frac{(|k_3|M_1+|k_1|M_2)T^{\upsilon_1+1}}{|k_2k_3-k_1k_4|\Gamma(\upsilon_1+2)} \\
&+ \frac{C_1T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)} + \frac{\chi_1T^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + [1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]M_3 + \frac{2M_4|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)} \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{M_1}{|k_1|} + \frac{|k_1k_2|M_2+|k_2k_3|M_1}{|k_1(k_2k_3-k_1k_4)|}] \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{|k_3|M_1+|k_1|M_2}{|k_2k_3-k_1k_4|}] + \frac{T^{\kappa_2+\upsilon_2}C_2}{\Gamma(\kappa_2+\upsilon_2+1)} + \frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2+1)}, \\
\rho_2 &= \frac{\varrho_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1+1)} + \frac{(|k_1k_2|\varrho_2+|k_2k_3|\varrho_1)T^{\upsilon_1}}{|k_1(k_2k_3-k_1k_4)|\Gamma(\upsilon_1+1)} + \frac{(|k_3|\varrho_1+|k_1|\varrho_2)T^{\upsilon_1+1}}{|k_2k_3-k_1k_4|\Gamma(\upsilon_1+2)} \\
&+ \frac{T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)} + [1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\varrho_3 + \frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\varrho_5 \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\varrho_1}{|k_1|} + \frac{|k_1k_2|\varrho_2+|k_2k_3|\varrho_1}{|k_1(k_2k_3-k_1k_4)|}] \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{|k_3|\varrho_1+|k_1|\varrho_2}{|k_2k_3-k_1k_4|}], \\
\rho_3 &= [1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\varrho_4 + \frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\varrho_6.
\end{aligned}
$$

Let $B_r = \{(z_1(t), z_2(t)) \in Z\times Z : \sup_{t\in[0,T]}|(z_1(t), z_2(t))| = \|(z_1,z_2)\| \le r\}$.
We show that $\Phi B_r \subset B_r$.
By assumption (H), for $(z_1,z_2)\in B_r$, $t\in[0,T]$, we have

$$
\begin{aligned}
\sup_{t\in[0,T]}|\Psi_i(t,z_1(t),z_2(t))| &\le \sup_{t\in[0,T]}|\Psi_i(t,z_1(t),z_2(t)) - \Psi_i(t,0,0)| + \sup_{t\in[0,T]}|\Psi_i(t,0,0)| \\
&\le C_i(\sup_{t\in[0,T]}|z_1(t)| + \sup_{t\in[0,T]}|z_2(t)|) + \aleph_i \\
&\le C_ir + \aleph_i,\ i = 1,2
\end{aligned}
$$

which lead to

$$
\begin{aligned}
\|\lambda_{11}\| &\le M_1r + \varrho_1\aleph_1, \\
\|\lambda_{22}\| &\le M_2r + \varrho_2\aleph_1, \\
\|\zeta_{11}\| &\le M_3r + \varrho_3\aleph_1 + \varrho_4\aleph_2, \\
\|\zeta_{22}\| &\le M_4r + \varrho_5\aleph_1 + \varrho_6\aleph_2.
\end{aligned}
$$

Hence

$$\begin{aligned}
\|\Phi_1(z_1,z_2)\| &= \sup_{t\in[0,T]} |\Phi_1(z_1,z_2)(t)| \\
&= \sup_{t\in[0,T]} \Big|[\frac{\lambda_{11}}{k_1} + \frac{k_1k_2\lambda_{22} - k_2k_3\lambda_{11}}{k_1(k_2k_3 - k_1k_4)}]\frac{t^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{\lambda_{11}k_3 - \lambda_{22}k_1}{k_2k_3 - k_1k_4}\frac{t^{(\upsilon_1+1)}}{\Gamma(\upsilon_1+2)} \\
&+ \int_0^t \frac{(t-s)^{\kappa_1+\upsilon_1-1}}{\Gamma(\kappa_1+\upsilon_1)}\Psi_1(s,z_1(s),z_2(s))ds - \chi_1\int_0^t \frac{(t-s)^{\upsilon_1-1}}{\Gamma(\upsilon_1)}z_1(s)ds\Big| \\
&\le \Big[\frac{\|\lambda_{11}\|}{|k_1|} + \frac{|k_1k_2|\|\lambda_{22}\| + |k_2k_3|\|\lambda_{11}\|}{|k_1(k_2k_3 - k_1k_4)|}\Big]\frac{T^{\upsilon_1}}{\Gamma(\upsilon_1+1)} \\
&+ \Big[\frac{\|\lambda_{11}\||k_3| + \|\lambda_{22}\||k_1|}{|k_2k_3 - k_1k_4|}\Big]\frac{T^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} + \frac{T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}[C_1r + \aleph_1] + \chi_1\frac{T^{\upsilon_1}r}{\Gamma(\upsilon_1+1)} \\
&\le \Big[\frac{M_1r + \varrho_1\aleph_1}{|k_1|} + \frac{|k_1k_2|(M_2r + \varrho_2\aleph_1) + |k_2k_3|(M_1r + \varrho_1\aleph_1)}{|k_1(k_2k_3 - k_1k_4)|}\Big]\frac{T^{\upsilon_1}}{\Gamma(\upsilon_1+1)} \\
&+ \Big[\frac{(M_1r + \varrho_1\aleph_1)|k_3| + (M_2r + \varrho_2\aleph_1)|k_1|}{|k_2k_3 - k_1k_4|}\Big]\frac{T^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} \\
&+ \frac{T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}[C_1r + \aleph_1] + \chi_1\frac{T^{\upsilon_1}r}{\Gamma(\upsilon_1+1)} \\
&\le \Big\{\frac{M_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1+1)} + \frac{(|k_1k_2|M_2 + |k_2k_3|M_1)T^{\upsilon_1}}{|k_1(k_2k_3 - k_1k_4)|\Gamma(\upsilon_1+1)} + \frac{(|k_3|M_1 + k_1M_2)T^{\upsilon_1+1}}{|k_2k_3 - k_1k_4|\Gamma(\upsilon_1+2)} \\
&+ \frac{C_1T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)} + \frac{\chi_1T^{\upsilon_1}}{\Gamma(\upsilon_1+1)}\Big\}r \\
&+ \Big\{\frac{\varrho_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1+1)} + \frac{(|k_1k_2|\varrho_2 + |k_2k_3|\varrho_1)T^{\upsilon_1}}{|k_1(k_2k_3 - k_1k_4)|\Gamma(\upsilon_1+1)} + \frac{(|k_3|\varrho_1 + |k_1|\varrho_2)T^{\upsilon_1+1}}{|k_2k_3 - k_1k_4|\Gamma(\upsilon_1+2)} \\
&+ \frac{T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}\Big\}\aleph_1,
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi_2(z_1,z_2)\| &= \sup_{t\in[0,T]} |\Phi_2(z_1,z_2)(t)| \\
&= \sup_{t\in[0,T]} \Big|[\frac{t^{\upsilon_2}}{T^{\upsilon_2}} - \frac{B_1Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)} + \frac{B_1t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\zeta_{11} + [\frac{B_2Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)} - \frac{B_2t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\zeta_{22} \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1}t^{\upsilon_2}}{T^{\upsilon_2}\Gamma(\upsilon_1+1)} - \frac{B_3Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)} + \frac{B_3t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\lambda_{11}}{k_1} + \frac{k_1k_2\lambda_{22} - k_2k_3\lambda_{11}}{k_1(k_2k_3 - k_1k_4)}] \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1+1}t^{\upsilon_2}}{T^{\upsilon_2}\Gamma(\upsilon_1+2)} - \frac{B_4Tt^{\upsilon_2}}{\Gamma(\upsilon_2+2)} + \frac{B_4t^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\lambda_{11}k_3 - \lambda_{22}k_1}{k_2k_3 - k_1k_4}] \\
&+ \int_0^t \frac{(t-s)^{\kappa_2+\upsilon_2-1}}{\Gamma(\kappa_2+\upsilon_2)}\Psi_2(s,z_1(s),z_2(s))ds - \chi_2\int_0^t \frac{(t-s)^{\upsilon_2-1}}{\Gamma(\upsilon_2)}z_2(s)ds\Big| \\
&\le [1 + \frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\|\zeta_{11}\| + \frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\|\zeta_{22}\| \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\|\lambda_{11}\|}{|k_1|} + \frac{|k_1k_2|\|\lambda_{22}\| + |k_2k_3|\|\lambda_{11}\|}{|k_1(k_2k_3 - k_1k_4)|}] \\
&+ [\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\|\lambda_{11}\|k_3 + \|\lambda_{22}\||k_1|}{|k_2k_3 - k_1k_4|}] \\
&+ \frac{T^{\kappa_2+\upsilon_2}}{\Gamma(\kappa_2+\upsilon_2+1)}[C_2r + \aleph_2] + \frac{\chi_2T^{\upsilon_2}r}{\Gamma(\upsilon_2+1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq && [1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][M_3 r+\varrho_3\aleph_1+\varrho_4\aleph_2]+\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}[M_4 r+\varrho_5\aleph_1+\varrho_6\aleph_2]\\
&+ && [\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{M_1 r+\varrho_1\aleph_1}{|k_1|}+\frac{|k_1k_2|[M_2r+\varrho_2\aleph_1]+|k_2k_3|[M_1r+\varrho_1\aleph_1]}{|k_1(k_2k_3-k_1k_4)|}]\\
&+ && [\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)}+\frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{[M_1r+\varrho_1\aleph_1]k_3+[M_2r+\varrho_2\aleph_1]|k_1|}{|k_2k_3-k_1k_4|}]\\
&+ && \frac{T^{\kappa_2+\upsilon_2}}{\Gamma(\kappa_2+\upsilon_2+1)}[C_2r+\aleph_2]+\frac{\chi_2T^{\upsilon_2}r}{\Gamma(\upsilon_2+1)}\\
&\leq && \Big\{[1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]M_3+\frac{2M_4|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}+[\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\\
&\times && [\frac{M_1}{|k_1|}+\frac{|k_1k_2|M_2+|k_2k_3|M_1}{|k_1(k_2k_3-k_1k_4)|}]+[\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)}+\frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{|k_3|M_1+k_1M_2}{|k_2k_3-k_1k_4|}]\\
&+ && \frac{T^{\kappa_2+\upsilon_2}C_2}{\Gamma(\kappa_2+\upsilon_2+1)}+\frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2+1)}\Big\}r\\
&+ && \Big\{[1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\varrho_3+\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\varrho_5+[\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{2B_3T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\\
&\times && [\frac{\varrho_1}{k_1}+\frac{k_1k_2\varrho_2+|k_2k_3|\varrho_1}{|k_1(k_2k_3-k_1k_4)|}]+[\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)}+\frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{|k_3|\varrho_1+|k_1|\varrho_2}{|k_2k_3-k_1k_4|}]\Big\}\aleph_1\\
&+ && \Big\{[1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\varrho_4+\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\varrho_6\Big\}\aleph_2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|\Phi(z_1,z_2)\| &\leq && \Big\{\frac{M_1T^{\upsilon_1}}{k_1\Gamma(\upsilon_1+1)}+\frac{(|k_1k_2|M_2+|k_2k_3|M_1)T^{\upsilon_1}}{|k_1(k_2k_3-k_1k_4)|\Gamma(\upsilon_1+1)}+\frac{(|k_3|M_1+|k_1|M_2)T^{\upsilon_1+1}}{|k_2k_3-k_1k_4|\Gamma(\upsilon_1+2)}\\
&+ && \frac{C_1T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}+\frac{\chi_1T^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+[1+\frac{2B_1T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]M_3+\frac{2M_4|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\\
&+ && [\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{M_1}{|k_1|}+\frac{|k_1k_2|M_2+|k_2k_3|M_1}{|k_1(k_2k_3-k_1k_4)|}]\\
&+ && [\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)}+\frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{|k_3|M_1+|k_1|M_2}{|k_2k_3-k_1k_4|}]+\frac{T^{\kappa_2+\upsilon_2}C_2}{\Gamma(\kappa_2+\upsilon_2+1)}+\frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2+1)}\Big\}r\\
&+ && \Big\{\frac{\varrho_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1+1)}+\frac{(|k_1k_2|\varrho_2+|k_2k_3|\varrho_1)T^{\upsilon_1}}{|k_1(k_2k_3-k_1k_4)|\Gamma(\upsilon_1+1)}+\frac{(|k_3|\varrho_1+|k_1|\varrho_2)T^{\upsilon_1+1}}{|k_2k_3-k_1k_4|\Gamma(\upsilon_1+2)}\\
&+ && \frac{T^{\kappa_1+\upsilon_1}}{\Gamma(\kappa_1+\upsilon_1+1)}+[1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\varrho_3+\frac{2B_2T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\varrho_5\\
&+ && [\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)}+\frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\varrho_1}{|k_1|}+\frac{|k_1k_2|\varrho_2+|k_2k_3|\varrho_1}{|k_1(k_2k_3-k_1k_4)|}]\\
&+ && [\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)}+\frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{|k_3|\varrho_1+|k_1|\varrho_2}{|k_2k_3-k_1k_4|}]\Big\}\aleph_1\\
&+ && \Big\{[1+\frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}]\varrho_4+\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}\varrho_6\Big\}\aleph_2\\
&= && \rho_1r+\rho_2\aleph_1+\rho_3\aleph_2\leq r.
\end{aligned}
$$

For (z_1,z_2), $(\bar{z_1},\bar{z_2})\in Z\times Z$ and for any $t\in[0,T]$, we get

$$
\|\Phi_1(\bar{z_1},\bar{z_2})-\Phi_1(z_1,z_2)\| \leq \frac{T^{\upsilon_1}}{\Gamma(\upsilon_1+1)}\Big[\frac{M_1\|\bar{z_1}-z_1\|+\varrho_1\|\bar{z_2}-z_2\|}{k_1}
$$

$$
\begin{aligned}
&+ \quad \frac{[|k_1k_2|M_2 + |k_2k_3|M_1]\|\bar{z_1} - z_1\| + [|k_1k_2|\varrho_2 + |k_2k_3|\varrho_1]\|\bar{z_2} - z_2\|]}{|k_1(k_2k_3 - k_1k_4)|}\Big] \\
&+ \quad \Big[\frac{[|k_3|M_1 + |k_1|M_2]\|\bar{z_1} - z_1\| + [|k_3|\varrho_1 + |k_1|\varrho_2]\|\bar{z_2} - z_2\|}{|k_2k_3 - k_1k_4|}\Big]\frac{T^{(\upsilon_1+1)}}{\Gamma(\upsilon_1 + 2)} \\
&+ \quad \frac{T^{\kappa_1+\upsilon_1}C_1[\|\bar{z_1} - z_1\| + \|\bar{z_2} - z_2\|]}{\Gamma(\kappa_1 + \upsilon_1 + 1)} + \frac{T^{\upsilon_1}\chi_1\|\bar{z_1} - z_1\|}{\Gamma(\upsilon_1 + 1)} \\
&\leq \quad \Big\{\frac{M_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1 + 1)} + \frac{[|k_1k_2|M_2 + |k_2k_3|M_1]T^{\upsilon_1}}{|k_1(k_2k_3 - k_1k_4)|\Gamma(\upsilon_1 + 1)} \\
&+ \quad \frac{[|k_3|M_1 + |k_1|M_2]T^{(\upsilon_1+1)}}{|k_2k_3 - k_1k_4|\Gamma(\upsilon_1 + 2)} + \frac{T^{\kappa_1+\upsilon_1}C_1}{\Gamma(\kappa_1 + \upsilon_1 + 1)} + \frac{T^{\upsilon_1}\chi_1}{\Gamma(\upsilon_1 + 1)}\Big\}\|\bar{z_1} - z_1\| \\
&+ \quad \Big\{\frac{\varrho_1T^{\upsilon_1}}{|k_1|\Gamma(\upsilon_1 + 1)} + \frac{[|k_1k_2|\varrho_2 + |k_2k_3|\varrho_1]T^{\upsilon_1}}{|k_1(k_2k_3 - k_1k_4)|\Gamma(\upsilon_1 + 1)} \\
&+ \quad \frac{[|k_3|\varrho_1 + |k_1|\varrho_2]T^{(\upsilon_1+1)}}{|k_2k_3 - k_1k_4|\Gamma(\upsilon_1 + 2)} + \frac{T^{\kappa_1+\upsilon_1}C_1}{\Gamma(\kappa_1 + \upsilon_1 + 1)}\Big\}\|\bar{z_2} - z_2\| \\
&\leq \quad \hbar_1(\|\bar{z_1} - z_1\| + \|\bar{z_2} - z_2\|), \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\|\Phi_2(\bar{z_1}, \bar{z_2}) - \Phi_2(z_1, z_2)\| &\leq \quad \Big(1 + \frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)}\Big)\Big([\frac{\delta_2\chi_1\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1 + 1)} + C_1\varrho_3 + C_2\varrho_4]\|\bar{z_1} - z_1\| \\
&+ \quad [\frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2 + 1)} + C_1\varrho_3 + C_2\varrho_4]\|\bar{z_2} - z_2\|\Big) \\
&+ \quad \Big(\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)}\Big)\Big([\frac{\epsilon_2\chi_1\xi_2^{\upsilon_1-1}}{\Gamma(\upsilon_1)} + C_1\varrho_5 + C_2\varrho_6]\|\bar{z_1} - z_1\| \\
&+ \quad [\frac{\chi_2T^{\upsilon_2-1}}{\Gamma(\upsilon_2)} + C_1\varrho_5 + C_2\varrho_6]\|\bar{z_2} - z_2\|\Big) \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1 + 1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)})(\frac{M_1\|\bar{z_1} - z_1\| + \varrho_1\|\bar{z_2} - z_2\|}{|k_1|} \\
&+ \quad \frac{k_1k_2[M_2\|\bar{z_1} - z_1\| + \varrho_2\|\bar{z_2} - z_2\|] + |k_2|k_3[M_1\|\bar{z_1} - z_1\| + \varrho_1\|\bar{z_2} - z_2\|]}{|k_1(k_2k_3 - k_1k_4)|}) \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1 + 2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)}) \\
&\times \quad (\frac{[M_1\|\bar{z_1} - z_1\| + \varrho_1\|\bar{z_2} - z_2\|]|k_3| + [M_2\|\bar{z_1} - z_1\| + \varrho_2\|\bar{z_2} - z_2\|]|k_1|}{|k_2k_3 - k_1k_4|}) \\
&+ \quad \frac{T^{\kappa_2+\upsilon_2}C_2[\|\bar{z_1} - z_1\| + \|\bar{z_2} - z_2\|]}{\Gamma(\kappa_2 + \upsilon_2 + 1)} + \frac{\chi_2T^{\upsilon_2}\|\bar{z_2} - z_2\|}{\Gamma(\upsilon_2 + 1)} \\
&\leq \quad \Big\{[1 + \frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)}][\frac{\delta_2\chi_1\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1 + 1)} + C_1\varrho_3 + C_2\varrho_4] \\
&+ \quad [\frac{2|B_2|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)}][\frac{\epsilon_2\chi_1\xi_2^{\upsilon_1-1}}{\Gamma(\upsilon_1)} + C_1\varrho_5 + C_2\varrho_6] \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1 + 1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)})(\frac{M_1}{|k_1|} + \frac{|k_1k_2|M_2 + |k_2k_3|M_1}{|k_1(k_2k_3 - k_1k_4)|}) \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1 + 2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)})(\frac{M_1|k_3| + M_2|k_1|}{|k_2k_3 - k_1k_4|}) + \frac{T^{\kappa_2+\upsilon_2}C_2}{\Gamma(\kappa_2 + \upsilon_2 + 1)}\Big\}\|\bar{z_1} - z_1\| \\
&+ \quad \Big\{[1 + \frac{2|B_1|T^{\upsilon_2+1}}{\Gamma(\upsilon_2 + 2)}][\frac{\chi_2T^{\upsilon_2}}{\Gamma(\upsilon_2 + 1)} + C_1\varrho_3 + C_2\varrho_4]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad [\frac{2B_2T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)}][\frac{\chi_2 T^{\upsilon_2-1}}{\Gamma(\upsilon_2)} + C_1\varrho_5 + C_2\varrho_6] \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1}}{\Gamma(\upsilon_1+1)} + \frac{2|B_3|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{\varrho_1}{|k_1|} + \frac{|k_1k_2|\varrho_2 + |k_2k_3|\varrho_1}{|k_1(k_2k_3-k_1k_4)|}) \\
&+ \quad (\frac{\delta_2\eta_2^{\upsilon_1+1}}{\Gamma(\upsilon_1+2)} + \frac{2|B_4|T^{\upsilon_2+1}}{\Gamma(\upsilon_2+2)})(\frac{\varrho_1|k_3| + \varrho_2|k_1|}{|k_2k_3-k_1k_4|}) \\
&+ \quad \frac{T^{\kappa_2+\upsilon_2}C_2}{\Gamma(\kappa_2+\upsilon_2+1)} + \frac{\chi_2 T^{\upsilon_2}}{\Gamma(\upsilon_2+1)}\Big\}\|\bar{z_2} - z_2\| \\
&\leq \quad \hbar_2(\|\bar{z_1} - z_1\| + \|\bar{z_2} - z_2\|). \qquad (8)
\end{aligned}
$$

From (7) and (8), we obtain

$$\|\Phi(\bar{z_1},\bar{z_2}) - \Phi(z_1,z_2)\| \quad \leq \quad (\hbar_1 + \hbar_2)(\|\bar{z_1} - z_1\| + \|\bar{z_2} - z_2\|).$$

Since $\hbar_1 + \hbar_2 < 1$, therefore, Φ is a contraction operator. So, from Banach's fixed point theorem, the operator Φ has a unique fixed point, which is the unique solution of system (1), (2). This completes the proof.

4. Example

Consider the following coupled system of nonlinear fractional Langevin equation

$$
\begin{cases}
{}^cD_{0+}^{1.5}({}^cD_{0+}^{1.5} + 0.1)z_1(t) = \frac{t^2|z_1(t)|^2}{5(t+3)^2)(1+|z_1(t)|^2)} + \frac{1}{30}e^{z_2(t)}, \ t \in J := [0, 0.01], \\
{}^cD_{0+}^{1.5}({}^cD_{0+}^{1.5} + 0.1)z_2(t) = \frac{1}{50}cos(2\pi z_1(t)) + \frac{t^2|z_2(t)|^2}{25(t+1)^2)(1+|z_2(t)|^2)}, \ t \in J := [0, 0.01]
\end{cases} \qquad (9)
$$

subject to the following coupled boundary conditions

$$
\begin{cases}
z_1(0) = 0, \ z_1(0.01) = 0.03z_1(0.02), \ z_1'(0.01) = 0.05z_1'(0.04) \\
z_2(0) = 0, \ z_2(0.01) = 0.03z_1(0.02), \ z_2'(0.01) = 0.05z_1'(0.04)
\end{cases} \qquad (10)
$$

Here, $\kappa_1 = \kappa_2 = \upsilon_1 = \upsilon_2 = 1.5, \ \chi_1 = \chi_2 = 0.1, \ \delta_1 = \delta_2 = 0.03, \ \epsilon_1 = \epsilon_2 = 0.05,$ $\eta_1 = \eta_2 = 0.02, \ \xi_1 = \xi_2 = 0.04, \ \Psi_1(t, z_1(t), z_2(t)) = \frac{t^2|z_1(t)|^2}{5(t+3)^2)(1+|z_1(t)|^2)} + \frac{1}{30}e^{z_2(t)}$ and $\Psi_2(t, z_1(t), z_2(t)) = \frac{1}{50}cos(2\pi z_1(t)) + \frac{(t)^2|z_2(t)|^2}{25(t+1)^2)(1+|z_2(t)|^2)}$. Take $C_1 = C_2 = 0.01$. With given data, we find that

$$
\begin{aligned}
k_1 &= 6.88\times10^{-4}, \ k_2 = 2.4\times10^{-6}, \ k_3 = 0.1, \ k_4 = 4.5\times10^{-4} \\
\varrho_1 &= 2.066\times10^{-7}, \ \varrho_2 = 1.3\times10^{-5}, \ \varrho_3 = 4\times10^{-8}, \varrho_4 = 1.66\times10^{-7}, \\
\varrho_5 &= 8\times10^{-6}, \varrho_4 = 5\times10^{-6} \\
|B_1| &= 199350, \ |B_2| = 3322.5, \ |B_3| = 3.78\times10^{-4}, \ |B_4| = 2.83\times10^{-6}, \\
M_1 &= 7.588\times10^{-4}, \ M_2 = 2.26\times10^{-3}, \\
\hbar_1 &= 7.43\times10^{-4}, \quad \hbar_2 = 8.3255\times10^{-4}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&|\Psi_1(t, u_1, u_2)) - \Psi_1(t, v_1, v_2)| \leq 0.01(|u_1 - v_1| + |u_2 - v_2|), \\
&|\Psi_2(t, u_1, u_2)) - \Psi_2(t, v_1, v_2)| \leq 0.01(|u_1 - v_1| + |u_2 - v_2|),
\end{aligned}
$$

and $\hbar_1 + \hbar_2 = 7.43\times10^{-4} + 8.3255\times10^{-4} = 1.576\times10^{-3} < 1$. Thus all conditions of theorem 3.1 are satisfied. Hence, the system (9), (10) has a unique solution on $[0, 0.01]$.

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