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An Extension of the Generalized Admissible S-Algorithm for Multivalued Type Contractions in Graphic Banach Spaces with Application to Image Recovery Problem

A. M. Saddeek[a], N. Hussain[b]

[a]*Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt*
[b]*Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Abstract. The purpose of this paper is to introduce a new extension of the generalized admissible S-algorithm for approximating common fixed point of three multivalued mappings satisfying two general classes of contraction conditions in a uniformly convex Banach space endowed with a graph. As an application of our result we establish the solution of image recovery problem in Hilbert space setting.

1. Introduction

The development of algorithms for approximating common fixed points of nonlinear contraction mappings has historically been a significant enterprise.
The Ishikawa iteration algorithm associated with two mappings, e.g. ([12, 15, 29], [14] in single valued and multivalued setting, respectively) is one of the most popular methods to approximate a common fixed point of a pair of mappings.
The Ishikawa iteration algorithm in the form of three single valued mappings has been considered by Ghosh and Debnath [12] and Rashwan and Saddeek [30]. Originally, the Ishikawa iteration algorithm for a single valued mapping has been introduced by Ishikawa [18].
It is worth mentioning that modifying the Ishikawa iteration scheme is an important approach for approximating common fixed points of three contraction mappings.
In 2015, Saluja [36] introduced the modified S-algorithm for two single valued mappings. Originally, the S-algorithm of a single valued mapping has been introduced by Agarwal et al. [2].
Subsequently, by using the concept of admissible mappings, Saddeek and Ahmed [35] introduced a new generalized S-algorithm for three single valued contraction mappings (one of them is known as generalized weakly contraction, e.g. [13]).
In 2018, Saddeek and Ahmed [35] studied the convergence of the sequence generated by the new generalized S-algorithm method to common fixed points of these types of mappings in uniformly convex Banach spaces. These results extend and unify a number of existing results, see ([5, 11, 15, 26, 30, 31, 38]).
Recently, the combination of graph theory and fixed point theory has emerged as a new direction for the

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Email addresses: saddeek@aun.edu.eg (A. M. Saddeek), nhusain@kau.edu.sa (N. Hussain)

study of existence of the fixed points in partially ordered spaces, e.g. ([10, 24, 25, 28, 39]).
Approximating common fixed points for generalized multivalued contraction mappings in uniformly convex Banach spaces endowed with a directed graph is a fascinating aspect that has a wide range of applications in several practical areas.
In this paper, we extend the approach of Saddeek and Ahmed [35] to more generalized multivalued contractions on a uniformly convex Banach space with a graph. Furthermore, we apply the new result to solve the image recovery problem.

Definition 1.1. *[32] A map $\varphi : [0, +\infty) \to [0, +\infty)$ is called a comparison function if the following axioms hold: (i) φ is monotone increasing, (ii) $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \geq 0$, where φ^n is the nth iterate of φ.*

It is well known that if φ is a comparison function, then $\varphi(t) < t$ for each $t > 0$ and φ is continuous at 0. For further details of φ on partial ordered spaces, the reader is refered to Hussain et al. [16].

Let C be a nonempty convex subset of a real Banach space X. Suppose that R, S and T are three self mappings of C into itself. Let $m(x, y)$ (resp., $n(x, y)$) denote the maximum (resp., minimum) of $M^3_T(x, y)$ (resp., $N^2_{R,S}(x, y)$), where

$$M^3_T(x, y) = \{\|x - y\|, \frac{[\|x - Tx\| + \|y - Ty\|]}{2}, \frac{[\|x - Ty\| + \|y - Tx\|]}{2}\},$$

$$N^2_{R,S}(x, y) = \{\|Sx - y\|, \|x - Ry\|\}.$$

We use $F(T)$ (resp., $F(R, S, T)$) to denote the set of fixed points of T (resp., common fixed points of R, S, T).

Definition 1.2. *[13] A map $T : C \to C$ is said to be a generalized weakly contraction if there exists a comparison function φ such that*

$$\|Tx - Ty\| \leq \varphi(m(x, y)), \ \forall x, y \in C. \tag{1.1}$$

Let us assume that the mappings R and S satisfy the following conditions:

$$\|Sx - Ry\| \leq n(x, y), \ \forall x, y \in C. \tag{1.2}$$

$$\lim_{n\to\infty} \|Rx_n - Sx_n\| = 0, \forall x_n \in C, n \geq 0. \tag{1.3}$$

For $n \geq 0$, let $\{\alpha^{(i)}_n\}$ and $\{\beta^{(i)}_n\}$, $i = 1, 2, 3$, be two real sequences in $[0, 1]$ satisfying the following conditions: (c_1) there are constants $a, b \in (0, 1)$ with $0 < a \leq \alpha^{(i)}_n \leq b < 1$, $\sum_{i=1}^3 \alpha^{(i)}_n = 1$, (c_2) $0 \leq \beta^{(i)}_n \leq 1$, $\sum_{i=1}^3 \beta^{(i)}_n = 1$ and (c_3) $\limsup_{n\to\infty} \beta^{(i)}_n < 1$.

Rashwan and Saddeek [30] proved that if R, S and T are three self mappings of a closed convex subset C of a uniformly convex Banach space X such that
(i) T is a generalized weakly contraction mapping with a comparison function $\varphi(t) = t$,
(ii) R and S satisfy the conditions (1.2) and (1.3),
(iii) $F(R, S, T) \neq \emptyset$,
(iv) one of $R(C), S(C)$ and $T(C)$ is relatively compact.

Then the sequence $\{x_n\}$ defined iteratively by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha^{(1)}_n Rx_n + \alpha^{(2)}_n Sx_n + \alpha^{(3)}_n Ty_n, \ n \geq 0 \\ y_n = \beta^{(1)}_n Rx_n + \beta^{(2)}_n Sx_n + \beta^{(3)}_n Tx_n \ \ n \geq 0, \end{cases} \tag{1.4}$$

converges to an element of $F(R,S,T)$.

The iteration algorithm (1.4) is another type of generalization of the Ishikawa iteration algorithm. Indeed, if $R = S$ (resp., $R = S = I$), then the iteration algorithm (1.4) reduces to the the generalized Ishikawa (resp., standard Ishikawa) iteration process, e.g. Huang and Jeng [15] and Ishikawa [18], respectively.

The result of Rashwan and Saddeek [30] provides a unifying framework for some fixed point results of Huang and Jeng [15], Rhoades [31], Osilike [26], Tiwary and Debnath [38].

In 2018, Saddeek and Ahmed [35] proposed the following modified iteration algorithm:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n^{(1)} R x_n + \alpha_n^{(2)} S x_n + \alpha_n^{(3)} T y_n, \ n \geq 0 \\ y_n = (1-\beta_n^{(3)}) x_n + \beta_n^{(3)} T x_n, \ \ n \geq 0, \end{cases} \tag{1.5}$$

where $\{\alpha_n^{(i)}\}$, $i = 1,2,3$ and $\{\beta_n^{(3)}\}$ are real sequences in $(0,1)$ satisfying the following conditions:
$(c_1^{'})$ there are constants $a, b \in (0,1)$ with $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $\sum_{i=1}^{3} \alpha_n^{(i)} = 1$,
$(c_2^{'})$ there is a constant $\beta \in (0,1)$ with $0 \leq \beta_n^{(3)} \leq \beta < 1$ and
$(c_3^{'})$ $\limsup_{n\to\infty} \beta_n^{(3)} \leq 1$.

Especially, if $R = S = T$ (resp., $R = S = I$), then the iteration process (1.5) reduces to the $\mathbb{S}$-iteration (resp., Ishikawa iteration) process, e.g. Agarwal et al. [2] and Ishikawa [18], respectively.

Recently, Rus [33] suggested a new approach to fixed point iteration methods for certain classes of mappings. The novelty of the proposed approach is the usage of admissible perturbations theory. It is worth to mention that this theory opened a new topic of research that unifies the most prominent and useful aspects of the iteration techniques for approximating fixed points of single and multivalued self mappings.
Let X_1 be a nonempty set and let X_2 denote the vector space over the field of real numbers.

Definition 1.3. *[33] A map $G_1 : X_1 \times X_1 \to X_1$ is called an admissible map if*
(i) $G_1(x,x) = x$,
(ii) $G_1(x,y) = x \Rightarrow x = y$, $\forall\, x, y \in X_1$.

Definition 1.4. *[35] A map $G_2 : X_1 \times X_1 \times X_1 \to X_1$ is called a generalized admissible map if*
(i) $G_2(x,x,x) = x$,
(ii) $G_2(x,y,z) = x \Rightarrow x = y = z$, $\forall\, x, y, z \in X_1$.

Definition 1.5. *[33] Let $T : X_1 \to X_1$ and $G_1 : X_1 \times X_1 \to X_1$ be two mappings. Then the mapping $T_{G_1} : X_1 \to X_1$ defined by $T_{G_1}(x) = G_1(x, Tx)$ for each $x \in X_1$ is called an admissible perturbation of T corresponding to G_1 if the mapping G_1 is admissible.*

Definition 1.6. *[35] Let $R, S, T : X_1 \to X_1$ and $G_2 : X_1 \times X_1 \times X_1 \to X_1$ be nonlinear self mappings. Then the mapping $T_{G_2}^{R,S} : X_1 \times X_1 \to X_1$ defined by $T_{G_2}^{R,S}(x,y) = G_2(Rx, Sx, Ty)$ for each $x, y \in X_1$ is called a generalized admissible perturbation of R, S and T corresponding to G_2 if the mapping G_2 is generalized admissible.*

Clearly, if $T_{G_2}^{R,S}$ maps X_1 to itself (i.e, the case of one variable) and $R = S = I$, then the Definition 1.6 reduces to the Definition 1.5.

Example 1.1. *[35] Let X_1 be a convex subset of X_2 and let $R, S, T : X_1 \to X_1$ be nonlinear self mappings. Then the mapping $G_n^{'} : X_1 \times X_1 \times X_1 \to X_1$, $n \geq 0$ defined by*

$$T_{G_n^{'}}^{R,S}(\cdot, \diamond) = G_n^{'}(R(\cdot), S(\cdot), T(\diamond)) = \alpha_n^{(1)} R(\cdot) + \alpha_n^{(2)} S(\cdot) + \alpha_n^{(3)} T(\diamond), \ \ \forall n \geq 0,$$

where $\alpha_n^{(i)} \in [0,1)$, $i = 1,2,3$, is a generalized admissible perturbation for R, S, T and $F(T_{G_n^{'}}^{R,S}) = F(R,S,T)$.

Example 1.2. *[5] Let X_1 be as in Example 1.1 and let T be a nonlinear self mapping of X_1 into itself. Then the mapping $G_n : X_1 \times X_1 \to X_1, n \geq 0$ defined by*

$$T_{G_n}(\cdot) = (1 - \beta_n^{(3)})I(\cdot) + \beta_n^{(3)}T(\cdot), \quad \forall n \geq 0,$$

where $\beta_n^{(3)} \in (0, 1]$ is an admissible (Mann) perturbation for T and $F(T_{G_n}) = F(T)$.

For more examples, one can see Rus [33] and Berinde et al. [4].

Based on the notions of admissible and generalized admissible mappings, authors in [35] introduced the following generalized $G'\mathfrak{S}$-algorithm:

Algorithm 1.1. For an arbitrary initial guess $x_0 \in X_2$, let $\{x_n\}$ be a sequence generated iteratively by

$$\begin{cases} x_{n+1} = G_n'(Rx_n, Sx_n, Ty_n), \\ y_n = G_n(x_n, Tx_n), \ n \geq 0, \end{cases} \tag{1.6}$$

where $R, S, T : X_2 \to X_2$ are nonlinear mappings, $G_n' : X_2 \times X_2 \times X_2 \to X_2$ is a generalized admissible mapping and $G_n : X_2 \times X_2 \to X_2$, is an admissible mapping.

This algorithm includes as special cases interesting algorithms studied in ([2, 5, 12, 18, 21, 26, 30, 33, 38]).

The following definition of Saddeek and Ahmed [35] is a generalization of the concepts of sequentially affine Lipschitzian mappings and affine Lipschitzian mappings, se also ([3, 5]):

Definition 1.7. *Suppose X_2 is a real normed space. A mapping $G_n' : X_2 \times X_2 \times X_2 \to X_2$, $n \geq 0$ is called generalized sequentially affine Lipschitzian if the following axioms hold:*
(i) G_n', $n \geq 0$ is generalized admissible,
(ii) $\exists\{\alpha_n^{(i)}\} \subset [0, 1]$, $i = 1, 2, 3$ such that

$$\|G_n'(x_1, y_1, z_1) - G_n'(x_2, y_2, z_2)\| \leq \|\alpha_n^{(1)}(x_1 - x_2) + \alpha_n^{(2)}(y_1 - y_2) + \alpha_n^{(3)}(z_1 - z_2)\|,$$

for all $x_j, y_j, z_j \in X_2$, $j = 1, 2$.

Using this concept, Saddeek and Ahmed [35], proved a strong convergence theorem by the generalized $G'\mathfrak{S}$-algorithm for finding an element in $F(R, S, T)$, where R, S and T are three self mappings defined on a closed convex subset C of a uniformly convex Banach space satisfying the conditions (1.1)-(1.3) and one of $R(C)$, $S(C)$ and $T(C)$ is assumed to be relatively compact.

The following Lemma which is due to Dotson [9] has played a fundamental role in proving the main result in [35]:

Lemma 1.1. *Suppose that X is a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1, n \geq 0$. Also, suppose that $\{\mu_n\}$ and $\{\nu_n\}$ are two sequences in X such that* $\limsup_{n\to\infty} \|\mu_n\| \leq 1$, $\limsup_{n\to\infty} \|\nu_n\| \leq 1$ *and* $\lim_{n\to\infty} \|(1 - t_n)\mu_n + t_n\nu_n\| = 1$. *Then* $\lim_{n\to\infty} \|\mu_n - \nu_n\| = 0$.

Let (X, d) be a metric space, $CB(X)$ stands for the collection of all nonempty, bounded, closed subsets of X, $H(A, B)$, is the Hausdorff metric on $CB(X)$. Let $R, S, T : X \to CB(X)$ be multivalued mappings. The set $F_T = \{x \in X : x \in T(x)\}$ denotes the fixed point set of T and $F_R \cap F_S \cap F_T$ will be denoted by $\widetilde{\mathfrak{F}}$.

The following Lemma is due to Nadler [23]:

Lemma 1.2. *Let $A, B \in CB(X)$ and $a \in A$. Then, for each $\varepsilon > 0$, there exists $b \in B$ such that* $d(a, b) \leq H(A, B) + \varepsilon$.

We now reformulate the general multivalued versions of Definitions 1.3, 1.4 and 1.7 and the Definition of sequentially affine Lipschitzian mapping.

Definition 1.8. *Let $G_1 : X_1 \times CB(X_1) \to CB(X_1)$ be a mapping such that (i) and (ii) of Definition 1.3 hold for all $x \in X_1$ and $y \in CB(X_1)$. Then G_1 is called an admissible on $X_1 \times CB(X_1)$.*

It is worth mentioning here that the admissible perturbation $T_{G_1} : X_1 \to CB(X_1)$ of $T : X_1 \to CB(X_1)$ corresponding to G_1 (i.e., $T_{G_1}(x) = G_1(x, Tx) = \{G_1(x, y) : y \in T(x)\}$ has the following properties (see Lemma 5.1 of Rus [33]) :
(i) $F_{T_{G_1}} = F_T$,
(ii) $\{x \in X_1 : \{x\} = T_{G_1}(x)\} = \{x \in X_1 : \{x\} = T(x)\}$.

The property (ii) is known as the strict fixed point property.

Definition 1.9. *Let X_2 be a real normed space and let $G_n^2 : X_2 \times CB(X_2) \to CB(X_2)$ be a multivalued admissible mapping. If there is $\{\beta_n^{(i)}\} \subset [0, 1]$, $i = 1, 2$ such that*

$$\begin{cases} H(G_n^2(x_1, A), G_n^2(x_2, B)) \le \beta_n^{(1)} d(x_1, x_2) + \beta_n^{(2)} H(A, B), \\ \|\tilde{\xi} - \tilde{\eta}\| \le \|\beta_n^{(1)}(x_1 - x_2) + \beta_n^{(2)}(y_1 - y_2)\|, \ n \ge 0, \end{cases}$$

for all $x_j \in X_2, A, B \in CB(X_2), y_1 \in A, y_2 \in B, \tilde{\xi} \in G_n^2(x_1, A), \tilde{\eta} \in G_n^2(x_2, B)$, whenever $\sum_{j=1}^{2} \beta_n^{(j)} = 1$, then G_n^2 is called sequentially affine Lipschitzian mapping on $X_2 \times CB(X_2)$.

Definition 1.10. *Let $G_2 : CB(X_1) \times CB(X_1) \times CB(X_1) \to CB(X_1)$ be a mapping such that*
(i) $G_2(A, A, A) = A$, $\forall\, A \in CB(X_1)$,
(ii) $G_2(A, B, D) = A \Rightarrow A = B = D$, $\forall\, A, B, D \in CB(X_1)$.
Then G_2 is called a generalized admissible mapping on $CB(X_1) \times CB(X_1) \times CB(X_1)$.

Definition 1.11. *Let X_2 be a real normed space and let $G_n^{'1} : CB(X_2) \times CB(X_2) \times CB(X_2) \to CB(X_2), n \ge 0$ be a mapping such that (i) of Definition 1.7 holds and there is a $\{\alpha_n^{(i)}\} \subset [0, 1]$, $i = 1, 2, 3$ with*

$$\begin{cases} H(G_n^{'1}(A_1, B_1, D_1), G_n^{'2}(A_2, B_2, D_2)) \le \alpha_n^{(1)} H(A_1, A_2) + \alpha_n^{(2)} H(B_1, B_2) + \alpha_n^{(3)} H(D_1, D_2), \\ \|\xi - \eta\| \le \|\alpha_n^{(1)}(a_1 - a_2) + \alpha_n^{(2)}(b_1 - b_2) + \alpha_n^{(3)}(d_1 - d_2)\|, \ n \ge 0, \end{cases}$$

for all $a_j \in A_j, b_j \in B_j, d_j \in D_j, A_j, B_j, D_j \in CB(X_2)$, $j = 1, 2, \xi \in G_n^{'1}(A_1, B_1, D_1), \eta \in G_n^{'1}(A_2, B_2, D_2)$, whenever $\sum_{i=1}^{3} \alpha^{(i)} = 1$, then $G_n^{'1}$ is called a generalized sequentially affine Lipschitzian mapping on $CB(X_2) \times CB(X_2) \times CB(X_2)$.

Let us now describe the generalized $G'\mathbb{S}$-algorithm associated with three multivalued mappings as follows:

Definition 1.12. *Let X_2 be a real vector space, $R, S, T : X_2 \to CB(X_2)$, $G_n^{'1} : CB(X_2) \times CB(X_2) \times CB(X_2) \to CB(X_2)$, $G_n^{'2} : X_2 \times CB(X_2) \to CB(X_2)$ be mappings such that $G_n^{'1}$ is generalized admissible on $CB(X_2) \times CB(X_2) \times CB(X_2)$ and $G_n^{'2}$ is admissible on $X_2 \times CB(X_2)$.*

The extended generalized $G'\mathbb{S}$−algorithm starting at $x_0 \in X_2$ and associated with the multivalued mappings R, S and T is defined as follows:

$$\begin{cases} x_{n+1} \in G_n^{'1}(Rx_n, Sx_n, Ty_n), \ n \ge 0 \\ y_n \in G_n^{'2}(x_n, Tx_n), \ n \ge 0, \end{cases} \tag{1.7}$$

If (X_2, d) is a metric space, then as a consequence of Lemma 1.2, one can select the $a_n \in Rx_n, b_n \in Sx_n, c_n \in Ty_n$ and $d_n \in Tx_n$ to satisfy the following inequalities:

$$\begin{cases} d(a_n, d_n) \le H(Rx_n, Tx_n) + \varepsilon_n^{(1)}, \\ d(b_n, d_n) \le H(Sx_n, Tx_n) + \varepsilon_n^{(2)}, \\ d(c_n, d_n) \le H(Ty_n, Tx_n) + \varepsilon_n^{(3)}, \end{cases} \tag{1.8}$$

with $\lim_{n\to\infty} \varepsilon_n^{(i)} = 0, i = 1, 2, 3$.

Recently, a new approach in metric fixed point theory has investigated by using a graph on metric spaces, e.g. ([17, 19]).

According to Diestel [8], a graph $\mathbb{G}$ is a data structure consisting of a nonempty set V of vertices and a set E of edges, that connect some of them. A directed graph is a graph in which $E \subseteq \{(u, v) : (u, v) \in V \times V,\ u \neq v\}$ is a set of directed edges. In $\mathbb{G}$ if an edge is drawn from vertex to itself (i.e., in the form (v, v)), then it is called a loop on V.

Let (X, d) be a metric space and $\triangle$ be the diagonal of the cartesian product $X \times X$ (i.e., $\triangle = \{(x, x) : x \in X\}$). Let $\mathbb{G}$ be a directed graph, such that $V(\mathbb{G}) = X$ (i.e., the set of vertices of $\mathbb{G}$ coincides with X) and $\triangle \subseteq E(\mathbb{G})$ (i.e., the set edges contains all loops). In order to identify $\mathbb{G}$ with the pair $(V(\mathbb{G}), E(\mathbb{G}))$, we need to consider that $\mathbb{G}$ has no parallel edges. Let $E(\mathbb{G}^{-1}) = \{(x, y) : (y, x) \in E(\mathbb{G})\}$. Let n be a non-negative integer and $\mathbb{G}$ be a directed graph. If x_0 and x_n are vertices in $\mathbb{G}$, then a path of length n in $\mathbb{G}$ from x_0 to x_n is a sequence $\{x_i\}$ of $n + 1$ vertices such that $(x_{i-1}, x_i) \in E(\mathbb{G})$ when $1 \leq i \leq n$. If there is a path between any two vertices, then $\mathbb{G}$ is called connected. A weighted graph is a graph $\mathbb{G}$ together with a weight function $w : E \to [0, \infty)$. Here, we represent the graph $\mathbb{G}$ as a weighted graph by assigning to every edge a weight function equal to the distance between its vertices.
In the Hausdorff metric on X, the Hausdorff weight function $H(A, B) \neq 0, \forall A, B \in CB(X)$ and whenever $H(A, B) = 0$ for some $A, B \in CB(X)$, it follows that $A = B$.

Definition 1.13. *Let $(X, \|\ .\ \|)$ be a Banach space endowed with a graph $\mathbb{G}$. For $x \in X$ and $A \in CB(X)$, let $D(x, A) = \inf\{\| x - y \| : y \in A\}$. Let $T : X \to CB(X)$ be a multivalued mapping such that for each $x, y \in X$, $z \in T(x)$, $\acute{z} \in T(y)$ with $(x, y) \in E(\mathbb{G})$, we have $(z, \acute{z}) \in E(\mathbb{G})$ (i.e., T preserves edges of $\mathbb{G}$). We say that T is an $E(\mathbb{G})$-generalized weakly contraction if there exists a comparison function φ such that for all $x, y \in X$ with $(x, y) \in E(\mathbb{G})$, the following inequality holds:*

$$H(Tx, Ty) \leq \varphi(\max\{\| x - y \|, \frac{[D(x, Tx) + D(y, Ty)]}{2}, \frac{[D(x, Ty) + D(y, Tx)]}{2}\}). \tag{1.9}$$

Based on the above definition we can rewrite the conditions (1.2) and (1.3) in the multivalued case as follows:

$$H(Sx, Ry) \leq \min\{D(Sx, y), D(x, Ry)\}, \tag{1.10}$$

for all $x, y \in X$ with $(x, y) \in E(\mathbb{G})$.

$$\lim_{n\to\infty} H(Rx_n, Sx_n) = 0, \tag{1.11}$$

for all $\{x_n\} \in X$ with $(x_n, x_{n+1}) \in E(\mathbb{G})$, $n \geq 0$.

Definition 1.14. *([20, 22]) Let $(X, \|\ .\ \|)$ be a normed space endowed with a graph $\mathbb{G}$. A sequence $\{x_n\} \in X$, $n \geq 0$ with $(x_n, x_{n+1}) \in E(\mathbb{G})$ is said to be:*
(1) strongly $E(\mathbb{G})$-convergent if and only if there is $x \in X$ with $(x_n, x) \in E(\mathbb{G})$ such that
$\lim_{n\to\infty} \| x_n - x \| = 0$;
(2) $E(\mathbb{G})$-Cauchy if and only if for $(x_n, x_m) \in E(\mathbb{G})$, we have $\lim_{n\to\infty} sup_{m>n} \| x_n - x_m \| = 0$;
(3) $(X, \|\ .\ \|)$ is $E(\mathbb{G})$-complete if every $E(\mathbb{G})$-Cauchy sequence in X is strongly $E(\mathbb{G})$-convergent.

Definition 1.15. *Let $(X, \|\ .\ \|)$ be the same as in Definition 1.14 and $T : X \to CB(X)$ be a multivalued mapping. Then $T(X)$ is $E(\mathbb{G})$-relatively compact if for any sequence $\{x_n\} \in X$ with $y_n \in Tx_n, n \geq 0$ and $(x_n, y_n) \in E(\mathbb{G})$ there is a subsequence $\{x_{n_k}\}, k \geq 0$ of $\{x_n\}$ with $y_{n_k} \in Tx_{n_k}, (x_{n_k}, y_{n_k}) \in E(\mathbb{G})$ such that $\{y_{n_k}\}$ is strongly $E(\mathbb{G})$-convergent.*

Definition 1.16. *[1] For $\phi \neq A, B \in CB(X), (A, B) \subset E(\mathbb{G})$ means that there is an edge between some $a \in A$ and $b \in B$.*

The following two Lemmas are crucial in proving our main result:

Lemma 1.3. *[37] Let a real sequence $\{a_n\}$ satisfy the following relation:*

$$a_{n+1} \leq a_n + b_n, \forall n \geq 1,$$

where $a_n, b_n \geq 0$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.4. *[7] Let $P(X)$ be the collection of all nonempty bounded and closed subsets of X. Let $T : X \to P(X)$ be a multivalued mapping with $F_T \neq \phi$ and let $P_T : K \to CB(X)$ be a multivalued mapping given by $P_T(x) = \{y \in T(x) : \|x - y\| = D(x, Tx)\}, x \in X$. Then the following statements hold: (i) P_T is a multivalued mapping from X to $P(X)$;*
(ii) $F_T = F_{P_T}$;
(iii) $P_T(p) = \{p\}$ for all $p \in F_T$;
(iv) For any $x \in X$, $P_T(x)$ is a closed subset of $T(x)$;
(v) $D(x, Tx) = D(x, P_T(x))$ for all $x \in X$.

2. Main result

The main result of the present paper is as follows:

Theorem 2.1. *Suppose that X is a real uniformly convex Banach space and that $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ is a connected directed graph such that $\triangle \subseteq E(\mathbb{G})$ and $E(\mathbb{G})$ is convex. Suppose $C \subseteq X$ a nonempty, closed and convex set with $V(\mathbb{G}) = C$. Let $R, S, T : C \to P(C)$ be three multivalued mappings with $\phi \neq \tilde{\mathfrak{F}}$. Let $P_R, P_S, P_T : C \to CB(C)$ be three multivalued mappings such that P_T satisfies condition (1.9) (Replacing T by P_T) and P_R, P_S satisfy the conditions (1.10) and (1.11) (Replacing R and S by P_R and P_S, respectively) and that the range of one of these mappings is $E(\mathbb{G})$-relatively compact in $CB(C)$. Let $G_n^{'1} : P(C)\times P(C)\times P(C) \to P(C), n \geq 0$ be a generalized admissible and generalized sequentially affine Lipschitzian mapping whenever $(A_1, A_2), (B_1, B_2), (D_1, D_2), G_n^{'1}(A_1, B_1, D_1)$, $G_n^{'1}(A_2, B_2, D_2)$ are in $E(\mathbb{G})$ for any $A_j, B_j, D_j \in P(C), j = 1, 2$. Assume that $G_n^{'2} : C \times P(C) \to P(C), n \geq 0$ is an admissible and sequentially affine Lipschitzian mapping whenever $(x_1, x_2), (y_1, y_2), G_n^{'2}(x_1, y_1), G_n^{'2}(x_2, y_2)$ are in $E(\mathbb{G})$ for any $x_j \in C, y_j \in P(C), j = 1, 2$. Let $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(j)}\}$, $i = 1, 2, 3, j = 1, 2$ be two sequences in $[0, 1]$ such that (i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $\sum_{i=1}^{3} \alpha_n^{(i)} = 1$; (ii) $\sum_{j=1}^{2} \beta_n^{(j)} = 1, \limsup_{n\to\infty} \beta_n^{(2)} < \frac{1}{3}$.*
For arbitrary chosen $x_0 \in C$, let $\{x_n\}$ be the extended generalized $G^{'}\mathbb{S}$−algorithm defined by

$$\begin{cases} x_{n+1} \in G_n^{'1}(P_R(x_n), P_S(x_n), P_T(y_n)), \\ y_n \in G_n^{'2}(x_n, P_T(x_n)),\ n \geq 0. \end{cases} \tag{2.1}$$

Let $p \in \tilde{\mathfrak{F}}$ be such that (x_n, p) and (x_n, x_{n+1}) are in $E(\mathbb{G})$. Then the sequence $\{x_n\}$ strongly $E(\mathbb{G})$-converges to some point p of $\tilde{\mathfrak{F}}$.

Proof. By virtue of Lemma 1.4, we have that $P_R(p) = P_S(p) = P_T(p) = \{p\}$ and $F_R = F_{P_R}, F_S = F_{P_S}, F_T = F_{P_T}$ for each $p \in \tilde{\mathfrak{F}}$.
According to Lemma 1.2, (2.1), the admissibility, and the sequentiality affine Lipschitzian of $\{G_n^{'2}\}$, we deduce that $G_n^{'2}(p, p) = p$ and

$$\begin{aligned} \|y_n - p\| &\leq H(G_n^{'2}(x_n), P_T(x_n)), G_n^{'2}(p, p)) + \varepsilon_n \\ &\leq \beta_n^{(1)}\|x_n - p\| + \beta_n^{(2)}H(P_T(x_n), P_T(p)) + \varepsilon_n. \end{aligned} \tag{2.2}$$

Together with (1.9), (ii) and $\varphi(t) < t$ for all $t > 0$, we obtain

$$\begin{aligned} \|y_n - p\| &\leq \beta_n^{(1)}\|x_n - p\| + \beta_n^{(2)}\varphi(m(x_n, p)) + \varepsilon_n \\ &\leq \sum_{j=1}^{2} \beta_n^{(j)}\|x_n - p\| + \varepsilon_n \\ &= \|x_n - p\| + \varepsilon_n, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} m(x_n,p) &= \max\{\|x_n-p\|, \frac{D(x_n,P_T(x_n))}{2}, \frac{[\|x_n-p\|+D(p,P_T(x_n)]}{2}\} \\ &= \|x_n-p\|. \end{aligned}$$

On the other hand, it follows from (1.9) and (1.10) that

$$H(P_T(x_n),P_T(p)) \leq \|x_n-p\|, \tag{2.4}$$

$$H(P_S(x_n),P_S(p)) \leq \|x_n-p\|, \tag{2.5}$$

$$H(P_R(x_n),P_R(p)) \leq \|x_n-p\|. \tag{2.6}$$

Applying Lemma 1.2 once again and using (2.1), (2.3)-(2.6), (i) and the definition of $G_n^{'1}$, we obtain

$$\begin{aligned} \|x_{n+1}-p\| &\leq H(G_n^{'1}(P_R(x_n),P_S(x_n),P_T(y_n)),G_n^{'1}(P_R(p),P_S(p),P_T(p))+\varepsilon_n \\ &\leq \alpha_n^{(1)}H(P_R(x_n),P_R(p))+\alpha_n^{(2)}H(P_S(x_n),P_S(p))+\alpha_n^{(3)}H(P_T(y_n),P_T(p))+\varepsilon_n \\ &\leq \sum_{i=1}^{3}\alpha_n^{(i)}\|x_n-p\|+(1+\alpha_n^{(3)})\varepsilon_n \\ &= \|x_n-p\|+(1+b)\varepsilon_n. \end{aligned} \tag{2.7}$$

Choose $\{\varepsilon_n\}$ such that $\sum_{n=1=1}^{\infty}\varepsilon_n<\infty$ (such a choice is possible, because $\{\varepsilon_n\}$ arbitrary).
So applying Lemma 1.3 with $a_n=\|x_n-p\|$ and $b_n=(1+b)\varepsilon_n$, we conclude that $\lim_{n\to\infty}\|x_n-p\|$ exists. Suppose the $\lim_{n\to\infty}\|x_n-p\|=c\geq 0$. If $c=0$, then we are done.
Now, let $c>0$ and set $u_n\in P_R(x_n), v_n\in P_S(x_n), w_n\in P_T(y_n)$, $\mu_n=\frac{\alpha_n^{(1)}}{1-\alpha_n^{(3)}}\frac{u_n-p}{\|x_n-p\|}+\frac{\alpha_n^{(2)}}{1-\alpha_n^{(3)}}\frac{v_n-p}{\|x_n-p\|}$ and $\nu_n=\frac{w_n-p}{\|x_n-p\|}$ such that $(u_n,p),(v_n,p),(w_n,p),(\mu_n,\nu_n)$ are in the convex set $E(\mathbb{G})$.
It follows from (2.3)-(2.6) and (i) that

$$\begin{aligned} \|\mu_n\| &= \frac{\alpha_n^{(1)}}{1-\alpha_n^{(3)}}\frac{\|u_n-p\|}{\|x_n-p\|}+\frac{\alpha_n^{(2)}}{1-\alpha_n^{(3)}}\frac{\|v_n-p\|}{\|x_n-p\|} \\ &\leq \frac{\alpha_n^{(1)}}{1-\alpha_n^{(3)}}\frac{1}{\|x_n-p\|}H(P_R(x_n),P_R(p))+\frac{\alpha_n^{(2)}}{1-\alpha_n^{(3)}}\frac{1}{\|x_n-p\|}H(P_S(x_n),P_S(p)) \\ &\leq \frac{\alpha_n^{(1)}}{1-\alpha_n^{(3)}}++\frac{\alpha_n^{(2)}}{1-\alpha_n^{(3)}}=1, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \|\nu_n\| &= \frac{\|w_n-p\|}{\|x_n-p\|}\leq\frac{1}{\|x_n-p\|}H(P_T(y_n)P_T(p)) \\ &\leq \frac{1}{\|x_n-p\|}\|y_n-p\|\leq 1+\frac{\varepsilon_n}{\|x_n-p\|}. \end{aligned} \tag{2.9}$$

Taking the limit supremum of both sides of (2.8) and (2.9) and using (i), $\lim_{n\to\infty}\varepsilon_n=0$ and $\lim_{n\to\infty}\|x_n-p\|=c>0$, we have $\limsup_{n\to\infty}\|\mu_n\|\leq 1$ and $\limsup_{n\to\infty}\|\nu_n\|\leq 1$ and

$$\lim_{n\to\infty}\|(1-\alpha_n^{(3)})\mu_n+\alpha_n^{(3)}\nu_n\|\leq\lim_{n\to\infty}(\sum_{i=1}^{3}\alpha_n^{(i)}+\frac{\alpha_n^{(3)}\varepsilon_n}{\|x_n-p\|})=1,$$

which implies that

$$\lim_{n\to\infty} \|(1-\alpha_n^{(3)})\mu_n + \alpha_n^{(3)}\nu_n\| \le 1. \tag{2.10}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \|\alpha_n^{(1)}(u_n - p) + +\alpha_n^{(2)}(v_n - p) + \alpha_n^{(3)}(w_n - p)\| \\ &= \|(1-\alpha_n^{(3)})\mu_n + \alpha_n^{(3)}\nu_n\|\|x_n - p\|. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \|(1-\alpha_n^{(3)})\mu_n + \alpha_n^{(3)}\nu_n\| \ge 1. \tag{2.11}$$

Thus, from (2.10) and (2.11), we conclude

$$\lim_{n\to\infty} \|(1-\alpha_n^{(3)})\mu_n + \alpha_n^{(3)}\nu_n\| = 1.$$

Hence, by Lemma 1.1, we obtain

$$\lim_{n\to\infty} \|\frac{1}{1-\alpha_n^{(3)}}(\alpha_n^{(1)}u_n + \alpha_n^{(2)}v_n) - w_n\| = 0. \tag{2.12}$$

Now consider that $P_T(C)$ is $E(\mathbb{G})$-relatively compact. Then there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a point $q \in P(C)$ with $(y_{n_k}, w_{n_k}), (q, w_{n_k}) \in E(\mathbb{G})$, $w_{n_k} \in P_T(y_{n_k})$ for $k \ge 0$ such that

$$\lim_{k\to\infty} w_{n_k} = q. \tag{2.13}$$

This, together with (2.12), implies that

$$\lim_{k\to\infty} \frac{1}{1-\alpha_{n_k}^{(3)}}(\alpha_{n_k}^{(1)}u_{n_k} + \alpha_{n_k}^{(2)}v_{n_k}) = q. \tag{2.14}$$

Moreover, since P_R and P_S satisfy the condition (1.11), it follows from (2.12) that

$$\lim_{n\to\infty} \|u_n - w_n\| = 0, \tag{2.15}$$

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{2.16}$$

Hence, it follows from (2.13), (2.15) and (2.16) that

$$\lim_{k\to\infty} u_{n_k} = q, \tag{2.17}$$

$$\lim_{k\to\infty} v_{n_k} = q. \tag{2.18}$$

On the other hand, for each $n \ge 0$ with $(x_{n+1}, u_n), (u_n, v_{n+1}), (v_{n+1}, x_{n+1}) \in E(\mathbb{G})$, we have

$$\begin{aligned} \|v_{n+1} - x_{n+1}\| &\le \|x_{n+1} - u_n\| + \|u_n - v_{n+1}\| \\ &\le 2\|x_{n+1} - u_n\|. \end{aligned} \tag{2.19}$$

Since

$$\begin{aligned} \|x_{n+1} - u_n\| &\le \|\alpha_n^{(1)}(u_n - u_n) + \alpha_n^{(2)}(v_n - u_n) + \alpha_n^{(3)}(w_n - u_n)\| \\ &\le \alpha_n^{(2)}\|v_n - u_n\| + \alpha_n^{(3)}\|w_n - u_n\|. \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20) and using (i), we get

$$\|v_{n+1} - x_{n+1}\| \le 2b[\|v_n - u_n\| + \|w_n - u_n\|]. \tag{2.21}$$

Taking the limit in (2.21) as $n \to \infty$ and using (1.11) (with $u_n \in P_R(x_n), v_n \in P_S(x_n)$) and (2.15), we obtain

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{2.22}$$

By (2.18) and (2.22), we have

$$\lim_{k\to\infty} x_{n_k} = q. \tag{2.23}$$

Since P_T is an $E(\mathbb{G})$-generalized weakly contraction, it follows for all $\tilde{w}_{n_k} \in P_T(x_{n_k}), w_{n_k} \in P_T(y_{n_k})$ with $(x_{n_k}, \tilde{w}_{n_k}), (x_{n_k}, w_{n_k}), (w_{n_k}, \tilde{w}_{n_k}) \in E(\mathbb{G})$ that

$$\begin{aligned} \|x_{n_k} - \tilde{w}_{n_k}\| &\leq \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - \tilde{w}_{n_k}\| \\ &\leq \varphi(\max\{\|y_{n_k} - x_{n_k}\|, \frac{[\|y_{n_k} - w_{n_k}\| + \|x_{n_k} - \tilde{w}_{n_k}\|]}{2}, \\ &\quad \frac{[\|y_{n_k} - \tilde{w}_{n_k}\| + \|x_{n_k} - w_{n_k}\|]}{2}\}) + \|x_{n_k} - w_{n_k}\|. \end{aligned} \tag{2.24}$$

Using again the admissibility and the sequential Lipschitzian property of $\{G_n^{'2}\}$, with (2.1), we get

$$\|y_{n_k} - x_{n_k}\| \leq \beta_{n_k}^{(2)} \|x_{n_k} - \tilde{w}_{n_k}\|,$$

$$\begin{aligned} \|y_{n_k} - w_{n_k}\| &\leq \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - w_{n_k}\| \\ &\leq \beta_{n_k}^{(2)} \|x_{n_k} - \tilde{w}_{n_k}\| + \|x_{n_k} - w_{n_k}\|, \end{aligned}$$

and

$$\begin{aligned} \|y_{n_k} - \tilde{w}_{n_k}\| &\leq \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - \tilde{w}_{n_k}\| \\ &\leq (1 + \beta_{n_k}^{(2)}) \|x_{n_k} - \tilde{w}_{n_k}\|. \end{aligned}$$

Substituting into (2.24) and taking into account $\varphi(t) < t$ for each $t > 0$, we obtain

$$\begin{aligned} \|x_{n_k} - \tilde{w}_{n_k}\| &\leq \varphi(\max\{\beta_{n_k}^{(2)} \|x_{n_k} - \tilde{w}_{n_k}\|, \frac{[(1 + \beta_{n_k}^{(2)})\|x_{n_k} - \tilde{w}_{n_k}\| + \|x_{n_k} - w_{n_k}\|]}{2}\}) \\ &+ \|x_{n_k} - w_{n_k}\| \\ &\leq \frac{[(1 + 3\beta_{n_k}^{(2)})\|x_{n_k} - \tilde{w}_{n_k}\| + 3\|x_{n_k} - w_{n_k}\|]}{2}, \end{aligned}$$

which implies that

$$\|x_{n_k} - \tilde{w}_{n_k}\| \leq \frac{3}{(1 - 3\beta_{n_k}^{(2)})} \|x_{n_k} - w_{n_k}\|. \tag{2.25}$$

This, together with (ii) and (2.13), implies that

$$\lim_{k\to\infty} \tilde{w}_{n_k} = q. \tag{2.26}$$

Let us now show that $q \in \tilde{\mathfrak{F}}$. From

$$\begin{aligned} D(q, P_T(q)) &\leq \|x_{n_k} - q\| + D(x_{n_k}, P_T(x_{n_k})) + H(P_T(x_{n_k}), P_T(q)) \\ &\leq \|x_{n_k} - q\| + \|x_{n_k} - \tilde{w}_{n_k}\| + H(P_T(x_{n_k}), P_T(q)), \end{aligned} \tag{2.27}$$

since $H(P_T(x_{n_k}), P_T(q)) \leq \|x_{n_k} - q\|$, passing to limit as $k \to \infty$ and using (2.23) and (2.26), we obtain $D(q, P_T(q)) = 0$.

Hence, by Lemma 1.4, (ii) and (iv), we conclude that $q \in F_{P_T} = F_T$.
Also,

$$\begin{aligned} D(q, P_S(q)) &\leq \|x_{n_k} - q\| + D(x_{n_k}, P_R(x_{n_k})) + H(P_R(x_{n_k}), P_S(q)) \\ &\leq \|x_{n_k} - q\| + \|x_{n_k} - u_{n_k}\| + \min\{D(x_{n_k}, P_S(q)), D(q, P_R(x_{n_k}))\}. \end{aligned} \tag{2.28}$$

If $\min\{D(x_{n_k}, P_S(q)), D(q, P_R(x_{n_k}))\} = D(x_{n_k}, P_S(q))$, then we have

$$D(x_{n_k}, P_S(q)) < D(q, P_R(x_{n_k})) \leq \|q - u_{n_k}\|. \tag{2.29}$$

If $\min\{D(x_{n_k}, P_S(q)), D(q, P_R(x_{n_k}))\} = D(q, P_R(x_{n_k})$, then from (2.28), we deduce

$$\begin{aligned} D(q, P_S(q)) &\leq \|x_{n_k} - q\| + \|x_{n_k} - u_{n_k}\| + D(q, P_R(x_{n_k})) \\ &\leq \|x_{n_k} - q\| + \|x_{n_k} - u_{n_k}\| + \|q - u_{n_k}\|. \end{aligned} \tag{2.30}$$

Letting $k \to \infty$ in (2.29) and (2.30) and using (2.17) and (2.23), yields $D(q, P_S(q)) = 0$.
Again, by Lemma 1.4 (ii) and (iv), we conclude $q \in F_{P_S} = F_S$.
Now, replacing the mappings $P_S(q), P_R(x_{n_k}), u_{n_k}$ in (2.28)-(2.30) by $P_R(q), P_S(x_{n_k}), v_{n_k}$, respectively and using the same argument again with the help of (2.18), yields $q \in F_{P_R} = F_R$. Hence, we have $q \in \tilde{\mathfrak{F}}$.
Now, using the inequality (2.7) with $p = q$, we deduce that the sequence $\{\|x_n - q\|\}$ is decreasing for all sufficiently large n. Since $\{\|x_{n_k} - q\|\}$ converges to 0, it follows that the whole sequence $\{\|x_n - q\|\}$ converges to 0, that is, $x_n \to q \in \tilde{\mathfrak{F}}$ as $n \to \infty$.
Similarly, we can also complete the proof of our theorem by considering either P_R or P_S is $E(\mathbb{G})$-relatively compact.

Remark 2.1. *Our main result is a generalized multivalued version of Theorem 2.1 of Saddeek and Ahmed [35] in a graph approach. As a consequence, it extends and improves the corresponding results of Huang and Jeng [15], Rashwan and Saddeek [30], Ganguly and Bandyopadhyay [11] Rhoades [31], Osilike [26], Tiwary and Debnath [38], Bunlue and Suantai [5], Petrusel and Rus [27] and the references therein.*

Example 2.1. *Let $X = C = [0,1]$ with the usual norm $\|x - y\| = |x - y|, \forall x, y \in X$. Let $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ be a connected directed graph with $V(\mathbb{G}) = X$. Assume that $E(\mathbb{G}) = X \times X$. Clearly, $E(\mathbb{G})$ is convex and $\triangle \subseteq E(\mathbb{G})$. Let $R, S, T : X \to P(X)$ be three mappings defined as*

$$T(x) = \begin{cases} \{\frac{1}{3}\}, & x \in [0, \frac{1}{3}] \\ [\frac{1}{3}, x], & x \in (\frac{1}{3}, \frac{1}{2}], \\ \{\frac{1}{2}\}, & x \in (\frac{1}{2}, 1), \\ \{0\}, & x = 1, \end{cases} \qquad \text{and} \qquad R(x) = S(x) = \begin{cases} [0, x], & x \in [0, \frac{2}{3}] \\ \{\frac{2}{3}\}, & x \in (\frac{2}{3}, 1]. \end{cases}$$

It is seen that $\phi \neq \tilde{\mathfrak{F}}$ and $R(X), S(X), T(X)$ are $E(\mathbb{G})$-relatively compact. If $(x, y) \in E(\mathbb{G})$, we have $(Tx, Ty) \in E(\mathbb{G})$. Then T preserves the edges of $\mathbb{G}$.
Let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined as $\varphi(t) = \frac{t}{2}, \forall t \in [0, +\infty)$. Then φ is a comparison function and the conditions (1.9)-(1.11) of Theorem 2.1 are satisfied with respect to R, S and T. Define the mappings $P_R, P_S, P_T : X \to CB(X)$ as in Lemma 1.4. Then, all these mappings satisfying the five statements of Lemma 1.4. For any $x, y \in X$ with $(x, y), (x_n, x_{n+1}) \in E(\mathbb{G})$, since $R(x), S(x)$ and $T(x)$ are nonempty bounded proximal subsets in C, there exist $y \in R(x), y_1 \in S(x)$ and $y_2 \in T(x)$ such that $\|x - y\| = D(x, R(x)), \|x - y_1\| = D(x, S(x)), \|x - y_2\| = D(x, T(x))$. So, by a similar way as given in Chang et al. [6], we can also prove that the mappings P_R, P_S and P_T satisfying the conditions (1.9)-(1.11) of Theorem 2.1. Since P_R, P_S, P_T are closed subsets of $R(x), S(x), T(x)$, respectively and $R(x), S(x)$ and $T(x)$ are $E(\mathbb{G})$-relatively compact, it also follows that P_R, P_S and P_T are $E(\mathbb{G})$-relatively compact in $CB(X)$. Let $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(j)}\}$, $i = 1, 2, 3$, $j = 1, 2$ be constant sequences in $[0, 1]$ such that for all $n \geq 0$, $\alpha_n^{(i)} = \frac{1}{3}$ and

$$\beta_n^{(j)} = \begin{cases} 1, & j = 1 \\ 0, & j = 2. \end{cases}$$

Then, the conditions (i) and (ii) of Theorem 2.1 are satisfied. Let $G_n'^1 : P(X) \times P(X) \times P(X) \to P(X)$, $G_n'^2 : X \times P(X) \to P(X), n \geq 0$ be defined by

$$\begin{cases} G_n'^1(P_R(x_n), P_S(x_n), P_T(y_n)) = \frac{P_R(x_n) + P_S(x_n) + P_T(y_n)}{3}, \\ G_n'^2(x_n, P_T(x_n)) = x_n, \ n \geq 0. \end{cases}$$

It is easy to check that $G_n'^1$ is a generalized admissible and generalized affine Lipschitzian mapping and $G_n'^2$ is admissible and sequentially affine Lipschitzian mapping.
Replacing the mappings R, S, T by P_R, P_S, P_T, respectively in $G_n'^1$ and $G_n'^2$ and chossing $x_0 = 1$, we can define a sequence $\{x_n\}$ as follows:
For $x_0 = 1$, we have $y_0 = 1$, $P_R(1) = P_S(1) = \{\frac{2}{3}\}, P_T(1) = \{0\}$. Taking $x_1 = \frac{1}{3}(1+\frac{1}{3}) \in \{\frac{4}{9}\}$, we have $y_1 = \frac{1}{3}(1+\frac{1}{3})$, $P_R(x_1) = P_S(x_1) = [0, \frac{4}{9}], P_T(x_1) = \{\frac{1}{3}\}$. Taking $x_2 = \frac{1}{3}(1+\frac{1}{3^2}) \in [\frac{1}{9}, \frac{11}{27}]$, we have $y_2 = \frac{1}{3}(1+\frac{1}{3^2})$, $P_R(x_2) = P_S(x_2) = [0, \frac{10}{27}], P_T(x_2) = \{\frac{1}{3}\}$. Taking $x_3 = \frac{1}{3}(1+\frac{1}{3^3}) \in [\frac{1}{9}, \frac{29}{81}]$, we have $y_3 = \frac{1}{3}(1+\frac{1}{3^3})$, $P_R(x_3) = P_S(x_3) = [0, \frac{28}{81}], P_T(x_2) = [\frac{1}{3}, \frac{28}{81}]$. Inductively, we can obtain $x_{n+1} = \frac{1}{3}(1+\frac{1}{3^n}), \forall n \geq 0$. Taking the limit as $n \to \infty$, yields $\frac{1}{3} \in \tilde{\mathfrak{F}}$.

3. Application to image recovery

A multivalued mapping $T : C \to CB(C)$ is said to be $SKC-$type in the terminology of Chang et al. [6] if for all $x, y \in C$ with $\frac{1}{2}D(x, Tx) \leq\| x-y \|$ implies that (1.9) holds whenever $\varphi(t) = t$. The set-valued mapping $P_C : X \to C$ $(P_C(x) = \{z \in C :\| x-z \|= D(x, C)\})$ is said to be the metric projection from X onto C. Metric projection has important applications in the optimization, computational mathematics, theory of equation and control theory.

It is well known that if C is a closed convex subset of a uniformly convex Banach space X, then the metric projection P_C is single valued $KSC-$type mapping, relatively compact and $F_{P_C} \neq \phi$ (see, [6, 40]).

A fundamental property of P_C is that it is nonexpansive ($\|P_C(x) - P_C(y)\| \leq \|x - y\|, \forall x, y \in X$). Furthermore if X is a Hilbert space, then the projection mapping P_C is characterized by $\frac{1}{2}\|x - P_C(x)\| \leq \|x - p\|, \forall x \in X, p \in F_{P_C}$ (see, [34]) and $\|P(x) - p\| \leq min\{\|P(x) - p\|, \|x - p\|\}, \forall x \in X, p \in F_P$.

The image recovery problem considered here is to find the nearest point in the intersection of any two nonempty, closed and convex subsets of a Hilbert space by using the corresponding metric projection mapping of each subset.

Theorem 3.1. *Let X be a Hilbert space and let $C_i, i = 1, 2$ be nonempty, closed and convex subsets of X such that $\bigcap_{i=1}^2 C_i \neq \phi$. Let $\mathbb{G}, E(\mathbb{G}), V(\mathbb{G})$ and $\triangle$ be the same as in Theorem 2.1 and suppose P_{C_1} and P_{C_2} are a pair of metric projections on C_1 and C_2, respectively. Suppose that for each vertices x and y, (x, y) is an edge there exists $i \in \{1, 2\}$ with $(P_{C_i}(x), P_{C_i}(y)) \in E(\mathbb{G})$. Let $G_n^i : C_i \times C_i \to C_i, n \geq 0,\ i = 1, 2$ be an admissible and sequentially affine Lipschitzian map whenever $(x_1, x_2), (y_1, y_2), G_n^1(P_{C_1}(x_1), P_{C_2}(y_1)), G_n^1(P_{C_1}(x_2), P_{C_2}(y_2)), G_n^2(x_1, P_{C_2}(x_1)), G_n^2(x_2, P_{C_2}(y_2)) \in E(\mathbb{G})$. Let $\{\alpha_n^{(j)}\}$ and $\{\beta_n^{(j)}\}$, $j = 1, 2$ be two real sequences in $[0, 1]$ such that $0 < a \leq \alpha_n^{(j)} \leq b < 1,\ \sum_{j=1}^2 \alpha_n^{(j)} = \sum_{j=1}^2 \beta_n^{(j)} = 1,\ \limsup_{n\to\infty} \beta_n^{(2)} < \frac{1}{3}, \forall n \geq 0$.*
For arbitrary chosen $x_0 \in X$, let $\{x_n\}$ be the algorithm defined by

$$\begin{cases} x_{n+1} = G_n^1(P_{C_1}(x_n), P_{C_2}(y_n)), \\ y_n = G_n^2(x_n, P_{C_2}(x_n)),\ n \geq 0. \end{cases} \tag{3.1}$$

If $p \in \bigcap_{i=1}^2 F_{P_{C_i}}$, (x_n, p) and (x_n, x_{n+1}) are in $E(\mathbb{G})$, then the sequence $\{x_n\}$ strongly $E(\mathbb{G})$-converges to a fixed point of point $C_1 \bigcap C_2$.

Proof X being a Hilbert space is uniformly convex, so P_{C_2} is a single valued $SKC-$type mapping and preserves the edges of $\mathbb{G}$. Thus P_{C_2} is $E(\mathbb{G})$- generalized weakly contraction with $\varphi(t) = t$. Further, we have $F_{P_{C_1}} = C_1, F_{P_{C_2}} = C_2$ and $\|P_{C_1}(x) - p\| \leq min\{\|P_{C_1}(x) - p\|, \|x - p\|\}, \forall x \in X, p \in P_{C_1}$. Thus, replacing $G_n'^1, G_n'^2$ by G_n^1, G_n^2, respectively and letting $P_R = P_S = P_{C_1}$ and $P_T = P_{C_2}$ in Theorem 2.1. The desired conclusion follows immediately.

4. Conclusions

In this paper, we present the generalized admissible $\mathbb{S}$-algorithm to approximate some common fixed points of two general classes of multivalued contraction conditions in Banach spaces with graphs. We have

six significant contributions:
(1) we introduce two general classes of multivalued contraction conditions in uniformly convex Banach spaces endowed with graphs;
(2) we explore some general concepts of multivalued admissible mappings;
(3) we provide a more general admissible $\mathbb{S}$-algorithm involving multivalued mappings;
(4) we study the strong convergence of the proposed algorithm to a common fixed point of three multivalued mappings under certain assumptions in the framework of uniformly convex Banach spaces with graphs;
(5) we give an example to illustrate the efficiency of the proposed algorithm;
(6) we apply our results to solve the image recovery problem in a Hilbert space by the metric projections.

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