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Generalized Analytic Feynman Integrals via the Operators and its Applications

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Abstract. In this paper, we introduce a new concept of a generalized analytic Feynman integral combining the bounded linear operators on abstract Wiener space. We then obtain some Feynman integration formulas involving the generalized first variation. These formulas are more generalized forms rather than the formulas studied in previous papers. Finally, we establish a generalized Cameron-Storvick theorem, and give some examples to illustrate the usefulness of our results and formulas.

1. Introduction

Let H be a real separable infinite-dimensional Hilbert space with the inner product $\langle\cdot,\cdot\rangle_H$ and norm $|\cdot|_H = \sqrt{\langle\cdot,\cdot\rangle_H}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gaussian cylinder set measure ν_0 on H. Let B denote the completion of H with respect to $\|\cdot\|_0$, and $\mathbf{i}$ denote the natural injection from H into B. The adjoint operator $\mathbf{i}^*$ of $\mathbf{i}$ is one to one and maps B^* continuously onto a dense subset H^*, where B^* and H^* are topological duals of B and H, respectively. By identifying H^* with H and B^* with $\mathbf{i}^*B^*$, we have a triple $B^* \subset H^* \approx H \subset B$. By a well-known result of Gross [11], $\nu_0 \circ \mathbf{i}^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (B, H, ν) is called an abstract Wiener space, for a more detailed study of the abstract Wiener space see [4, 5, 9–13, 16, 17, 19].

Let $\mathcal{M}(H)$ be the space of all complex-valued Borel measures on H. Under the total variation norm and with convolution as multiplication, $\mathcal{M}(H)$ is a commutative Banach algebra with identity. The Fourier transform of f in $\mathcal{M}(H)$ is defined by

$$\hat{f}(v) = \int_H \exp\{i\langle h, v\rangle\} df(h), \quad v \in H. \tag{1}$$

The set of all functionals of the form (1) is denoted by $\mathcal{F}(H)$ and is called the Fresnel class of H. It is known that each functional of the form (1) can be extended to B uniquely by

$$F(x) = \int_H \exp\{i(h, x)^\sim\} df(h), \quad x \in B, \tag{2}$$

where $(\cdot,\cdot)^\sim$ is a stochastic inner product between H and B. Then the Fresnel class $\mathcal{F}(B)$ of B is the space of all functionals of the form (2). It is also known that two Fresnel classes $\mathcal{F}(H)$ and $\mathcal{F}(B)$ are isometric.

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Let $\mathcal{F}(B^2)$ be the space of all s-equivalence classes of functionals which have the form

$$F(x_1, x_2) = \int_H \exp\left\{i\sum_{j=1}^{2}(h, x_j)^\sim\right\}df(h), \quad x_1, x_2 \in B, \tag{3}$$

for some $f \in \mathcal{M}(H)$. This class is a Banach algebra [2, 16]. Let A_1 and A_2 be bounded, nonnegative self adjoint operators on H. In [16] G. Kallianpur and C. Bromley introduced a larger class $\mathcal{F}_{A_1,A_2}$ of functionals of the form

$$F(x_1, x_2) = \int_H \exp\{i(A_1^{1/2}h, x_1)^\sim + i(A_2^{1/2}h, x_2)^\sim\}df(h), \quad x_1, x_2 \in B, \tag{4}$$

and proved the existence of the analytic Feynman integral for functionals in $\mathcal{F}_{A_1,A_2}$. The map $f \mapsto [F]$ defined by (4) establishes an algebraic isomorphism between $\mathcal{M}(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of $A_1 + A_2$ is dense in H. In this case, $\mathcal{F}_{A_1,A_2}$ becomes a Fresnel class under the norm $\|F\| = \|f\|$. Moreover, the two Fresnel classes $\mathcal{F}(H)$ and $\mathcal{F}_A \equiv \mathcal{F}_{A_1,A_2}$ are also homeomorphic in this case that $A = A_1 - A_2$ where $A_1 = A_+$ and $A_2 = A_-$. In many papers, fundamental theories of the analytic Feynman integrals were studied and developed for functionals in $\mathcal{F}(B)$ and $\mathcal{F}_{A_1,A_2}$ involving the Cameron-Storvick theorem [2–5, 10, 16, 17]. These generalizations are very important subject to study the quantum mechanics.

In this paper, we define a more generalized analytic Feynman integral combined with the bounded linear operators. Its existence is established for functionals in a Fresnel class. We then introduce the generalized first variation combined with bounded linear operators, and establish some Feynman integration formulas. Finally, we obtain a Cameron-Storvick theorem with respect to the generalized analytic Feynman integral with some examples.

2. Definitions and preliminaries

In this section we list some definitions and preliminaries to understand this paper.

A subset E of an abstract Wiener product space B^2 is said to be scale-invariant measurable provided $\{(\rho_1x_1, \rho_2x_2) : (x_1, x_2) \in E\}$ is abstract Wiener measurable for every $\rho_1 > 0$ and $\rho_2 > 0$, and a scale-invariant measurable set N of B^2 is said to be scale-invariant null provided $(\nu \times \nu)(\{(\rho_1x_1, \rho_2x_2) : (x_1, x_2) \in N\}) = 0$ for any $\rho_1, \rho_2 > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F on B^2 is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho_1\,\cdot\,, \rho_2\,\cdot\,)$ is measurable for any $\rho_1, \rho_2 > 0$. If two functionals F and G on B^2 are equal s-a.e., i.e., for any $\rho_1, \rho_2 > 0$, $(\nu \times \nu)(\{(x_1, x_2) \in B \times B : F(\rho_1x_1, \rho_2x_2) \neq G(\rho_1x_1, \rho_2x_2)\}) = 0$, then we say that two functionals F and G are coincided s-a.e. [16].

Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal set in H with e_j's are in B^*. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)^\sim$ by

$$(h, x)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle_H (e_j, x), & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases},$$

where $(\cdot, \cdot)$ is the natural dual paring on $B^* \times B$. Then it is well known [16] that for each $h(\neq 0)$ in H, $(h, \cdot)^\sim$ exists for all $x \in B$, is a Gaussian random variable on B with mean zero and variance $|h|_H^2$ and is essentially independent of the choice of the complete orthonormal set. The following integration formula is used several times in this paper. For $h \in H$ and $x \in B$,

$$\int_B \exp\{i\rho(h, x)^\sim\}d\nu(x) = \exp\left\{-\frac{\rho^2}{2}|h|_H^2\right\}, \qquad \rho > 0. \tag{1}$$

Let X and Y be normed spaces and let $\mathcal{L}(X : Y)$ be the space of all bounded linear operators from X into Y. Hence the space $\mathcal{L}(B : B)$ is the set of all bounded linear operators from B to B.

We are ready to state the definition of generalized analytic Feynman integral combining the bounded linear operator.

Definition 2.1. *Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$ and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0$ and $Re(\lambda) \geq 0\}$. Give two operators S_1 and S_2 in $\mathcal{L}(B : B)$, let $F : B^2 \longrightarrow \mathbb{C}$ be a functional such that for each $\lambda_1 > 0$ and $\lambda_2 > 0$, the Wiener integral*

$$J(\lambda_1, \lambda_2) = \int_{B^2} F(\lambda_1^{-\frac{1}{2}} S_1 x_1, \lambda_2^{-\frac{1}{2}} S_2 x_2) d(\nu \times \nu)(x_1, x_2)$$

exists as a real number. If there exists a function $J^*(\lambda_1, \lambda_2)$ analytic in $\mathbb{C}_+ \times \mathbb{C}_+$ such that $J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$, then $J^*(\lambda_1, \lambda_2)$ is defined to be the generalized analytic Wiener integral of F over B^2 with parameters λ_1 and λ_2, and for $\lambda_1, \lambda_2 \in \mathbb{C}_+$ we write

$$J^*(\lambda_1, \lambda_2) = \int_{B^2}^{an_{\lambda_1,\lambda_2}^{S_1,S_2}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2).$$

Let q_1 and q_2 be nonzero real numbers and let F be a functional such that $J^*(\lambda_1, \lambda_2)$ exists for all $\lambda_1, \lambda_2 \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameters q_1, q_2 and we write

$$\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2) = \lim_{\substack{\lambda_1 \to -iq_1 \\ \lambda_2 \to -iq_2}} \int_{B^2}^{an_{\lambda_1,\lambda_2}^{S_1,S_2}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2)$$

where λ_j approaches $-iq_j$ through values in $\mathbb{C}_+$, $j = 1, 2$.

Remark 2.2. *When $S_1 = S_2 = I$, where I is the identity operator, our generalized analytic Feynman integral is the analytic Feynman integral, namely,*

$$\int_{B^2}^{anf_{q_1,q_2}^{I,I}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2) = \int_{B^2}^{anf_{q_1,q_2}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2).$$

For a more detailed study of the analytic Feynman integral, see [5, 8–10, 17].

For an operator T in $\mathcal{L}(H : H)$, the extension operator $\overline{T}$ of T on B always exists and is an element of $\mathcal{L}(B : H)$ and so its adjoint operator $\overline{T}^* \in \mathcal{L}(H : B^*)$. Since $B^* \subset H$, we can consider that $\overline{T}^* \in \mathcal{L}(H : H)$. In order to develop our theories, let $\mathbb{E}$ be the set of all extension operator of an operator in $\mathcal{L}(H : H)$, namely,

$$\mathbb{E} = \{\overline{T} : T \in \mathcal{L}(H : H)\}.$$

Then following proposition which play key roles in this paper. For each $h \in H, x \in B$ and $S \in \mathbb{E}$

$$(h, Sx)^\sim = (S^* h, x)^\sim. \tag{2}$$

3. Generalized analytic Feynman integrals

In this section we establish some generalized analytic Feynman integration formulas of functionals in $\mathcal{F}(B^2)$.

We first show that the generalized analytic Wiener integral of functionals in $\mathcal{F}(B^2)$ exist.

Lemma 3.1. *Let S_1 and S_2 in $\mathbb{E}$, and let F be an element of $\mathcal{F}(B^2)$. Then the generalized analytic Wiener integral $\int_{B^2}^{an_{\lambda_1,\lambda_2}^{S_1,S_2}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2)$ exists and is given by the formula*

$$\int_H \exp\Big\{-\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2\Big\} df(h). \tag{3}$$

Proof. For $\lambda_1 > 0$ and $\lambda_2 > 0$, using the Fubini theorem and equations (1) and (2), we have

$$\begin{aligned} J(\lambda_1, \lambda_2) &\equiv \int_{B^2} F(\lambda_1^{-\frac{1}{2}} S_1 x_1, \lambda_2^{-\frac{1}{2}} S_2 x_2) d(\nu \times \nu)(x_1, x_2) \\ &= \int_{B^2} \int_H \exp\Big\{ \sum_{j=1}^{2} i\lambda_j^{-\frac{1}{2}} (h, S_j x_j)^\sim \Big\} df(h) d(\nu \times \nu)(x_1, x_2) \\ &= \int_H \int_{B^2} \exp\Big\{ \sum_{j=1}^{2} i\lambda_j^{-\frac{1}{2}} (S_j^* h, x_j)^\sim \Big\} d(\nu \times \nu)(x_1, x_2) df(h) \\ &= \int_H \exp\Big\{ -\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2 \Big\} df(h). \end{aligned}$$

Note that for all $\lambda_1 > 0$ and $\lambda_2 > 0$,

$$|J(\lambda_1, \lambda_2)| \le \int_H \Big| \exp\Big\{ -\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2 \Big\} \Big| |df(h)| \le \|f\| < \infty.$$

Now let for $\lambda_1, \lambda_2 \in \mathbb{C}_+$,

$$J^*(\lambda_1, \lambda_2) = \int_H \exp\Big\{ -\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2 \Big\} df(h).$$

Then $J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0$ and $\lambda_2 > 0$. We left to show that the function $J^*(\lambda_1, \lambda_2)$ is analytic in $\mathbb{C}_+^2$. In order to do this, let Γ be any closed contour in $\mathbb{C}_+^2$. Then by using the Morera theorem and the Fubini theorem, we have

$$\begin{aligned} \int_\Gamma J^*(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 &= \int_\Gamma \int_H \exp\Big\{ -\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2 \Big\} df(h) d\lambda_1 d\lambda_2 \\ &= \int_H \int_\Gamma \exp\Big\{ -\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2 \Big\} d\lambda_1 d\lambda_2 df(h) \\ &= 0, \end{aligned}$$

because the function $\exp\Big\{ -\sum_{j=1}^{2} \frac{1}{2\lambda_j} |S_j^* h|_H^2 \Big\}$ is analytic in $\mathbb{C}_+^2$ as a function of (λ_1, λ_2). Hence we complete the proof of Lemma 3.1 as desired. □

In our next theorem, we establish a formula for the generalized analytic Feynman integral of functionals in $\mathcal{F}(B^2)$.

Theorem 3.2. *Let q_1 and q_2 be nonzero real numbers and let S_1, S_2 and F be as in Lemma 3.1 above. Then the generalized analytic Feynman integral $\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F(x_1, x_2) d(\nu \times \nu)(x_1, x_2)$ of F exists and is given by the formula*

$$\int_H \exp\Big\{ -\sum_{j=1}^{2} \frac{i}{2q_j} |S_j^* h|_H^2 \Big\} df(h). \tag{4}$$

Proof. In Lemma 3.1 above, the existence of generalized analytic Wiener integral was established. To complete the proof, it suffices to show that

$$\lim_{\substack{\lambda_1\to -iq_1\\ \lambda_2\to -iq_2}} J^*(\lambda_1,\lambda_2)=\int_H \exp\Big\{-\sum_{j=1}^{2}\frac{i}{2q_j}|S_j^*h|_H^2\Big\}df(h).$$

For given nonzero real numbers q_j, $j=1,2$, there exist sequences $\{\lambda_{nj}\}_{n=1}^{\infty}$, $j=1,2$, in $\mathbb{C}_+$ such that $\lambda_{nj}\to -iq_j$ as $n\to\infty$. By Lemma 3.1, $|J^*(\lambda_{l1},\lambda_{r2})|\le \|f\|$ for all $l,r=1,2,\cdots$. Hence using the dominated convergence theorem, for all nonzero real numbers q_1 and q_2,

$$\begin{aligned}\lim_{\substack{\lambda_1\to -iq_1\\ \lambda_2\to -iq_2}} J^*(\lambda_1,\lambda_2)&=\lim_{\substack{\lambda_{l1}\to -iq_1\\ \lambda_{r2}\to -iq_2}}\int_H \exp\Big\{-\frac{1}{2\lambda_{l1}}|S_1^*h|_H^2-\frac{1}{2\lambda_{r2}}|S_2^*h|_H^2\Big\}df(h)\\ &=\int_H \exp\Big\{-\sum_{j=1}^{2}\frac{i}{2q_j}|S_j^*h|_H^2\Big\}df(h),\end{aligned}$$

which establishes equation (4) as desired. Furthermore,

$$\Big|\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)\Big|\le \|f\|<\infty.$$

Hence we complete the proof of Theorem 3.2. □

From the results in Theorem 3.2 above together with some results in [16], we have the following equations :

(I) Using equation (4), we have

$$\begin{aligned}\int_{B^2}^{anf_{1,-1}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)&=\int_H \exp\Big\{-\frac{i}{2}|S_1^*h|_H^2+\frac{i}{2}|S_2^*h|_H^2\Big\}df(h)\\ &=\int_H \exp\Big\{-\frac{i}{2}\langle S_1S_1^*h,h\rangle_H+\frac{i}{2}\langle S_2S_2^*h,h\rangle_H\Big\}df(h).\end{aligned}\tag{5}$$

In particular, if S_1 and S_2 are unitary operators on H, then we have

$$\int_{B^2}^{anf_{1,-1}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)=\int_H \exp\Big\{-\frac{i}{2}\langle h,h\rangle_H+\frac{i}{2}\langle h,h\rangle_H\Big\}df(h)=f(H).$$

(II) If $S_1^*=A_1^{\frac12}$ and $S_2^*=A_2^{\frac12}$, where $A_j^{\frac12}$ is the nonnegative self-adjoint operator introduced by Kallianpur and Bromley in [16, Proposition 3.3], then we have

$$\begin{aligned}\int_{B^2}^{anf_{1,-1}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2)&=\int_H \exp\Big\{-\frac{i}{2}|A_1^{\frac12}h|_H^2+\frac{i}{2}|A_2^{\frac12}h|_H^2\Big\}df(h)\\ &=\int_H \exp\Big\{-\frac{i}{2}\langle A_1h,h\rangle_H+\frac{i}{2}\langle A_2h,h\rangle_H\Big\}df(h)\\ &=\int_H \exp\Big\{-\frac{i}{2}\langle Ah,h\rangle_H\Big\}df(h)\end{aligned}\tag{6}$$

where $A=A_1-A_2$.

(III) The facts (I) and (II) tell us that our formulas and results are more generalized formulas than the results in [16]. That is to say, many formulas and results of Kallianpur and Bromley are corollaries of our formulas and results.

4. Further generalized analytic Feynman integration formulas involving the generalized first variations

In this section we establish some generalized analytic Feynman integrals involving the generalized first variation.

Definition 4.1. *Let S_1 and S_2 be elements of $\mathcal{L}(B:B)$ and let F be a measurable functional on B^2. Then the generalized first variation $\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)$ of F is defined by the formula*

$$\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)=\frac{\partial}{\partial\alpha_1}F(x_1+\alpha_1S_1u_1,x_2)\Big|_{\alpha_1=0}+\frac{\partial}{\partial\alpha_2}F(x_1,x_2+\alpha_2S_2u_2)\Big|_{\alpha_2=0}, \tag{7}$$

for $x_1,x_2,u_1,u_2\in B$ if it exists.

In Theorem 4.2 below, we show that the generalized first variation of functionals in $\mathcal{F}(B^2)$ are elements of $\mathcal{F}(B^2)$.

Theorem 4.2. *Let S_1 and S_2 be elements of $\mathbb{E}$ and let F be an element of $\mathcal{F}(B^2)$. Let u_1 and u_2 be in H. Assume that*

$$\int_H |h|_H|df(h)|<\infty. \tag{8}$$

Then the generalized first variation $\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)$ of F exists, belongs to $\mathcal{F}(B^2)$ and is given by the formula

$$\int_H i\langle S_1^*h,u_1\rangle_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}df(h)+\int_H i\langle S_2^*h,u_2\rangle_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}df(h). \tag{9}$$

Proof. Using the dominated convergence theorem, equations (4) and (7), equation (9) is obtained as follows:

$$\begin{aligned}
&\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)\\
&=\frac{\partial}{\partial\alpha_1}\int_H\exp\Big\{i(h,x_1)^\sim+i\alpha_1\langle S_1^*h,u_1\rangle_H+i(h,x_2)^\sim\Big\}df(h)\Big|_{\alpha_1=0}\\
&\quad+\frac{\partial}{\partial\alpha_2}\int_H\exp\Big\{i(h,x_1)^\sim+i(h,x_2)^\sim+i\alpha_2\langle S_2^*h,u_2\rangle_H\Big\}df(h)\Big|_{\alpha_2=0}\\
&=\int_H i\langle S_1^*h,u_1\rangle_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}df(h)+\int_H i\langle S_2^*h,u_2\rangle_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}df(h).
\end{aligned}\tag{10}$$

In fact, using equation (10), the Hólder inequality and the assumption (8), we have

$$\begin{aligned}
\Big|\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)\Big| &\le \int_H|\langle S_1^*h,u_1\rangle_H||df(h)|+\int_H|\langle S_2^*h,u_1\rangle_H||df(h)|\\
&\le\int_H|S_1^*h|_H\,|u_1|_H|df(h)|+\int_H|S_2^*h|_H\,|u_2|_H|df(h)|\\
&\le\int_H\|S_1^*\|_{op}\,|h|_H\,|u_1|_H|df(h)|+\int_H\|S_2^*\|_{op}\,|h|_H\,|u_2|_H|df(h)|,\\
&\le 2M\int_H|h|_H|df(h)|<\infty
\end{aligned}$$

where $\|T\|_{op}$ denotes the operator norm of an operator T and $M=\max\{\|S_1^*\|_{op}|u_1|_H,\|S_2^*\|_{op}|u_2|_H\}$. Furthermore, we note that

$$\begin{aligned}
\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)&=\int_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}df_1(h)+\int_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}df_2(h)\\
&=\int_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}d\tilde{f}(h)
\end{aligned}$$

where $f_j, j=1,2$, are complex measures defined by

$$f_j(E)=\int_E i\langle S_j^*h,u_j\rangle_H df(h),$$

for $E\in\mathcal{B}(H)$ and $\tilde{f}$ is given as in the proof of Theorem 4.2. It means that the generalized first variation $\delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)$ is an element of $\mathcal{F}(B^2)$ and hence we complete the proof of Theorem 4.2. □

In Theorem 4.3, we give a formula for the generalized analytic Feynman integral involving the generalized first variation.

Theorem 4.3. *Let S_1,S_2,S_3 and S_4 be elements of $\mathbb{E}$ and let F be an element of $\mathcal{F}(B^2)$ such that the condition* (8) *is satisfied. Let $u_1,u_2\in H$. Then the generalized analytic Feynman integral $\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}}\delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)$ involving the generalized first variation exists and is given by the formula*

$$\int_H i\langle S_3^*h,u_1\rangle_H\exp\Big\{-\sum_{j=1}^2\frac{i}{2q_j}|S_j^*h|_H^2\Big\}df(h)+\int_H i\langle S_4^*h,u_2\rangle_H\exp\Big\{-\sum_{j=1}^2\frac{i}{2q_j}|S_j^*h|_H^2\Big\}df(h). \tag{11}$$

Proof. We proved that the generalized first variation $\delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)$ exists, belongs to $\mathcal{F}(B^2)$ and is given by the formula

$$\delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)=\int_H\exp\Big\{i\sum_{j=1}^2(h,x_j)^\sim\Big\}d\tilde{f}(h)$$

where $\tilde{f}$ is in the proof of Theorem 4.2. By using equations (4) and (9), we have

$$\begin{aligned}&\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}}\delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)\\&=\int_H\exp\Big\{-\sum_{j=1}^2\frac{i}{2q_j}|S_j^*h|_H^2\Big\}d\tilde{f}(h)\\&=\int_H i\langle S_3^*h,u_1\rangle_H\exp\Big\{-\sum_{j=1}^2\frac{i}{2q_j}|S_j^*h|_H^2\Big\}df(h)+\int_H i\langle S_4^*h,u_2\rangle_H\exp\Big\{-\sum_{j=1}^2\frac{i}{2q_j}|S_j^*h|_H^2\Big\}df(h).\end{aligned}$$

Hence we complete the proof of Theorem 4.3. □

5. Generalized Cameron-Storvick theorem

The Cameron-Storvick theorem says that the (analytic Feynman)Wiener integrals involving the first variation can be expressed by the ordinary forms without the concept of the first variation. It looks like the integration by parts formulas. Numerous constructions and theories regarding the Cameron-Storvick theorem have been studied and applied in various papers [4, 5, 16, 17, 21].

In this section, we establish a more generalized Cameron-Storvick theorem with respect to the our generalized analytic Feynman integral and the generalized first variation.

The following lemma is the basic translation theorem on abstract Wiener space.

Lemma 5.1. (Translation theorem) *Let F be an integrable functional on B and let $x_0\in H$. Then*

$$\int_B F(x+x_0)d\nu(x)=\exp\Big\{-\frac{1}{2}|x_0|_H^2\Big\}\int_B F(x)\exp\{(x_0,x)^\sim\}d\nu(x). \tag{12}$$

Using equation (12), we establish a translation theorem to obtain the generalized Cameron-Storvick theorem.

Lemma 5.2. (Translation theorem with respect to the operators) *Let S_1 and S_2 be elements of $\mathbb{E}$ with $S_1S_1^* = I$ on H. Let F be an integrable functional on B and let $x_0 \in H$. Then*

$$\int_B F(S_1x + S_2x_0)d\nu(x) = \exp\Big\{-\frac{1}{2}|S_1^*S_2x_0|_H^2\Big\}\int_B F(S_1x)\exp\{(S_1^*S_2x_0, x)^\sim\}d\nu(x). \tag{13}$$

Proof. We first note that for $x_0 \in H$, we have $S_2x_0 \in H$ and hence $S_1^*S_2x_0 \in H$. Next, equation (13) immediately follow from equation (12) by replacing F_{S_1} instead of F, where $F_{S_1}(x) = F(S_1x)$ with $F_{S_1}(x+\theta_0)$ and $\theta_0 = S_1^*S_2x_0 \in H$. □

Equation (14) is called the generalized Cameron-Storvick theorem.

Theorem 5.3. *Let $S_1, S_2, S_3, S_4, F, f, u_1$, and u_2 be as in Theorem 4.3 above. Then*

$$\begin{aligned}&\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} \delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)\\ &= 2\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2) - i\sum_{j=1}^{2} q_j \int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F_j(x_1,x_2)d(\nu\times\nu)(x_1,x_2).\end{aligned} \tag{14}$$

where

$$F_1(x_1,x_2) = (S_1S_3^*S_1h, x_1)^\sim F(x_1,x_2)$$

and

$$F_2(x_1,x_2) = (S_2S_4^*S_2h, x_2)^\sim F(x_1,x_2).$$

Proof. The existence of the right-hand side of equation (14) was established in Theorem 4.3 above. We only left to show that the equality in equation (14) holds. For $\lambda_1 > 0$ and $\lambda_2 > 0$, we have

$$\begin{aligned}&\int_{B^2}\delta^{S_3,S_4}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)\\ &= \int_{B^2}\Big[\frac{\partial}{\partial\alpha_1}F(\lambda_1^{-\frac12}S_1x_1+\alpha_1S_3u_1, \lambda_2^{-\frac12}S_2x_2)\Big|_{\alpha_1=0} + \frac{\partial}{\partial\alpha_2}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2+\alpha_2S_4u_2)\Big|_{\alpha_2=0}\Big]d(\nu\times\nu)(x_1,x_2).\end{aligned}$$

Now we apply equation (13) with respect to the first and the second arguments of F, we have

$$\begin{aligned}&\int_{B^2}\delta^{S_3,S_4}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)\\ &= \frac{\partial}{\partial\alpha_1}\Big[\exp\Big\{-\frac{\lambda_1\alpha_1^2}{2}|S_1^*S_3u_1|_H^2\Big\}\int_{B^2}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)\exp\{\lambda_1^{\frac12}\alpha_1(S_3^*S_1h,x)^\sim\}d(\nu\times\nu)(x_1,x_2)\Big]\Big|_{\alpha_1=0}\\ &+ \frac{\partial}{\partial\alpha_2}\Big[\exp\Big\{-\frac{\lambda_2\alpha_2^2}{2}|S_2^*S_4u_2|_H^2\Big\}\int_{B^2}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)\exp\{\lambda_2^{\frac12}\alpha_2(S_4^*S_2h,x)^\sim\}d(\nu\times\nu)(x_1,x_2)\Big]\Big|_{\alpha_2=0}\\ &= 2\int_{B^2}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)d(\nu\times\nu)(x_1,x_2) + \lambda_1^{\frac12}\int_{B^2}(S_3^*S_1h,x_1)^\sim F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)d(\nu\times\nu)(x_1,x_2)\\ &\quad + \lambda_2^{\frac12}\int_{B^2}(S_4^*S_2h,x_2)^\sim F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)d(\nu\times\nu)(x_1,x_2)\\ &= 2\int_{B^2}F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)d(\nu\times\nu)(x_1,x_2) + \lambda_1\int_{B^2}(S_3^*S_1h, \lambda_1^{-\frac12}S_1^*S_1x_1)^\sim F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)d(\nu\times\nu)(x_1,x_2)\\ &\quad + \lambda_2\int_{B^2}(S_4^*S_2h, \lambda_2^{-\frac12}S_2^*S_2x_2)^\sim F(\lambda_1^{-\frac12}S_1x_1, \lambda_2^{-\frac12}S_2x_2)d(\nu\times\nu)(x_1,x_2)\end{aligned}$$

$$= 2\int_{B^2} F(\lambda_1^{-\frac{1}{2}}S_1x_1, \lambda_2^{-\frac{1}{2}}S_2x_2)d(\nu\times\nu)(x_1,x_2) + \lambda_1\int_{B^2}(S_1S_3^*S_1h, \lambda_1^{-\frac{1}{2}}S_1x_1)^{\sim}F(\lambda_1^{-\frac{1}{2}}S_1x_1, \lambda_2^{-\frac{1}{2}}S_2x_2)d(\nu\times\nu)(x_1,x_2)$$
$$+ \lambda_2\int_{B^2}(S_2S_4^*S_2h, \lambda_2^{-\frac{1}{2}}S_2x_2)^{\sim}F(\lambda_1^{-\frac{1}{2}}S_1x_1, \lambda_2^{-\frac{1}{2}}S_2x_2)d(\nu\times\nu)(x_1,x_2).$$

It can be analytically continued in $(\lambda_1, \lambda_2) \in \mathbb{C}_+^2$ by similar methods in the proof of Theorem 3.2, and thus, letting $\lambda_j \to -iq_j$, $j = 1, 2$, we have

$$\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} \delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)$$
$$= 2\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2) - iq_1\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F_1(x_1,x_2)d(\nu\times\nu)(x_1,x_2) - iq_2\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F_2(x_1,x_2)d(\nu\times\nu)(x_1,x_2).$$

Hence we have the desired results. □

From Theorem 5.3, we have the following corollary.

Corollary 5.4. *(I) If $S_1 = S_3$ and $S_2 = S_4$, then*

$$\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} \delta^{S_1,S_2}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)$$
$$= 2\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2) - i\sum_{j=1}^{2} q_j\int_{B^2}^{anf_{q_1,q_2}^{S_1,S_2}} G_j(x_1,x_2)d(\nu\times\nu)(x_1,x_2)$$

where

$$G_1(x_1,x_2) = (S_1h, x_1)^{\sim}F(x_1,x_2)$$

and

$$G_2(x_1,x_2) = (S_2h, x_2)^{\sim}F(x_1,x_2).$$

(II) If $S_1 = S_2 = S_3 = S_4 \equiv S$ and $q_1 = q_2 \equiv q$, then

$$\int_{B^2}^{anf_{q,q}^{S,S}} \delta^{S,S}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)$$
$$= 2\int_{B^2}^{anf_{q,q}^{S,S}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2) - iq\sum_{j=1}^{2}\int_{B^2}^{anf_{q,q}^{S,S}} L_j(x_1,x_2)d(\nu\times\nu)(x_1,x_2) \tag{15}$$

where

$$L_1(x_1,x_2) = (Sh, x_1)^{\sim}F(x_1,x_2)$$

and

$$L_2(x_1,x_2) = (Sh, x_2)^{\sim}F(x_1,x_2).$$

(III) If $S_i = I$ on H for all $i = 1, 2, 3, 4$, then our generalized analytic Feynman integral is the analytic Feynman integral and hence we have

$$\int_{B^2}^{anf_{q_1,q_2}} \delta^{S_3,S_4}F(x_1,x_2|u_1,u_2)d(\nu\times\nu)(x_1,x_2)$$
$$= 2\int_{B^2}^{anf_{q_1,q_2}} F(x_1,x_2)d(\nu\times\nu)(x_1,x_2) - \sum_{j=1}^{2} iq_j\int_{B^2}^{anf_{q_1,q_2}} K_j(x_1,x_2)d(\nu\times\nu)(x_1,x_2)$$

where

$$K_1(x_1,x_2) = (h,x_1)^\sim F(x_1,x_2)$$

and

$$K_2(x_1,x_2) = (h,x_2)^\sim F(x_1,x_2).$$

6. Possible results

Though the Sections 3-5, we generalize various formulas for the Feynman integrals and the integration by parts formulas combining bounded linear operators. We close this paper by giving some possible examples for the operators through subsequent remarks.

Remark 6.1. *We note that* $\overline{H}^{\|\cdot\|_0} = B$ *and* $\overline{B^*}^{\|\cdot\|_H} = H$. *For any* $h \in H$, *there exists a sequence* $\{e_n\}_{n=1}^\infty$ *in* B^* *so that* $\|e_n - h\|_H \to 0$ *as* $n \to \infty$. *This convergence is independent for the choice of* $\{e_n\}_{n=1}^\infty$ *in* B^*. *Now, let*

$$H_n(x) = \exp\{(e_n,x)\}$$

and

$$H(x) = \exp\{(h,x)^\sim\}.$$

Then $H_n(x)$ *converges to* $H(x)$ *for* ν*-a.e.* $x \in B$ *by the Kolmogorov theorem. From this observation, we see that*

$$\lim_{n\to\infty}\int_{B^*}\exp\Big\{i\sum_{j=1}^{2}(e_n,x_j)\Big\}df(v) = \int_H \exp\Big\{i\sum_{j=1}^{2}(h,x_j)^\sim\Big\}df(v)$$

for a.e. $(x_1,x_2) \in B^2$. *Hence our results and formulas can be obtained for the functionals of the form*

$$\int_{B^*}\exp\Big\{i\sum_{j=1}^{2}(h,x_j)\Big\}df(v)$$

for a.e. $(x_1,x_2) \in B^2$.

Remark 6.2. *We give an example of abstract Wiener space, and introduce some operators.*

(i) The Hilbert space

$$C_0' \equiv C_0'[0,T] = \{v : [0,T] \to \mathbb{R} : v(t) = \int_0^t z_v(s)ds, z_v \in L_2[0,T]\}$$

with the norm $\|\cdot\|^2_{C_0'[0,T]} = \int_0^t z_v^2(s)ds$ *is being used to explain various theories in mathematics fields. Its completion with respect to the measurable norm* $\|v\|_{C_0[0,T]} = \sup_{t\in[0,T]}|v(t)|$ *is the classical Wiener space* $C_0[0,T]$. *That is to say,* $(C_0'[0,T], C_0[0,T], m_w)$ *is an example of abstract Wiener space. Let* $A_1 : C_0'[0,T] \to C_0'[0,T]$ *be the linear operator defined by*

$$(A_1 w)(t) = \int_0^t w(s)ds. \tag{16}$$

Then we see that the adjoint operator A_1^* *of* A_1 *is given by*

$$A_1^* w(t) = w(T)t - \int_0^t w(s)ds = \int_0^t [w(T) - w(s)]ds$$

and the linear operator $P = A_1^* A_1$ is given by

$$Pw(t) = \int_0^T \min\{s,t\} w(s)ds.$$

Furthermore, we see that P is a self-adjoint operator on $C_0'[0,T]$ and that

$$(w_1, Aw_2)_{C_0'} = (A_1 w_1, A_1 w_2)_{C_0'} = \int_0^T w_1(s) w_2(s) ds$$

for all $w_1, w_2 \in C_0'[0,T]$. Hence P is a positive definite operator, i.e., $(w, Aw)_{C_0'} \geq 0$ for all $w \in C_0'[0,T]$. One can show that the orthonormal eigenfunction $\{e_m\}$ of P are given by

$$e_m(t) = \frac{\sqrt{2T}}{(m-\frac{1}{2})\pi} \sin\Big(\frac{(m-\frac{1}{2})\pi}{T} t\Big) \equiv \int_0^t \alpha_m(s)ds$$

with corresponding eigenvalues $\{\beta_m\}$ given by

$$\beta_m = \Big(\frac{T}{(m-\frac{1}{2})\pi}\Big)^2.$$

Furthermore, it can be shown that $\{e_m\}$ is a basis of $C_0'[0,T]$ and so $\{\alpha_m\}$ is a basis of $L_2[0,T]$, and that P is a trace class operator and so A_1 is a Hilbert-Schmidt operator on $C_0'[0,T]$. In fact, the trace of P is given by $TrP = \frac{1}{2}T^2 = \int_0^T t dt$.

(ii) *We next consider the multiplication operator A_2 which plays an important role in physics (quantum theories), see [18]. We define a multiplication operator A_2 with $t \in [0,T]$ on $C_0'[0,T]$ by*

$$(A_2(x))(t) \equiv A_2(x(t)) = tx(t). \tag{17}$$

Then we have $A_2(xy) = tx(t)y(t)$ and $xA_2(y) = x(t)ty(t)$. Also, one can easily check that $A_2^* v(t) = tv(t)$ for all $v \in C_0'$. Note that, the expected value or corresponding mean value is

$$E(x) \equiv \int_0^T t|x(t)|^2 dt = \int_0^T A_2(|x|^2)(t)dt,$$

where x is the state function of a particle in quantum mechanics and $\int_0^T |x(t)|^2 dt$ is the probability that the particle will be founded in $[0,T]$.

Remark 6.3. *We give another example of abstract Wiener space.*

(i) *Let $H \equiv l^2$ be the space of all sequences of real numbers with $\sum_{n=1}^{\infty} x_n^2 < \infty$. That is*

$$H \equiv l^2 = \Big\{(x_n) : \sum_{n=1}^{\infty} x_n^2 < \infty\Big\}.$$

Its completion with respect to the measurable norm $\|(x_n)\|_0 = \sum_{n=1}^{\infty} \frac{1}{n^2} x_n^2$ is

$$B = \Big\{(x_n) : \sum_{n=1}^{\infty} \frac{1}{n^2} x_n^2 < \infty\Big\}.$$

Also, note that its dual space is

$$B^* = \Big\{(x_n) : \sum_{n=1}^{\infty} n^2 x_n^2 < \infty\Big\}.$$

(ii) Let $R : B \to H$ be a linear operator defined by

$$R((x_n)) = \left(\frac{1}{n}x_n\right).$$

Now, let $A_3 = R|_H$. Then $A_3 \in \mathcal{L}(H : H)$, A_3 is a self-adjoint operator and Hilbert-Schdmit operator on H.

Remark 6.4. *Using the concept of ν-lifting on abstract Wiener space, the operators A_1, A_2 and A_3 can be extended on B, for more detailed study for the m-lifting see [5, 8, 11, 12, 18, 19].*

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