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A Normal Distribution on Time Scales with Application

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Abstract. We introduce a new normal distribution on time scales. Based on this generalized normal distribution, a Brownian motion is introduced and its quadratic variation is derived.

1. Introduction

Probability theory on time scales is still in development, see [20] for general theory, [4, 13, 18] for investigations of time scales analogues of distributions, [1, 2, 5, 10–12, 14, 19] for stochastic calculus on time scales, and see [16, 17] for papers on stochastic time scales, meaning the time scale is generated by sampling a random variable. See also [7, 15] as general references to relevant material.

One major deficit in probability theory on time scales is the lack of a normal distribution. The first attempt at such a distribution on time scales appears in [6] which presented a possible Gaussian bell as the solution to the dynamic equation $y^{\Delta} = \ominus(t \odot 1)y$ with initial condition $y(t_0) = 1$, but it was shown that this solution is not even square integrable on all isolated time scales.

We investigate a different approach for the normal distribution on time scales which is defined for all time scales with a finite lower bound and all isolated time scales. Therefore, our distribution will generalize the truncated normal distribution to all time scales and generalize the whole normal distribution for isolated time scales.

After proving its existence and the existence of its expected value and variance, we then show how it can be used as a basis for a Brownian motion on time scales. We emphasize that our construction is different than that in [1, 8, 9] which used the classical normal distribution restricted to a time scale to define a Brownian motion instead of a normal distribution dependent on the time scale for its definition.

2. A Generalized Normal Distribution

Throughout this paper, we will use the notation and basic definitions of classical probability theory. In this section, we will introduce a normal distribution for time scales. Let $\mathbb{T}$ be a time scale with forward

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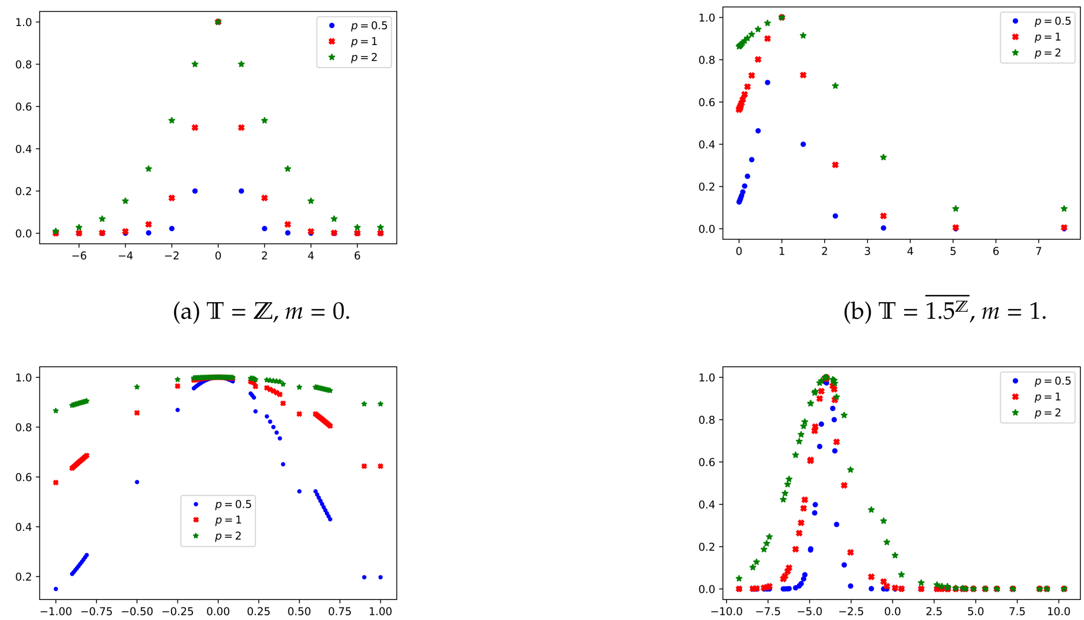
jump operator and delta differentiation operator σ and Δ, respectively. Suppose that $m \in \mathbb{T}$, $p \in \mathbb{R}$, and $p \neq 0$. For convenience of notation, let $w_1(t) = \dfrac{t-m}{p^2}$ and $w_2(t) = -w_1(t)$, and define

$$q(t) = \begin{cases} (\ominus w_1)(t), & t \geq m, \\ (\ominus w_2)(t), & t < m. \end{cases}$$

With this q, we define

$$N_1(t; m, p) = \begin{cases} e_q(\sigma(t), m), & t \geq m, \\ e_q(m, t), & t < m. \end{cases}$$



(a) $\mathbb{T} = \mathbb{Z}$, $m = 0$.

(b) $\mathbb{T} = \overline{1.5^{\mathbb{Z}}}$, $m = 1$.

(c) $\mathbb{T}$ is comprised of a mixture of isolated points and intervals, $m = 0$.

(d) $\mathbb{T}$ is generated by sampling a normal distribution, $m = -4$.

Figure 1: Plots of the normal distribution on two different time scales.[1)]

Example 2.1. *Let $\mathbb{T} = \mathbb{Z}$, $m = 0, p = 1$. Then*

$$N_1(t; 0, 1) = \begin{cases} \displaystyle\prod_{s \in [0, \sigma(t))} \frac{1}{1+s} = \frac{1}{(t+1)!}, & t \geq 0, \\ \displaystyle\prod_{s \in [t, 0)} \frac{1}{1-s} = \frac{1}{(|t|+1)!}, & t < 0. \end{cases}$$

[1)]Figures made using the `timescalecalculus` Python package, commit `96e1faa`, see `https://github.com/tomcuchta/timescalecalculus`

t	$N_1(t;0,1)$	t	$N_1(t;0,1)$
-3	$\frac{1}{24}$	1	$\frac{1}{2}$
-2	$\frac{1}{6}$	2	$\frac{1}{6}$
-1	$\frac{1}{2}$	3	$\frac{1}{24}$
0	1	4	$\frac{1}{120}$

In this case, we obtain integrability over the time scale and one can compute

$$\int_{\mathbb{Z}} N_1(t;0,1)\Delta t = 1 + \sum_{k=-\infty}^{-1} \frac{1}{|k|!} + \sum_{k=1}^{\infty} \frac{1}{k!} = 2e - 3.$$

Lemma 2.2. *If $t > m$, then $t \mapsto N_1(t;m,p)$ is strictly decreasing and if $t < m$, then $t \mapsto N_1(t;m,p)$ is strictly increasing.*

Proof. At right-dense points $t \neq m$, e_q is clearly decreasing, so assume that $\mu(t) > 0$ for the remainder of the proof. First compute

$$1 + \mu(t)q(t) = \begin{cases} \dfrac{p^2}{p^2 + \mu(t)(t-m)}, & t \geq m, \\ \dfrac{p^2}{p^2 + \mu(t)(m-t)}, & t < m, \end{cases}$$

and from both pieces, we observe that $1 + \mu(t)q(t) > 0$. Hence $q \in \mathcal{R}^+$ and so $e_q > 0$. For all $t > m$, we see $0 < 1 + \mu(t)q(t) = \dfrac{p^2}{p^2 + \mu(t)(t-m)} < 1$. From here, we conclude that $t \mapsto e_q(t,m)$ is a nonincreasing function. If $t < m$, then

$$N^{\Delta}(t;m,p) = e_q^{\Delta}(m,t) = e_{\ominus q}^{\Delta}(t,m) = (\ominus q)(t)e_q(m,t) = \frac{m-t}{p^2}e_q(m,t) > 0,$$

completing the proof. $\square$

In the next result, we use the notation $N_1(t;m,p,\mathbb{T})$ to denote the normal distribution on the time scale $\mathbb{T}$.

Lemma 2.3. *Let $L \in \mathbb{T}$ be a limit point of $\mathbb{T}$ with $L > m$. Let also $V \subset \mathbb{R}$ be an open set containing L, with $u = \rho(\min(V \cap \mathbb{T})) > m$, and $w = \sigma(\max(V \cap \mathbb{T}))$. Then*

$$\int_u^w N_1(\tau;m,p,\mathbb{T})\Delta\tau < \int_u^w N_1(\tau;m,p,\mathbb{T} \setminus V)\Delta\tau.$$

Proof. By Lemma 2.2, we see that $t \mapsto N_1(t;m,p,\mathbb{T})$ is decreasing on (u,w), for all the points $t_1 < t_2 < t_3 < \ldots \in V \cap \mathbb{T}$:

$$N_1(u;m,p,\mathbb{T}) > N_1(t_1;m,p,\mathbb{T}) > N_1(t_2;m,p,\mathbb{T}) > \ldots > N_1(w;m,p,\mathbb{T}).$$

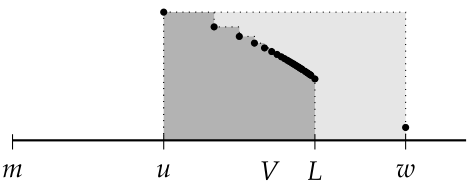


Figure 2: A left-dense limit point L. The lightly shaded area corresponds to $\int_u^v N_1(\tau; m, p, \mathbb{T} \setminus V)\Delta\tau$, while the darkly shaded area corresponds to $\int_u^v N_1(\tau; m, p, \mathbb{T})\Delta\tau$.

We compute

$$\int_u^w N_1(\tau; m, p, \mathbb{T} \setminus V)\Delta\tau = \mu(u)e_q(u, m; \mathbb{T} \setminus V) = (w-u)e_q(u, m; \mathbb{T} \setminus V).$$

On the other hand, since $N_1(u; m, p, \mathbb{T}) = N_1(u; m, p, \mathbb{T} \setminus V)$,

$$\begin{aligned}\left|\int_u^w N_1(\tau; m, p, \mathbb{T})\Delta\tau\right| &\leq \int_u^w \left|N_1(\tau; m, p, \mathbb{T})\right|\Delta\tau \\ &< \int_u^w \left|N_1(u; m, p, \mathbb{T})\right|\Delta\tau \\ &= (w-u)\left|N_1(u; m, p, \mathbb{T} \setminus V)\right| \\ &= \int_u^w N_1(\tau; m, p, \mathbb{T} \setminus V)\Delta\tau,\end{aligned}$$

completing the proof for $u > m$. □

Lemma 2.4. *If $\mathbb{T}$ is a time scale of isolated points, then $t \mapsto e_q(t, m)$ is integrable on $[m, \infty) \cap \mathbb{T}$ and on $(-\infty, m] \cap \mathbb{T}$.*

Proof. We necessarily have $\mu(t) \geq c$ for some fixed positive real number c, else the time scale is no longer contains only isolated points. Express the relevant portion of the time scale as $\widetilde{\mathbb{T}} = \{m = t_0, t_1, t_2, \ldots\}$ with $t_{k+1} > t_k$ for all $k = 0, 1, 2, \ldots$. We will show that

$$\left|\int_m^\infty e_q(\sigma(\tau), m)\Delta\tau\right| < \infty. \tag{1}$$

We compute

$$\begin{aligned}\int_m^\infty e_q(\sigma(\tau), m)\Delta\tau &= \sum_{k=0}^\infty \mu(t_k)e_q(t_{k+1}, m) \\ &= \sum_{k=0}^\infty \mu(t_k)\prod_{\ell=0}^k \left[1 + \ominus\left(\frac{t_\ell - m}{p^2}\right)\mu(t_\ell)\right] \\ &= \sum_{k=0}^\infty \mu(t_k)\prod_{\ell=0}^k \left[1 - \frac{(t_\ell - m)\mu(t_\ell)}{p^2 + \mu(t_\ell)(t_\ell - m)}\right] \\ &= \sum_{k=0}^\infty \mu(t_k)\prod_{\ell=0}^k \frac{1}{p^2 + \mu(t_\ell)(t_\ell - m)}.\end{aligned}$$

By the ratio test,

$$\lim_{k\to\infty}\left|\frac{\mu(t_{k+1})\prod_{\ell=0}^{k+1}\frac{1}{p^2+\mu(t_\ell)(t_\ell-m)}}{\mu(t_k)\prod_{\ell=0}^{k}\frac{1}{p^2+\mu(t_\ell)(t_\ell-m)}}\right| = \lim_{k\to\infty}\left|\frac{\mu(t_{k+1})}{\mu(t_k)(p^2+\mu(t_{k+1})(t_{k+1}-m))}\right|$$

$$= \lim_{k\to\infty}\left|\frac{1}{\mu(t_k)\left(\frac{p^2}{\mu(t_{k+1})}+(t_{k+1}-m)\right)}\right|.$$

which equals zero since the graininess cannot approach zero in a time scale of isolated points and hence (1) converges. The proof for integrability on $(-\infty, m]\cap\mathbb{T}$ is the same. □

Theorem 2.5. *If* $\mathbb{T}$ *is a time scale with the property that* $\inf\mathbb{T}\neq-\infty$*, then* $\left|\int_{\mathbb{T}} N_1(\tau;m,p,\mathbb{T})\Delta\tau\right|<\infty$.

Proof. Since $\inf\mathbb{T}>-\infty$, we know that $\int_{\inf\mathbb{T}}^{m} N_1(t;m,p,\mathbb{T})\Delta t$ is finite since it is an integral of a continuous function on a compact domain. If $\sup\mathbb{T}<\infty$, then the proof is trivial since the whole time scale is compact. Assume that $\sup\mathbb{T}=\infty$. Lemma 2.3 shows that there is a time scale $\mathbb{T}^+_{\text{iso}}$ with $\inf\mathbb{T}^+_{\text{iso}}=m$ containing only isolated points where for all $t\in\mathbb{T}^+_{\text{iso}}$,

$$N_1(t;m,p,\mathbb{T})\leq N_1(t;m,p,\mathbb{T}^+_{\text{iso}}). \tag{2}$$

Finally, Lemma 2.4 shows that the upper bound in (2) is integrable, completing the proof. □

Theorem 2.5 shows that $\lim_{t\to\infty}N_1(t;m,p)=0$ and Lemma 2.4 additionally shows that for isolated time scales, $\lim_{t\to-\infty}N_1(t;m,p)=0$. Now we are prepared to define the normal distribution on time scales. Since $q\in\mathcal{R}^+$, Theorem 2.5 shows that if $\inf\mathbb{T}>-\infty$, then $C_{\mathbb{T}}=\int_{\mathbb{T}}N_1(t;m,p,\mathbb{T})\Delta t$ is finite and positive. Lemma 2.4 shows the same for isolated time scales.

Definition 2.6. *We define the probability density of the time scale normal distribution with parameters m and p by* $N(t;m,p)=\dfrac{N_1(t;m,p,\mathbb{T})}{C_{\mathbb{T}}}$. *The support of this random variable is the entire time scale. We say that a random variable with this density is time scale normally distributed.*

Example 2.7. *Let* $\mathbb{T}=\mathbb{Z}$ *with* $m=0$ *and* $p=1$. *Let X be normally distributed with density* $N(t;0,1)=\dfrac{1}{2e-3}N_1(t;0,1)$*, where* N_1 *is defined as in Example 2.1. We may compute the expected value*

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k\in\mathbb{Z}}kN(k;0,1)\\ &= \frac{1}{2e-3}\left[\sum_{k=-\infty}^{-1}\frac{k}{(|k|+1)!}+\sum_{k=0}^{\infty}\frac{k}{(k+1)!}\right]=0.\end{aligned}$$

Similarly, compute the variance

$$\begin{aligned}\operatorname{Var}[X] &= \frac{1}{2e-3}\sum_{k\in\mathbb{Z}} k^2 N(k;0,1)\\ &= \frac{1}{2e-3}\left[\sum_{k=-\infty}^{-1}\frac{k^2}{(|k|+1)!}+\sum_{k=0}^{\infty}\frac{k^2}{(k+1)!}\right]\\ &= \frac{(e-1)+(e-1)}{2e-3}=\frac{2e-2}{2e-3}\approx 1.4104.\end{aligned}$$

On the other hand, consider $\mathbb{T}=\overline{1.5^{\mathbb{Z}}}$ *with* $m=1$ *and* $p=1$. *In this case,* $\mu(t)=0.5t$ *and*

$$q(t)=\begin{cases}\dfrac{1-t}{1+0.5t(t-1)}, & t\geq 1,\\[2mm] \dfrac{t-1}{1+0.5t(1-t)}, & t<1.\end{cases}$$

Therefore if $t=1.5^k\in\mathbb{T}$ *for some* $k\in\mathbb{Z}$, *then*

$$\begin{aligned}N_1(t;1,1) &= \begin{cases} e_q(\sigma(t),1)=\displaystyle\prod_{j=0}^{k}\left(1+\frac{0.5(1.5^j)(1-1.5^j)}{1+0.5(1.5^j)(1.5^j-1)}\right), & t\geq 1,\\ e_q(1,t)=\displaystyle\prod_{j=k}^{-1}\frac{1}{1+\frac{0.5(1.5^j)(1.5^j-1)}{1+0.5(1.5^j)(1-1.5^j)}}, & t<1.\end{cases}\\ &= \begin{cases}\displaystyle\prod_{j=0}^{k}\frac{1}{1+0.5(1.5^j)(1.5^j-1)}, & t\geq 1,\\ \displaystyle\prod_{j=k}^{-1}\Big(1+0.5(1.5^j)(1-1.5^j)\Big), & t<1.\end{cases}\end{aligned}$$

We compute the normalizing constant

$$\begin{aligned}C_{\mathbb{T}} &= \int_{\overline{1.5^{\mathbb{Z}}}} N_1(t;1,1)\\ &= \sum_{k=-\infty}^{\infty} 0.5(1.5)^k N_1(1.5^k;1,1)\\ &= 0.5\sum_{k=-\infty}^{-1}\frac{1}{(1.5)^{|k|}}\prod_{j=k}^{-1}\Big(1+0.5(1.5^k)(1-1.5^k)\Big)+0.5\sum_{k=0}^{\infty}(1.5)^k\prod_{j=0}^{k}\left(\frac{1}{1+0.5(1.5^j)(1.5^j-1)}\right)\\ &\approx 1.23+1.50=2.73.\end{aligned}$$

Therefore we have the $\mathbb{T}=\overline{1.5^{\mathbb{Z}}}$ *normal distribution* $N(t;1,1)\approx\dfrac{1}{2.73}N_1(t;1,1)$. *Estimation of the mean and variance follow as*

$$\mathbb{E}[X]=\frac{1}{2.73}\sum_{k=-\infty}^{\infty}(1.5)^{2k}N(1.5^k;1,1)\approx 2.68$$

and

$$\mathrm{Var}[X] = \frac{1}{2.73}\sum_{k=-\infty}^{\infty}(1.5)^{3k}N(1.5^k;1,1) \approx 5.09.$$

Theorem 2.8. *If* $\inf\mathbb{T} > -\infty$ *and* X *is a random variable with density* $N(t;m,p)$*, then it has expected value* $\mathbb{E}[X] = m+\frac{p^2}{C_{\mathbb{T}}}e_{w_2}(\inf\mathbb{T},m)$ *and it has variance* $\mathrm{Var}[X] = \nu_1+\nu_2-\frac{p^4}{C_{\mathbb{T}}^2}(e_{w_2}(\inf\mathbb{T},m))^2$*, where* $\nu_1 := \frac{1}{C_{\mathbb{T}}}\int_{\inf\mathbb{T}}^{m}(t-m)^2e_q(m,t)\Delta t$ *and* $\nu_2 := \frac{1}{C_{\mathbb{T}}}\int_m^\infty (t-m)^2e_q(\sigma(t),m)\Delta t$.

Proof. Compute

$$\begin{aligned}
\int_m^\infty (t-m)N(t;m,p)\Delta t &= \frac{1}{C_{\mathbb{T}}}\int_m^\infty (t-m)e_q(\sigma(t),m)\Delta t\\
&= \frac{1}{C_{\mathbb{T}}}\int_m^\infty (t-m)(1+\mu(t)(\ominus w_1)(t))e_{\ominus w_1}(t,m)\Delta t\\
&= \frac{1}{C_{\mathbb{T}}}\int_m^\infty \frac{p^2(t-m)}{p^2+(t-m)\mu(t)}e_{\ominus w_1}(t,m)\Delta t\\
&= \frac{-p^2}{C_{\mathbb{T}}}\int_m^\infty e^{\Delta}_{\ominus w_1}(t,m)\Delta t\\
&= -\frac{p^2}{C_{\mathbb{T}}}e_{\ominus w_1}(t,m)\Big|_m^\infty = \frac{p^2}{C_{\mathbb{T}}}.
\end{aligned}$$

Similarly, since $w_2 = -w_1 = \frac{m-t}{p^2}$, we get

$$\begin{aligned}
\int_{\inf\mathbb{T}}^m (t-m)N(t;m,p) &= \frac{1}{C_{\mathbb{T}}}\int_{\inf\mathbb{T}}^m (t-m)e_{\ominus w_2}(m,t)\Delta t\\
&= \frac{-p^2}{C_{\mathbb{T}}}\int_{\inf\mathbb{T}}^m \frac{m-t}{p^2}e_{w_2}(t,m)\Delta t\\
&= \frac{-p^2}{C_{\mathbb{T}}}\int_{\inf\mathbb{T}}^m e^{\Delta}_{w_2}(t,m)\Delta t = -\frac{p^2}{C_{\mathbb{T}}}\Big(1-e_{w_2}(\inf\mathbb{T},m)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X] &= \int_{\inf\mathbb{T}}^\infty tN(t;m,p)\Delta t = \int_{\inf\mathbb{T}}^\infty (t-m+m)N(t;m,p)\Delta t\\
&= \left(\int_{\inf\mathbb{T}}^m + \int_m^\infty\right)(t-m)N(t;m,p)\Delta t + m\int_{\inf\mathbb{T}}^\infty N(t;m,p)\Delta t\\
&= -\frac{p^2}{C_{\mathbb{T}}}\Big(1-e_{w_2}(\inf\mathbb{T},m)\Big) + \frac{p^2}{C_{\mathbb{T}}} + m\\
&= m + \frac{p^2}{C_{\mathbb{T}}}e_{w_2}(\inf\mathbb{T},m),
\end{aligned}$$

completing the proof for the expected value. It is clear that $\nu_1 < \infty$ since $[\inf\mathbb{T},m]$ is compact. Now compute

$$\begin{aligned}
\nu_2 &:= \int_m^\infty (t-m)^2e_{\ominus w_1}(\sigma(t),m)\Delta t\\
&= \int_m^\infty (t-m)^2\frac{p^2}{p^2+(t-m)\mu}e_{\ominus w_1}(t,m)
\end{aligned}$$

$$
\begin{aligned}
&= p^2 \int_m^\infty (m-t)(\ominus w_1) e_{\ominus w_1}(t,m)\Delta t \\
&= p^2 \int_m^\infty (m-t) e^{\Delta}_{\ominus w_1}(t,m)\Delta t \\
&= p^2 \int_m^\infty e_{\ominus w_1}(\sigma(t),m)\Delta t \\
&\leq p^2 \int_m^\infty e_{\ominus w_1}(t,m)\Delta t.
\end{aligned}
$$

Finally, we compute

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_{\inf \mathbb{T}}^\infty t^2 N(t;m,p)\Delta t \\
&= \frac{1}{C_{\mathbb{T}}} \int_{\inf \mathbb{T}}^\infty (t-m+m)^2 N_1(t;m,p)\Delta t \\
&= \frac{1}{C_{\mathbb{T}}} \int_{\inf \mathbb{T}}^\infty \Big[(t-m)^2 + 2m(t-m) + m^2\Big] N_1(t;m,p)\Delta t \\
&= \nu_1 + \nu_2 + 2m\mathbb{E}[X] - m^2 \\
&= \nu_1 + \nu_2 + 2m\Big(m + \frac{p^2}{C_{\mathbb{T}}} e_{w_2}(\inf \mathbb{T}, m)\Big) + m^2 \\
&= \nu_1 + \nu_2 + m^2 + 2m\frac{p^2}{C_{\mathbb{T}}} e_{w_2}(\inf \mathbb{T}, m).
\end{aligned}
$$

Therefore,

$$
\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \nu_1 + \nu_2 - \frac{p^4}{C_{\mathbb{T}}^2}(e_{w_2}(\inf \mathbb{T}, m))^2.
$$

□

By the same proof on an isolated time scale, we obtain the following result whose proof we omit due to similarity with the proof of Theorem 2.8.

Theorem 2.9. *If $\mathbb{T}$ is an isolated time scale with $\inf \mathbb{T} = -\infty$ and X is a random variable with density $N(t;m,p)$, then it has expected value $\mathbb{E}[X] = m$ and it has variance $\mathrm{Var}[X] = \nu_1 + \nu_2$.*

3. Brownian Motion on General Time Scale

In this section, we define a time scales Brownian motion and examine some of its properties.

Definition 3.1. *A time scale Brownian motion (or time scale Wiener process) is a stochastic process $\{W_t\}_{t\in\mathbb{T}}$ with the following properties:*

1. *if $t_0 \leq t_1 < t_2 < \ldots < t_n$ for $t_i \in \mathbb{T}, i = 0, \ldots, n$, the random variables $W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent,*
2. *for all $t \geq s \geq t_0$, $W_t - W_s$ is time scale normally distributed with mean 0 and*

$$
\mathrm{Var}\left[W_t - W_s - \sum_{\substack{u\in[s,t)\cap\mathbb{T} \\ \sigma(u)>u}} \Big(W_{\sigma(u)} - W_u\Big) \right] = t - s - \sum_{\substack{u\in[s,t)\cap\mathbb{T} \\ \sigma(u)>u}} \mu(u),
$$

3. *the process starts almost surely at* 0*, i.e.* $W_{t_0} = 0$ *with probability one, and*

4. *the paths of the process* W_t *are all continuous.*

We point out here that the set of all right-scattered points of $\mathbb{T}$ is at most countable [3, Lemma 3.1], so we do not have to consider uncountable sums. Immediately from the definition, we observe that

$$\mathbb{E}[W_t] = 0, \quad \mathrm{Var}[W_t] = t + h_{t_0}(t),$$

where

$$h_{t_0}(t) = \mathbb{E}\left[\sum_{\substack{u\in[t_0,t)\cap\mathbb{T}\\ \sigma(u)>u}} \left(\left(W_{\sigma(u)} - W_u\right)^2 - \mu(u)\right)\right]. \tag{3}$$

When $\mathbb{T} = \mathbb{R}$, note that Definition 3.1 part 2 reduces so that both summations are zero and the variance becomes simply $t - s$, consistent with the classical Brownian motion.

Theorem 3.2. *If* $\{W_t\}_{t\in\mathbb{T}}$ *is a time scale Brownian motion, then for* $s, t \in \mathbb{T}$ *with* $s, t \geq t_0$*,*

$$\mathbb{E}[W_t W_s] = \min\{s, t\} + h_{t_0}(\min\{s, t\}).$$

Proof. Without loss of generality, assume that $t \geq s \geq t_0$. By Definition 3.1, $W_t - W_s$ is independent of W_s, so compute

$$\begin{aligned}
\mathbb{E}[W_t W_s] &= \mathbb{E}\left[(W_t - W_s + W_s)W_s\right]\\
&= \mathbb{E}\left[W_s^2 + (W_t - W_s)W_s\right]\\
&= \mathbb{E}[W_s^2] + \mathbb{E}\left[(W_t - W_s)W_s\right]\\
&= s + \mathbb{E}\left[\sum_{\substack{u\in[t_0,s)\cap\mathbb{T}\\ \sigma(u)>u}} \left((W_{\sigma(u)} - W_u)^2 - \mu(u)\right)\right] + \mathbb{E}(W_t - W_s)\mathbb{E}(W_s)\\
&= \min\{s, t\} + \mathbb{E}\left[\sum_{\substack{u\in[t_0,\min\{s,t\})\cap\mathbb{T}\\ \sigma(u)>u}} \left((W_{\sigma(u)} - W_u)^2 - \mu(u)\right)\right],
\end{aligned}$$

completing the proof. □

Now we are prepared to define a time scale Paley-Wiener-Zygmund integral.

Definition 3.3. *Suppose that* $g\colon [a, b] \cap \mathbb{T} \to \mathbb{R}$ *is* Δ*-integrable,* g^Δ *exists and it is continuous on* $[a, b] \cap \mathbb{T}$*, and* $g(a) = g(b) = 0$*. We define a time scale Paley-Wiener-Zygmund integral as follows:*

$$\int_a^b g\Delta W_t = -\int_a^b g^\Delta(t) W_t \Delta t.$$

Lemma 3.4. *If* $t \geq t_0$ *is right-scattered, then*

$$h_{t_0}^\Delta(t) = \mathbb{E}\left[\frac{\left(W_{\sigma(t)} - W_t\right)^2}{\mu(t)} - 1\right], \tag{4}$$

where h_{t_0} *is given by* (3)*.*

Proof. Since t is right-scattered, we have

$$\begin{aligned} h_{t_0}^{\Delta}(t) &= \frac{h_{t_0}(\sigma(t)) - h_{t_0}(t)}{\mu(t)} \\ &= \frac{1}{\mu(t)}\left(\mathbb{E}\left[\sum_{\substack{u\in[t_0,\sigma(t))\cap\mathbb{T}\\ \sigma(u)>u}}\left(\left(W_{\sigma(u)} - W_u\right)^2 - \mu(u)\right)\right] - \mathbb{E}\left[\sum_{\substack{u\in[t_0,t)\cap\mathbb{T}\\ \sigma(u)>u}}\left(\left(W_{\sigma(u)} - W_u\right)^2 - \mu(u)\right)\right]\right) \\ &= \frac{1}{\mu(t)}\mathbb{E}\left[\left(\left(W_{\sigma(t)} - W_t\right)^2 - \mu(t)\right)\right], \end{aligned}$$

completing the proof. □

Since $\int_a^b g\Delta W_t$ is a random variable, it is of interest to compute its expected value and the expected value of its square.

Theorem 3.5. *If $g\colon [a,b]\cap\mathbb{T}\to\mathbb{R}$ is Δ-integrable, g^{Δ} exists and is continuous on $[a,b]\cap\mathbb{T}$, and $g(a) = g(b) = 0$, then*

(i) $\mathbb{E}\left[\int_a^b g\Delta W_t\right] = 0$, *and*

(ii) if $h_a(t)$ given by (3) *is Δ-differentiable for all $t\in[a,b]\cap\mathbb{T}$, then*

$$\mathbb{E}\left[\left(\int_a^b g\Delta W_t\right)^2\right] = \int_a^b \left(g(\sigma(t))\right)^2\left(1 + h_a^{\Delta}(t)\right)\Delta t.$$

In particular, if all points of $\mathbb{T}$ are right-scattered, then

$$\mathbb{E}\left[\left(\int_a^b g\Delta W_t\right)^2\right] = \sum_{t\in[a,b)}\left(g(\sigma(t))\right)^2\mathbb{E}\left[\left(W_{\sigma(t)} - W_t\right)^2\right].$$

Proof. Definition 3.1 and Definition 3.3 imply

$$\begin{aligned} \mathbb{E}\left[\int_a^b g\Delta W_t\right] &= \mathbb{E}\left[-\int_a^b g^{\Delta}(t)W_t\Delta t\right] \\ &= -\int_a^b g^{\Delta}(t)\mathbb{E}[W_t]\Delta t = 0, \end{aligned}$$

completing the proof of (*i*). For (*ii*), Definition 3.3 and Theorem 3.2 imply

$$\begin{aligned} \mathbb{E}\left[\left(\int_a^b g\Delta W_t\right)^2\right] &= \mathbb{E}\left[\int_a^b g^{\Delta}(t)W_t\Delta t\int_a^b g^{\Delta}(s)W_s\Delta s\right] \\ &= \int_a^b\int_a^b g^{\Delta}(t)g^{\Delta}(s)\mathbb{E}\left[W_tW_s\right]\Delta s\Delta t \\ &= \int_a^b\int_a^b g^{\Delta}(t)g^{\Delta}(s)\left(\min\{s,t\} + \mathbb{E}\left[\sum_{\substack{u\in[a,\min\{s,t\})\cap\mathbb{T}\\ \sigma(u)>u}}\left(\left(W_{\sigma(u)} - W_u\right)^2 - \mu(u)\right)\right)\Delta s\Delta t\right] \\ &= \int_a^b\int_a^b g^{\Delta}(t)g^{\Delta}(s)\min\{s,t\}\Delta s\Delta t + \int_a^b\int_a^b g^{\Delta}(t)g^{\Delta}(s)h_a\left(\min\{s,t\}\right)\Delta s\Delta t =: L_1 + L_2. \end{aligned} \tag{5}$$

For L_1, we apply integration by parts twice, the hypothesis, and the fundamental theorem of calculus to compute

$$\begin{aligned} L_1 &= \int_a^b g^\Delta(t)\left(\int_a^t sg^\Delta(s)\Delta s + \int_t^b tg^\Delta(s)\Delta s\right)\Delta t \\ &= \int_a^b g^\Delta(t)\left(tg(t) - ag(a) - \int_a^t g(\sigma(s))\Delta s + tg(b) - tg(t)\right)\Delta t \\ &= \int_a^b g^\Delta(t)\left(-\int_a^t g(\sigma(s))\Delta s\right)\Delta t \\ &= \int_a^b (g(\sigma(t)))^2\,\Delta t. \end{aligned}$$

Similarly for L_2, we compute

$$\begin{aligned} L_2 &= \int_a^b\int_a^b g^\Delta(t)g^\Delta(s)h_a(\min\{s,t\})\Delta s\Delta t \\ &= \int_a^b g^\Delta(t)\left(\int_a^t g^\Delta(s)h_a(s)\Delta s + \int_t^b g^\Delta(s)h_a(t)\Delta s\right)\Delta t \\ &= \int_a^b g^\Delta(t)\left(\int_a^t g^\Delta(s)h_a(s)\Delta s + h_a(t)\int_t^b g^\Delta(s)\Delta s\right)\Delta t \\ &= \int_a^b g^\Delta(t)\left(g(t)h_a(t) - \int_a^t g(\sigma(s))h_a^\Delta(s)\Delta s - g(t)h_a(t)\right)\Delta t \\ &= -\int_a^b g^\Delta(t)\int_a^t g(\sigma(t))h_a^\Delta(s)\Delta s\Delta t \\ &= \int_a^b \left(g(\sigma(t))\right)^2 h_a^\Delta(t)\Delta t. \end{aligned}$$

Now suppose that all points of $\mathbb{T}$ are right-scattered. Then, we apply (4) to (5) and compute

$$\mathbb{E}\left[\left(\int_a^b g\Delta W_t\right)^2\right] = L_1 + L_2 = \int_a^b \left(g(\sigma(t))\right)^2 \mathbb{E}\left[\frac{1}{\mu(t)}\left(W_{\sigma(t)} - W_t\right)^2\right]\Delta t.$$

Now writing the integral as a sum completes the proof. □

Theorem 3.5 assumes that the function g is integrable on $[a,b]\cap\mathbb{T}$. It is simple to extend to L^2 functions as we now explain. If $g \in L^2([a,b]\cap\mathbb{T})$ with $g(a) = g(b) = 0$ and h is Δ-differentiable for every $t \in (a,b)\cap\mathbb{T}$, then we can take a sequence of functions g_n as in Definition 3.3 such that as $n \to \infty$,

$$\int_a^b \left(g_n(\sigma(t)) - g(\sigma(t))\right)^2\left(1 + h_a^\Delta(t)\right)\Delta t \to 0.$$

By Theorem 3.5(*ii*), we get

$$\mathbb{E}\left[\left(\int_a^b g_m\Delta W_t - \int_a^b g_n\Delta W_t\right)^2\right] = \int_a^b \left(g_m(\sigma(t)) - g_n(\sigma(t))\right)^2\left(1 + h_a^\Delta(t)\right)\Delta t.$$

Therefore $\left\{\int_a^b g_n\Delta W_t\right\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in L^2 so we can define $\int_a^b g\Delta W_t = \lim_{n\to\infty}\int_a^b g_n\Delta W_t$, and the extended definition satisfies Theorem 3.5. We now use this concept to pursue quadratic variation.

Let $\mathcal{P}([a,b)\cap\mathbb{T}) = \left\{\mathcal{P}\colon \mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}\right\}$ be a collection of partitions. We include b as part of these partitions because the rightmost endpoint is absent when calculating Δ-integrals.

Lemma 3.6. *If $[a,b)\cap\mathbb{T}$ contains infinitely many points and for all $n\in\mathbb{N}_0$, $\mathcal{P}_n\in\mathcal{P}([a,b)\cap\mathbb{T})$ with $\mathcal{P}_n\subsetneq\mathcal{P}_{n+1}$, then*

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(W_{t_{i+1}}-W_{t_i}\right)^2 = b-a+h_a(b),$$

where the convergence is in L^2 sense.

Proof. Let $Q_n = \sum_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2$. For arbitrary $\delta>0$, we write

$$\begin{aligned} Q_n-(b-a)-h_a(b) &= \sum_{t_{k+1}-t_k\le\delta}\Big((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)\Big)\sum_{t_{k+1}-t_k>\delta}\Big((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)\Big)-h_a(b)\\ &=: A_1+A_2-h_a(b). \end{aligned}$$

Take the expected value of both sides and apply $(x+y)^2\le 2(x^2+y^2)$ to compute

$$\mathbb{E}\left[(Q_n-(b-a)-h_a(b))^2\right]\le 2\mathbb{E}\left[A_1^2\right]+2\mathbb{E}\left[(A_2-h_a(b))^2\right].$$

In the following, when $k\ne j$, the expectation is zero because the variables are independent, leading to the $k=j$ terms remaining, so compute

$$\begin{aligned} \mathbb{E}\left[A_1^2\right] &= \sum_{\substack{t_{k+1}-t_k\le\delta\\ t_{j+1}-t_j\le\delta}}\mathbb{E}\left[\Big((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)\Big)\Big(\big(W_{t_{j+1}}-W_{t_j}\big)^2-\big(t_{j+1}-t_j\big)\Big)\right]\\ &= \sum_{t_{k+1}-t_k\le\delta}\mathbb{E}\left[\Big((W_{t_{k+1}}-W_{t_k})^2-(t_{k+1}-t_k)\Big)^2\right]\\ &= \sum_{t_{k+1}-t_k\le\delta}\mathbb{E}\left[\left(\frac{(W_{t_{k+1}}-W_{t_k})^2}{t_{k+1}-t_k}-1\right)^2(t_{k+1}-t_k)^2\right]\to 0, \end{aligned}$$

as k goes to ∞. Moreover, when $t_{k+1}-t_k>\delta$ and k goes to ∞, and since δ is arbitrary we consider $\delta\to 0$ and observe that sums of the form $\sum_{t_{k+1}-t_k>\delta}(\cdot)$ limit to sums of the form $\sum_{\sigma(t)-t>0}(\cdot)$. Therefore we have $\mathbb{E}\left[(A_2-h_a(b))^2\right]\to 0$. Hence, $Q_n\to b-a+h_a(b)$ in L^2 sense, which completes the proof. □

Remark 3.7. *When the time scale is chosen as the set of real numbers, i.e. $\mathbb{T}=\mathbb{R}$, then the collection of right-scattered points is empty and $\sum_{i=0}^{\infty}(W_{t_{i+1}}-W_{t_i})^2$ converges to the length of the interval, $b-a$, in L^2 sense.*

Proposition 3.8. *If $[a,b)\cap\mathbb{T}$ contains infinitely many points, and for all $n\in\mathbb{N}_0$, $\mathcal{P}_n\in\mathcal{P}([a,b)\cap\mathbb{T})$ with $\mathcal{P}_n\subsetneq\mathcal{P}_{n+1}$, and $\tau_k\in[t_k,t_{k+1})\cap\mathbb{T}$ for $k\in\{0,\ldots,n-1\}$, then in L^2 sense,*

$$\sum_{k=0}^{n-1}W_{\tau_k}\left(W_{t_{k+1}}-W_{t_k}\right)\longrightarrow\frac{W_b^2-W_a^2}{2}+\left(\alpha-\frac{1}{2}\right)\left((b-a)+h_a(b)\right).$$

Proof. We express $\tau_k = (1-\alpha)t_k + \alpha t_{k+1}$ for some $\alpha \in [0,1)$ since $\tau_k \in [t_k, t_{k+1})$. The sum $\sum_{k=0}^{n-1} W_{\tau_k}\left(W_{t_{k+1}} - W_{t_k}\right)$ can be decomposed as follows:

$$\begin{aligned}\sum_{k=0}^{n-1} W_{\tau_k}\left(W_{t_{k+1}} - W_{t_k}\right) &= \sum_{k=0}^{n-1}\left(W_{\tau_k} - W_{t_k} + W_{t_k}\right)\left(W_{t_{k+1}} - W_{\tau_k} + W_{\tau_k} - W_{t_k}\right)\\ &= \sum_{k=0}^{n-1}\left(W_{\tau_k} - W_{t_k}\right)^2 + \sum_{k=0}^{n-1}\left(W_{\tau_k} - W_{t_k}\right)\left(W_{t_{k+1}} - W_{\tau_k}\right)\\ &+\frac{1}{2}\sum_{k=0}^{n-1}\left(W_{t_{k+1}}^2 - W_{t_k}^2\right) - \frac{1}{2}\sum_{k=0}^{n-1}\left(W_{t_{k+1}} - W_{t_k}\right)^2\\ &:= A + B + \frac{1}{2}C - \frac{1}{2}D.\end{aligned}$$

Observe that by telescoping sum, $C = \sum_{k=0}^{n-1}\left(W_{t_{k+1}}^2 - W_{t_k}^2\right) = W_b^2 - W_a^2$, using Lemma 3.6, we have $D = \sum_{k=0}^{n-1}\left(W_{t_{k+1}} - W_{t_k}\right)^2 \longrightarrow (b-a) + h_a(b)$, and similarly, $A = \sum_{k=0}^{n-1}\left(W_{\tau_k} - W_{t_k}\right)^2 \longrightarrow \alpha\Big((b-a) + h_a(b)\Big)$. By the independent increment property in Definition 3.1,

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=0}^{n-1}\left[\left(W_{\tau_k} - W_{t_k}\right)\left(W_{t_{k+1}} - W_{\tau_k}\right)\right]^2\right]\\ &= \sum_{k=0}^{n-1}\mathbb{E}\left[\left(W_{\tau_k} - W_{t_k}\right)^2\right]\mathbb{E}\left[\left(W_{t_{k+1}} - W_{\tau_k}\right)^2\right]\\ &= \sum_{k=0}^{n-1}\alpha(t_{k+1}-t_k)(1-\alpha)(t_{k+1}-t_k) + \sum_{k=0}^{n-1}\alpha(t_{k+1}-t_k)\mathbb{E}\left[\sum_{\substack{t\in[\tau_k,t_{k+1})\cap\mathbb{T}\\ \sigma(\tau_k)>\tau_k}}\Big((W_{\sigma(t)} - W_t)^2 - \mu(t)\Big)\right]\\ &+ \sum_{k=0}^{n-1}(1-\alpha)(t_{k+1}-t_k)\mathbb{E}\left[\sum_{\substack{t\in[t_k,\tau_k)\cap\mathbb{T}\\ \sigma(t_k)>t_k}}\Big((W_{\sigma(t)} - W_t)^2 - \mu(t)\Big)\right]\\ &+ \sum_{k=0}^{n-1}\mathbb{E}\left[\sum_{\substack{t\in[\tau_k,t_{k+1})\cap\mathbb{T}\\ \sigma(\tau_k)>\tau_k}}\Big((W_{\sigma(t)} - W_t)^2 - \mu(t)\Big)\right]\mathbb{E}\left[\sum_{\substack{t\in[t_k,\tau_k)\cap\mathbb{T}\\ \sigma(t_k)>t_k}}\Big((W_{\sigma(t)} - W_t)^2 - \mu(t)\Big)\right]\\ &\leq K\|\mathcal{P}_n\| \longrightarrow 0.\end{aligned}$$

Here, K is a constant depending on a, b, and α, and $\|\mathcal{P}_n\|$ is the norm of the partition $\mathcal{P}_n$. Hence

$$B = \sum_{k=0}^{n-1}\left(W_{\tau_k} - W_{t_k}\right)\left(W_{t_{k+1}} - W_{\tau_k}\right) \to 0,$$

which completes the proof. □

4. Conclusion

We defined a time scale normal distribution for isolated time scales and time scales such that $\inf \mathbb{T} > -\infty$. We investigated Brownian motion using this time scale normal distribution and deduced some of its properties. We gave definition for generalized Paley-Wiener-Zygmund integral and computed the quadratic variation of this time scale Brownian motion. Further work could include investigations of the cumulative distribution function and stochastic dynamic equations.

5. Competing Interests

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References

[1] Martin Bohner and Suman Sanyal. The stochastic dynamic exponential and geometric Brownian motion on isolated time scales. *Commun. Math. Anal.*, 8(3):120–135, 2010.
[2] Martin Bohner, Olexandr M. Stanzhytskyi, and Anastasiia O. Bratochkina. Stochastic dynamic equations on general time scales. *Electron. J. Differ. Equ.*, 2013:15, 2013.
[3] Alberto Cabada and Dolores R. Vivero. Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral; application to the calculus of Δ-antiderivatives. *Math. Comput. Modelling*, 43(1-2):194–207, 2006.
[4] Tom Cuchta, Robert Jon Niichel, and Sabrina Streipert. A Gompertz distribution for time scales. *Turk. J. Math.*, 45(1):185–200, 2021.
[5] Nguyen Huu Du and Nguyen Thanh Dieu. Stochastic dynamic equations on time scales. *Acta Mathematica Vietnamica*, 38(2):317–338, 2013.
[6] L. Erbe, A. Peterson, and M. Simon. Square integrability of Gaussian bells on time scales. *Comput. Math. Appl.*, 49(5-6):871–883, 2005.
[7] Lawrence C Evans. Partial differential equations. *Graduate Studies in Mathematics*, 19, 2002.
[8] David Grow and Suman Sanyal. Brownian motion indexed by a time scale. *Stochastic Anal. Appl.*, 29(3):457–472, 2011.
[9] David Grow and Suman Sanyal. The quadratic variation of Brownian motion on a time scale. *Stat. Probab. Lett.*, 82(9):1677–1680, 2012.
[10] Wenqing Hu. Itô's formula, the stochastic exponential, and change of measure on general time scales. *Abstract and Applied Analysis*, 2017:1–13, 2017.
[11] F. Korichi. Existence et unicité de la solution pour une classe d'équations stochastiques á coefficients définis par rapport á un emsemble fermé de temps. *Rend. Sem. Mat. Univ. Pol. Torino*, 73(2):227–243, 2015.
[12] Cristina Lungan and Vasile Lupulescu. Random dynamical systems on time scales. *Electron. J. Differ. Equ.*, 2012:14, 2012.
[13] Thomas Matthews. *Probability Theory on Time Scales and Applications to Finance and Inequalities*. Doctoral Thesis at Missouri University of Science and Technology, 2011.
[14] Nguyen Huu Du and Nguyen Thanh Dieu. The first attempt on the stochastic calculus on time scale. *Stochastic Anal. Appl.*, 29(6):1057–1080, 2011.
[15] Raymond Edward Alan Christopher Paley, Norbert Wiener, and Antoni Zygmund. Notes on random functions. *Mathematische Zeitschrift*, 37(1):647–668, 1933.
[16] Dylan Poulsen, John Davis, and Ian Gravagne. Optimal control on stochastic time scales. *IFAC-PapersOnLine*, 50:14861–14866, 07 2017.
[17] Dylan Poulsen and Nick Wintz. The Kalman filter on stochastic time scales. *Nonlinear Anal. Hybrid Syst.*, 33:151–161, 2019.
[18] Dylan R. Poulsen, Michael Z. Spivey, and Robert J. Marks. The Poisson process and associated probability distributions on time scales. In *2011 IEEE 43rd Southeastern Symposium on System Theory*, pages 49–54, 2011.
[19] Suman Sanyal. Stochastic Dynamic Equations. *Doctoral Thesis.*, 2008.
[20] Ünal Ufuktepe. Unification of probability theory on time scales. *Adv. Difference Equ.*, 2012:13, 2012. Id/No 211.