Filomat 36:16 (2022), 5337–5345
https://doi.org/10.2298/FIL2216337C

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inequalities of Generalized Euclidean Berezin Number

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Abstract. In this paper, we present several Berezin number inequalities involving extensions of Euclidean Berezin number for n operators. Among other inequalities for $(T_1, \dots, T_n) \in \mathbb{B}(\mathcal{H})$ we show that

$$\mathbf{ber}_{\mathbf{p}}^{p}(T_1, \dots, T_n) \leq \frac{1}{2^p}\mathbf{ber}\left(\sum_{i=1}^{n}(|T_i| + |T_i^*|)^p\right),$$

where $p > 1$.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle\, .\,, .\,\rangle$ and the corresponding norm $\|\,.\,\|$. An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and then we write $A \geq 0$.
A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions on a(nonempty) set Ω, which has the property that point evaluations are continuous i.e. for each $\lambda \in \Omega$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensure that for each $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)}e_n(z)$; (see [13, Problem 37]). For $\lambda \in \Omega$, let $\hat{k_\lambda} = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded linear operator A on $\mathcal{H}$, the function $\widetilde{A}$ defined on Ω by $\widetilde{A}(\lambda) = \langle A\hat{k_\lambda}, \hat{k_\lambda}\rangle$ is the Berezin symbol of A, which firstly have been introduced by Berezin [5, 6]. The Berezin set and the Berezin number of the operator A are defined by

$$\mathbf{Ber}(A) := \{\widetilde{A}(\lambda) : \lambda \in \Omega\} \qquad \text{and} \qquad \mathbf{ber}(A) := \sup\{|\widetilde{A}(\lambda)| : \lambda \in \Omega\},$$

respectively, (see [15]). The numerical radius of $A \in \mathbb{B}(\mathcal{H})$ is defined by $w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$. It is clear that

$$\mathbf{ber}(A) \leq w(A) \leq \|A\| \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 47A30, Secondary 47A12, 47A63, 47A99
Keywords. Berezin number, Berezin symbol, Euclidean operator.
Received: 14 February 2021; Revised: 08 November 2022; Accepted: 19 November 2022
Communicated by Dragan S. Djordjević
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for all $A \in \mathbb{B}(\mathcal{H})$. Moreover, the Berezin number of operators A, B satisfy the following properties:
(i) $\mathbf{ber}(\alpha \mathrm{A}) = |\alpha| \mathbf{ber}(\mathrm{A})$ for all $\alpha \in \mathbb{C}$;
(ii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$.
Let $A_i \in \mathbb{B}(\mathcal{H})$ $(1 \leq i \leq n)$. The generalized Euclidean Berezin number of $A_1, ..., A_n$ is defined in [1] as follows:

$$\mathbf{ber_p}(A_1, ..., A_n) := \sup_{\lambda \in \Omega} \left(\sum_{i=1}^{n} \left| \langle A_i \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^p \right)^{\frac{1}{p}}, \qquad (p \geq 1).$$

In the case $p = 2$, we have the Euclidean Berezin number and denote by

$$\mathbf{ber_e}(A_1, \ldots, A_n) := \sup_{\lambda \in \Omega} \left(\sum_{i=1}^{n} \left| \langle A_i \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^2 \right)^{\frac{1}{2}}.$$

For $p = 1$, we have $\mathbf{ber_1}(A_1, \ldots, A_n)$, such that if $A_1 = A_2 = \ldots = A_n = A$, then $\mathbf{ber_1}(A, \ldots, A) = n\mathbf{ber}(A)$.
The generalized Euclidean Berezin number $\mathbf{ber_p}(p \geq 1)$ has the following properties:
(i) $\mathbf{ber_p}(\alpha A_1, ..., \alpha A_n) = |\alpha| \mathbf{ber_p}(A_1, ..., A_n)$ for all $\alpha \in \mathbb{C}$;
(ii) $\mathbf{ber_p}(A_1 + B_1, ..., A_n + B_n) \leq \mathbf{ber_p}(A_1, ..., A_n) + \mathbf{ber_p}(B_1, ..., B_n)$;
(iii) $\mathbf{ber_p}(X^* A_1 X, ..., X^* A_n X) \leq \|X\| \mathbf{ber_p}(A_1, ..., A_n)$;
(iv) $\mathbf{ber_p}(A_1, A_2, ..., A_n) = \mathbf{ber_p}(A_1^*, A_2^*, ..., A_n^*)$,
where $A_i, B_i, X \in \mathbb{B}(\mathcal{H}(\Omega))(1 \leq i \leq n)$.
The proof of the properties (i) – (iv) immediately comes from definition of the generalized Berezin number.

Namely, the Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator(i.e., for all $\lambda \in \Omega, \widetilde{A}(\lambda) = \widetilde{B}(\lambda)$ implies $A = B$). For further information about Berezin symbol we refer the reader to [1, 9, 12, 16, 19–21] and references therein.

In this paper, we want to examine the properties of the generalized Euclidean Berezin number. Moreover, we obtain some generalization of the Euclidean Berezin number inequalities. For these goals, we will apply some methods from [18].

2. Main results

In this section we would like to check some properties about the generalized Euclidean Berezin number and then we state some inequalities related to this concept.
In the next lemma, we show that the generalized Euclidean Berezin number is weakly unitarily invariant.

Lemma 2.1. *Suppose $A_i, U_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \ldots, n)$ such that U_i's are unitarily invariant operators. Then*

$$\boldsymbol{ber}_p(U_1^* A_1 U_1, ..., U_n^* A_n U_n) = \boldsymbol{ber}_p(A_1, ..., A_n).$$

Proof. From the definition of the generalized Euclidean Berezin number, we have

$$\begin{aligned}
\mathbf{ber_p}(U_1^* A_1 U_1, ..., U_n^* A_n U_n) &= \sup_{\lambda} \left(\sum_{i=1}^{n} \left| \left\langle U_i^* A_i U_i \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^p \right)^{\frac{1}{p}} \\
&= \sup_{\lambda} \left(\sum_{i=1}^{n} \left| \left\langle A_i U_i \hat{k}_\lambda, U_i \hat{k}_\lambda \right\rangle \right|^p \right)^{\frac{1}{p}} \\
&= \sup_{\mu} \left(\sum_{i=1}^{n} \left| \left\langle A_i \hat{k}_\mu, \hat{k}_\mu \right\rangle \right|^p \right)^{\frac{1}{p}} \\
&= \mathbf{ber_p}(A_1, ..., A_n)
\end{aligned}$$

□

The next result follows from the Jensen's inequality, which asserts:

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} a_i^p \tag{2}$$

for any real positive sequence $(a_i)_i^n$ and $p \geq 1$.

Proposition 2.2. *Let $A_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \ldots, n)$. Then*
(i) $\boldsymbol{ber_p}(A_1, ..., A_n) \leq \boldsymbol{ber_1}(A_1, ..., A_n) \leq n^{1-\frac{1}{p}}\boldsymbol{ber_p}(A_1, ..., A_n)$ *for* $p \geq 1$*;*
(ii) $\boldsymbol{ber_p}(A_1, ..., A_n) \leq n^{\frac{1}{p}-\frac{1}{q}}\boldsymbol{ber_q}(A_1, ..., A_n)$ *for* $q \geq p \geq 1$.
In particular,

$$\boldsymbol{ber_e}(A_1, ..., A_n) \geq n^{\frac{1}{2}-\frac{1}{p}}\boldsymbol{ber_p}(A_1, ..., A_n).$$

Proof. The part (i) follows from (2) by letting $a_i = |\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle|(i = 1, \ldots, n)$.
For part (ii), from power inequality, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^q\right)^{\frac{1}{q}} \qquad (p \leq q),$$

if we take $a_i = |\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle|(i = 1, \ldots, n)$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} |\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{n}\sum_{i=1}^{n} |\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle|^q\right)^{\frac{1}{q}} \qquad (p \leq q)$$

Taking the supremum over $\lambda \in \Omega$, we deduce the desired result. □

To prove more the Berezin number inequalities, we need several well known lemmas.
The following lemma is a simple consequence of the classical Jensen and Young inequalities(see [14]).

Lemma 2.3. *Let $a, b \geq 0$ and $0 \leq \nu \leq 1$. Then*

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b \leq (\nu a^r + (1-\nu)b^r)^{\frac{1}{r}} \tag{3}$$

for $r \geq 1$.

The following lemma is known as the generalized mixed Schwarz inequality [17].

Lemma 2.4. *Let $A \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*
(a) *If $0 \leq \nu \leq 1$, then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\nu} x, x\rangle\langle |A^*|^{2(1-\nu)} y, y\rangle,$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A.
(b) *If f, g are nonnegative continuous functions on $[0, \infty)$ which are satisfying the relation $f(t)g(t) = t(t \in [0, \infty))$, then*

$$|\langle Ax, y\rangle| \leq \| f(|A|)x \| \|g(|A^*|)y\|$$

for all $x, y \in \mathcal{H}$.

The next lemma follows from the spectral theorem for positive operators and the Jensen's inequality(see [17]).

Lemma 2.5. *(McCarthy inequality). Let $A \in \mathbb{B}(\mathcal{H})$, $A \geq 0$ and $x \in \mathcal{H}$ be a unit vector. Then*
(a) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$;
(b) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.

Now, we are in the position to state our results.

Theorem 2.6. *Suppose that $(A_1, A_2, \dots, A_n) \in \mathbb{B}(\mathcal{H})^n$ and f, g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, \infty))$. Then*

$$\boldsymbol{ber}_{\boldsymbol{r}}^r(A_1, \dots, A_n) \leq \boldsymbol{ber}_{\boldsymbol{p}}(f^r(|A_1|), \dots, f^r(|A_n|))\boldsymbol{ber}_{\boldsymbol{q}}(g^r(|A_1^*|), \dots, g^r(|A_n^*|)), \tag{4}$$

in which $r \geq 2$, $p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From the Lemmas 2.4(b) and 2.5(a), we have

$$\begin{aligned}
|\langle A_i x, x\rangle|^r &\leq \|f(|A_i|)x\|^r \|g(|A_i^*|)x\|^r \\
&= \langle f^2(|A_i|)x, x\rangle^{\frac{r}{2}} \langle g^2(|A_i^*|)x, x\rangle^{\frac{r}{2}} \\
&\leq \langle f^r(|A_i|)x, x\rangle \langle g^r(|A_i^*|)x, x\rangle.
\end{aligned}$$

By taking the sum over all i from 1 to n, and applying the Holder's inequality, we have

$$\begin{aligned}
\sum_{i=1}^n |\langle A_i x, x\rangle|^r &\leq \sum_{i=1}^n \langle f^r(|A_i|)x, x\rangle \langle g^r(|A_i^*|)x, x\rangle \\
&\leq \sum_{i=1}^n (\langle f^r(|A_i|)x, x\rangle^p)^{\frac{1}{p}} \sum_{i=1}^n \left(\langle g^r(|A_i^*|)x, x\rangle^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, by taking the supremum over $\lambda \in \Omega$, we deduce the desired result. □

Corollary 2.7. *Let $(A_1, A_2, \dots, A_n) \in \mathbb{B}(\mathcal{H})^n$.Then*

$$\boldsymbol{ber}_{\boldsymbol{e}}^2(A_1, \dots, A_n) \leq \boldsymbol{ber}_{\boldsymbol{p}}(|A_1|, \dots, |A_n|)\boldsymbol{ber}_{\boldsymbol{q}}(|A_1^*|, \dots, |A_n^*|), \tag{5}$$

which $p \geq q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By putting $f(t) = g(t) = t^{\frac{1}{2}}$ and $r = 2$ in (4), we get the desired inequality. □

Remark 2.8. *By putting $p = q = 2$ in (4) for any $r \geq 2$, we have the following inequality:*

$$\boldsymbol{ber}_{\boldsymbol{r}}^r(A_1, \dots, A_n) \leq \boldsymbol{ber}_{\boldsymbol{e}}(f^r(|A_1|), \dots, f^r(|A_n|))\boldsymbol{ber}_{\boldsymbol{e}}(g^r(|A_1^*|), \dots, g^r(|A_n^*|)).$$

For the next result we applying the following inequality, which is found in [7]:

$$|\langle DCBAx, x\rangle|^2 \leq \langle A^*|B|^2 Ax, x\rangle \langle D|C^*|^2 D^* y, y\rangle, \tag{6}$$

where $A, B, C, D \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$.

Theorem 2.9. *Assume that $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \dots, n)$. Then*

$$\boldsymbol{ber}_{\boldsymbol{e}}^2(D_1C_1B_1A_1, \dots, D_nC_nB_nA_n) \leq \boldsymbol{ber}_{\boldsymbol{p}}(A_1^*|B_1|^2A_1, \dots, A_n^*|B_n|^2A_n)\boldsymbol{ber}_{\boldsymbol{q}}(D_1|C_1^*|^2D_1^*, \dots, D_n|C_n^*|^2D_n^*) \tag{7}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, then by applying (6) with putting $x = y = \hat{k}_\lambda$ and applying the Holder's inequality, we have

$$\begin{aligned}
&\sup_\lambda \left(\sum_{i=1}^n \left| \left\langle D_i C_i B_i A_i \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 \right) \\
&\quad \le \sup_\lambda \left(\sum_{i=1}^n \left\langle A_i^* |B_i|^2 A_i \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \left\langle D_i^* |C_i^*|^2 D_i^* \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) \\
&\quad \le \sup_\lambda \left(\sum_{i=1}^n \left(\langle A_i^* |B_i|^2 A_i \hat{k}_\lambda, \hat{k}_\lambda \rangle^p \right)^{\frac{1}{p}} \sum_{i=1}^n \left(\langle D_i^* |C_i^*|^2 D_i^* \hat{k}_\lambda, \hat{k}_\lambda \rangle^q \right)^{\frac{1}{q}} \right) \\
&\quad = \mathbf{ber_p}(A_1^*|B_1|^2 A_1, \dots, A_n^*|B_n|^2 A_n) \mathbf{ber_q}(D_1|C_1^*|^2 D_1^*, \dots, D_n|C_n^*|^2 D_n^*).
\end{aligned}$$

□

Corollary 2.10. *Let $B_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \dots, n)$. Then*

$$\boldsymbol{ber}_e^2(B_1^2, \dots, B_n^2) \le \boldsymbol{ber}_p(|B_1|^2, \dots, |B_n|^2)\boldsymbol{ber}_q(|B_1|^2, \dots, |B_n|^2) \tag{8}$$

for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
In particular for $p = q = 2$,

$$\boldsymbol{ber}_e(B_1^2, \dots, B_n^2) \le \boldsymbol{ber}_e(|B_1|^2, \dots, |B_n|^2). \tag{9}$$

Proof. By putting $C_i = B_i^*, A_i = U_i$ and $D_i = U_i^*$ in (7), where $U_i (i = 1, \dots, n)$ are unitaries, we have

$$\mathbf{ber_e^2}(U_1^* B_1^2 U_1, \dots, U_1^* B_n^2 U_1) \le \mathbf{ber_p}(U_1^*|B_1|^2 U_1, \dots, U_1^*|B_n|^2 U_1)\mathbf{ber_q}(U_1^*|B_1|^2 U_1, \dots, U_1^*|B_n|^2 U_1).$$

The result deduce from that fact $\mathbf{ber_p}(\cdot)$ is weakly unitarily invariant. □

Furuta [8] proved the following generalization of the Kato's inequality as follows:

$$|\langle A|A|^{\alpha+\beta-1}x, y\rangle|^2 \le \langle |A|^{2\alpha}x, x\rangle \langle |A|^{2\beta}y, y\rangle \tag{10}$$

for all $A \in \mathbb{B}(\mathcal{H})$, $x, y \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \ge 1$.
Applying (10), we have the next result for the Berezin number concept.

Proposition 2.11. *Assume that $A \in \mathbb{B}(\mathcal{H}(\Omega))$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \ge 1$. Then*

$$\boldsymbol{ber}(A|A|^{\alpha+\beta-1}) \le \frac{1}{2}(\boldsymbol{ber}(|A|^{2\alpha}) + \boldsymbol{ber}(|A|^{2\beta})). \tag{11}$$

Proof. If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, by applying (10) with putting $x = y = \hat{k}_\lambda$, and the arithmetic-geometric mean, we have

$$\begin{aligned}
|\langle A|A|^{\alpha+\beta-1}\hat{k}_\lambda, \hat{k}_\lambda\rangle| &\le (\langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle |A|^{2\beta}\hat{k}_\lambda, \hat{k}_\lambda\rangle)^{\frac{1}{2}} \\
&\le \frac{1}{2}(\langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |A|^{2\beta}\hat{k}_\lambda, \hat{k}_\lambda\rangle) \\
&\le \frac{1}{2}(\mathbf{ber}(|A|^{2\alpha}) + \mathbf{ber}(|A|^{2\beta}))
\end{aligned}$$

By taking the supremum over all $\lambda \in \Omega$, we get the desired result. □

Theorem 2.12. *Suppose that $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \dots, n)$. Then*

$$\boldsymbol{ber}_p^p(D_1C_1B_1A_1, \dots, D_nC_nB_nA_n) \\ \leq \frac{1}{2}\boldsymbol{ber}\left[\sum_{i=1}^{n}\left(\nu(A_i^*|B_i|^2A_i) + (1-\nu)(D_i|C_i^*|^2D_i^*)\right)^p + \left((1-\nu)(A_i^*|B_i|^2A_i) + \nu(D_i|C_i^*|^2D_i^*)\right)^p\right], \tag{12}$$

where $p > 1$ and $\nu \in [0, 1]$.

Proof. If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, by applying (6), and putting $x = y = \hat{k}_\lambda$, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left|\left\langle D_iC_iB_iA_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^p \\
&= \sum_{i=1}^{n}\left(\left\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^p \\
&\leq \frac{1}{2^p}\sum_{i=1}^{n}\left[\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle^\nu\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-\nu} + \langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-\nu}\langle D_i|C_i^*|^2D_i^*\hat{k}_{1-\lambda}, \hat{k}_\lambda\rangle^\nu\right]^p \\
&\qquad\qquad\left(\text{Since } \sqrt{ab} \leq \frac{a^\nu b^{1-\nu} + b^\nu a^{1-\nu}}{2}\right) \\
&\leq \frac{1}{2}\sum_{i=1}^{n}\left[\left(\nu\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\nu)\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^p\right. \\
&\qquad\left. + \left((1-\nu)\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle + \nu\langle D_i|C_i^*|^2D_i^*\hat{k}_{1-\lambda}, \hat{k}_\lambda\rangle\right)^p\right] \\
&\qquad\left(\text{by}\left(\frac{a+b}{2}\right)^p \leq \frac{a^p+b^p}{2}, \text{and Lemma 2.3}\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left\langle \nu(A_i^*|B_i|^2A_i) + (1-\nu)(D_i|C_i^*|^2D_i^*)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^p + \frac{1}{2}\sum_{i=1}^{n}\left\langle (1-\nu)(A_i^*|B_i|^2A_i) + \nu(D_i|C_i^*|^2D_i^*)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^p \\
&\leq \frac{1}{2}\sum_{i=1}^{n}\langle(\nu(A_i^*|B_i|^2A_i) + (1-\nu)(D_i|C_i^*|^2D_i^*))^p\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{2}\sum_{i=1}^{n}\langle((1-\nu)(A_i^*|B_i|^2A_i) + \nu(D_i|C_i^*|^2D_i^*))^p\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\qquad\text{(by Lemma 2.5(}a\text{))} \\
&= \frac{1}{2}\left\langle\sum_{i=1}^{n}\left(\nu A_i^*|B_i|^2A_i + (1-\nu)D_i|C_i^*|^2D_i^*\right)^p + \left((1-\nu)A_i^*|B_i|^2A_i + \nu D_i|C_i^*|^2D_i^*\right)^p\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\leq \frac{1}{2}\mathbf{ber}\left[\sum_{i=1}^{n}\left(\nu(A_i^*|B_i|^2A_i) + (1-\nu)(D_i|C_i^*|^2D_i^*)\right)^p + \left((1-\nu)(A_i^*|B_i|^2A_i) + \nu(D_i|C_i^*|^2D_i^*)\right)^p\right]
\end{aligned}$$

Now, by taking the supremum over $\lambda \in \Omega$, we deduce statement desired result. □

Corollary 2.13. *Let $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \dots, n)$. Then*

$$\begin{aligned}
\boldsymbol{ber}_p^p(D_1C_1B_1A_1, \dots, D_nC_nB_nA_n) &\leq \frac{1}{2^p}\boldsymbol{ber}\left(\sum_{i=1}^{n}\left(A_i^*|B_i|^2A_i + D_i|C_i^*|^2D_i^*\right)\right)^p \\
&\leq \frac{1}{2}\boldsymbol{ber}\left(\sum_{i=1}^{n}(A_i^*|B_i|^2A_i)^p + (D_i|C_i^*|^2D_i^*)^p\right),
\end{aligned} \tag{13}$$

where $p > 1$.

Proof. The inequality (13) immediately comes from (12) by putting $\nu = \frac{1}{2}$, and applying from that fact $(\frac{a+b}{2})^p \leq \frac{a^p+b^p}{2}$. □

In the next result, we state an extension of (11).

Corollary 2.14. *Assume that $(T_1, \ldots, T_n) \in \mathbb{B}(\mathcal{H}(\Omega))^n$. Then*

$$\textbf{\textit{ber}}_{\textbf{\textit{p}}}^p(T_1|T_1|^{\alpha+\beta-1}, \ldots, T_n|T_n|^{\alpha+\beta-1}) \leq \frac{1}{2^p}\textbf{\textit{ber}}\left(\sum_{i=1}^{n}(|T_i|^{2\alpha} + |T_i^*|^{2\beta})^p\right), \tag{14}$$

where $\alpha, \beta \in [0, 1]$, $p > 1$ such that $\alpha + \beta \geq 1$.

Proof. Let $T_i = U_i|T_i|(i = 1, \ldots, n)$ be the polar decomposition of T_i. Then by putting $D_i = U_i, B_i = 1_{\mathcal{H}(\Omega)}, C_i = |T_i|^\beta$ and $A_i = |T_i|^\alpha$ for all $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$, we have

$$D_iC_iB_iA_i = T_i|T_i|^{\alpha+\beta-1}, \quad A_i^*|B_i|^2A_i = |T_i|^{2\alpha}, \quad \text{and} \quad D_i|C_i^*|^2D_i^* = |T_i^*|^{2\beta}.$$

So by applying (13) we get the statement result. □

Remark 2.15. *By putting $\alpha = \beta = \frac{1}{2}$ in* (14)*, we have the following inequality:*

$$\textbf{\textit{ber}}_{\textbf{\textit{p}}}^p(T_1, \ldots, T_n) \leq \frac{1}{2^p}\textbf{\textit{ber}}\left(\sum_{i=1}^{n}(|T_i| + |T_i^*|)^p\right). \tag{15}$$

Theorem 2.16. *Suppose that $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathcal{H}(\Omega))(i = 1, \ldots, n)$. Then*

$$\begin{aligned}&\textbf{\textit{ber}}_{\textbf{\textit{1}}}(D_1C_1B_1A_1, \ldots, D_nC_nB_nA_n)\\ &\leq \textbf{\textit{ber}}\Big[\sum_{i=1}^{n}\frac{1}{2^p}\Big((A_i^*|B_i|^2A_i)^{(1-\nu)p} + (A_i^*|B_i|^2A_i)^{\nu p}\Big) + \frac{1}{2^q}\Big((D_i|C_i^*|^2D_i^*)^{\nu q} + (D_i|C_i^*|^2D_i^*)^{(1-\nu)q}\Big)\Big],\end{aligned} \tag{16}$$

where $p, q > 1$ and $\nu \in [0, 1]$ such that $p\nu \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$, by applying (6), and putting $x = y = \hat{k}_\lambda$, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left|\left\langle D_iC_iB_iA_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\\
&= \sum_{i=1}^{n}\left\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\\
&\leq \frac{1}{2}\sum_{i=1}^{n}\left(\left\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\nu}\left\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1-\nu} + \left\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1-\nu}\left\langle D_i|C_i^*|^2D_i^*\hat{k}_{1-\lambda}, \hat{k}_\lambda\right\rangle^{\nu}\right)\\
&\qquad\qquad\Big(\text{Since } \sqrt{ab} \leq \frac{a^\nu b^{1-\nu} + b^\nu a^{1-\nu}}{2}\Big)\\
&\leq \frac{1}{2}\Big((\sum_{i=1}^{n}\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle^{p\nu})^{\frac{1}{p}}(\sum_{i=1}^{n}\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-\nu)q})^{\frac{1}{q}}\Big)\\
&\qquad + \Big((\sum_{i=1}^{n}\langle A_i^*|B_i|^2A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle^{p(1-\nu)})^{\frac{1}{p}}(\sum_{i=1}^{n}\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\nu q})^{\frac{1}{q}}\Big)\\
&\qquad\qquad\Big(\text{by the Holder's inequality}\Big)
\end{aligned}$$

$$\leq \frac{1}{2}\Big(\sum_{i=1}^{n}\langle A_i^*|B_i|^2A_i\hat{k}_\lambda,\hat{k}_\lambda\rangle^{p\nu}+\langle A_i^*|B_i|^2A_i\hat{k}_\lambda,\hat{k}_\lambda\rangle^{p(1-\nu)}\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^{n}\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda,\hat{k}_\lambda\rangle^{(1-\nu)q}+\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda,\hat{k}_\lambda\rangle^{q\nu}\Big)^{\frac{1}{q}}$$

$$\leq \frac{1}{2p}\Big(\sum_{i=1}^{n}\langle A_i^*|B_i|^2A_i\hat{k}_\lambda,\hat{k}_\lambda\rangle^{p\nu}+\langle A_i^*|B_i|^2A_i\hat{k}_\lambda,\hat{k}_\lambda\rangle^{p(1-\nu)}\Big)$$

$$+\frac{1}{2q}\Big(\sum_{i=1}^{n}\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda,\hat{k}_\lambda\rangle^{(1-\nu)q}+\langle D_i|C_i^*|^2D_i^*\hat{k}_\lambda,\hat{k}_\lambda\rangle^{q\nu}\Big)$$

$$\Big(\text{by Lemma 2.3(a)}\Big)$$

$$\leq \sum_{i=1}^{n}\Big\langle\frac{1}{2p}\Big((A_i^*|B_i|^2A_i)^{p\nu}+(A_i^*|B_i|^2A_i)^{p(1-\nu)}\Big)+\frac{1}{2q}\Big((D_i|C_i^*|^2D_i^*)^{(1-\nu)q}+(D_i|C_i^*|^2D_i^*)^{\nu q}\Big)\hat{k}_\lambda,\hat{k}_\lambda\Big\rangle\Big).$$

$$\Big(\text{by Lemma 2.5(a)}\Big)$$

Now, by taking the supremum over $\lambda\in\Omega$, we statement the desired result. □

By putting $p=q=2$ in the inequality (16), we get the next result.

Corollary 2.17. *Let $A_i,B_i,C_i,D_i\in\mathbb{B}(\mathcal{H}(\Omega))(i=1,\dots,n)$. Then*

$$\begin{aligned}&\boldsymbol{ber_1}(D_1C_1B_1A_1,\dots,D_nC_nB_nA_n)\\&\leq\frac{1}{4}\boldsymbol{ber}\Big[\sum_{i=1}^{n}\Big((A_i^*|B_i|^2A_i)^{2(1-\nu)}+(A_i^*|B_i|^2A_i)^{2\nu}\Big)+\Big((D_i|C_i^*|^2D_i^*)^{2\nu}+(D_i|C_i^*|^2D_i^*)^{2(1-\nu)}\Big)\Big],\end{aligned}\tag{17}$$

where $\nu\in[0,1]$.

Corollary 2.18. *Assume that $(T_1,\dots,T_n)\in\mathbb{B}(\mathcal{H}(\Omega))^n$. Then*

$$\begin{aligned}&\boldsymbol{ber_1}(T_1|T_1|^{\alpha+\beta-1},\dots,T_n|T_n|^{\alpha+\beta-1})\\&\leq\boldsymbol{ber}\Big(\sum_{i=1}^{n}\frac{1}{2p}\Big(|T_i|^{2\alpha(1-\nu)p}+|T_i|^{2\alpha\nu p}\Big)+\frac{1}{2q}\Big(|T_i^*|^{2\beta\nu p}+|T_i^*|^{2\beta(1-\nu)p}\Big),\end{aligned}\tag{18}$$

where $p,q>1$ and $\alpha,\beta,\nu\in[0,1]$ such that $\alpha+\beta\geq1,p\nu\geq2$ and $\frac{1}{p}+\frac{1}{q}=1$

Proof. The proof of (18) is exactly the same as the proof of Corollary 2.14. □

Remark 2.19. *By putting $p=q=2$ in* (18)*, we have the following inequality:*

$$\begin{aligned}&\boldsymbol{ber_1}(T_1|T_1|^{\alpha+\beta-1},\dots,T_n|T_n|^{\alpha+\beta-1})\\&\leq\frac{1}{4}\boldsymbol{ber}\Big(\sum_{i=1}^{n}\Big(|T_i|^{4\alpha(1-\nu)}+|T_i|^{4\alpha\nu}\Big)+\Big(|T_i^*|^{4\beta\nu}+|T_i^*|^{4\beta(1-\nu)}\Big),\end{aligned}\tag{19}$$

In the next result, we have an upper bound for $\mathbf{ber_1}$.

Corollary 2.20. *Let $(T_1,\dots,T_n)\in\mathbb{B}(\mathcal{H}(\Omega))^n$. Then*

$$\boldsymbol{ber_1}(T_1,\dots,T_n)\leq\frac{1}{4}\boldsymbol{ber}\Big(\sum_{i=1}^{n}\Big(1_{\mathcal{H}(\Omega)}+|T_i|^2\Big)+\Big(|T_i^*|^2+1_{\mathcal{H}(\Omega)}\Big).\tag{20}$$

Proof. By putting $\alpha=\beta=\frac{1}{2}$ and $\nu=1$ in (19), we get the desired inequality. □

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