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Extensions of Soft Topologies

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Abstract. In this paper, we introduce the construction of extending a soft topological space with respect to a family of soft subsets from a given soft topological space. We focus on studying this extension when the family consists of a single soft set. We show that the extended soft topological space is not uniquely determined. We further study the conditions under which certain soft topological properties are shared between the extended soft topology and the original one. Lastly, applying a soft point theory, we see that the obtained results are parallel to those results that exist in classical topology, and by Terepeta's Theorem, our results are natural generalizations.

1. Introduction

The area of topology that deals with the fundamental set-theoretic definitions and constructions used in topology is known as general topology. Most other fields of topology, such as differential topology, geometric topology, and algebraic topology, are built on it. Soft topology is also a branch of topology that combines soft set theory and topology. It is motivated by the standard axioms of a classical topological space and is concerned with a structure on the set of all soft sets. A set of important properties was introduced to describe a universe of options as soft sets. Since its inception in 1999 by Molodtsov [14], soft set theory has been a booming topic of research and interaction with other fields. Shabir and Nazs' [17] work, in particular, helped to establish the subject of soft topology. Despite the fact that numerous studies followed their instructions and diverse notions appeared in soft settings, it is still possible to make significant contributions. In this note, we introduce the notion of extension of a soft topological space regarding a collection of soft subsets of the given soft space. But, here, we only study when the collection contains one soft set and call it a simple extension (for short, s-extension) of a soft topological space. The s-extension of crisp topological spaces is originally due to Levine [11]. The study of the preservation of soft topological properties under s-extension is made. Our results are built on the soft point theory given in [6]. By Theorem 1 in [18], the obtained results are generalizations of those that exist for crisp topology.

2. Preliminaries

Let $\mathcal{U}$ be an initial universe, $\mathcal{P}(\mathcal{U})$ be all subsets of $\mathcal{U}$ and E be a set of parameters. A collection $A_E = \{(e, A(e)) : e \in E\}$ is said to be a soft set [14] over $\mathcal{U}$, where $A : E \to \mathcal{P}(\mathcal{U})$ is a crisp map. The family of

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all soft sets on $\mathcal{U}$ is represented by $SP(\mathcal{U}, E)$. A soft element [15] is a soft set A_E over $\mathcal{U}$ in which $A_E = \{u\}$ for all $e \in E$, where $u \in \mathcal{U}$, and is denoted by $(\{u\}, E)$. A soft point [4], denoted by $u(e)$, is a soft set A_E over $\mathcal{U}$ in which $A(e) = \{u\}$ and $A(e') = \emptyset$ for each $e' \neq e$, $e' \in E$, where $e \in E$ and $u \in \mathcal{U}$. A statement $u(e)\widetilde{\in}A_E$ means that $u \in A(e)$. The soft set $\mathcal{U}_E \backslash A_E$ (or simply A^c_E) is the complement of A_E, where $A^c : E \to \mathcal{P}(\mathcal{U})$ is given by $A^c(e) = \mathcal{U}\backslash A(e)$ for all $e \in E$. A soft subset A_E over $\mathcal{U}$ is called null, denoted by $\widetilde{\Phi}$, if $A_E = \emptyset$ for any $e \in E$ and called absolute, denoted by $\widetilde{\mathcal{U}}$, if $A_E = \mathcal{U}$ for any $e \in E$. Notice that $\widetilde{\mathcal{U}}^c = \widetilde{\Phi}$ and $\widetilde{\Phi}^c = \widetilde{\mathcal{U}}$. It is said that A_{E_1} is a soft subset of B_{E_2} (written by $A_{E_1}\widetilde{\subseteq}B_{E_2}$, [13]) if $E_1 \subseteq E_2$ and $A(e) \subseteq B_E$ for any $e \in E_1$. We say $A_{E_1} = B_{E_2}$ if $A_{E_1}\widetilde{\subseteq}B_{E_2}$ and $B_{E_2}\widetilde{\subseteq}A_{E_1}$.

Maji et al. [13] gave definitions of soft union and soft intersection of two soft sets with respect to arbitrary subsets of E. However, as Ali et al. [3] and Terepeta [18] report, these definitions are inaccurate and confusing. As a result, we stick to Terepeta's [18] definitions.

Definition 2.1. Let $\{A^i_E : i \in I\}$ be a family of soft sets over $\mathcal{U}$, where I is any index set.

(i) The intersection of A^i_E, for $i \in I$, is a soft set A_E such that $A(e) = \bigcap_{i\in I} A^i(e)$ for each $e \in E$ and denoted by $A_E = \widetilde{\bigcap}_{i\in I}A^i_E$.
(ii) The union of A^i_E, for $i \in I$, is a soft set A_E such that $A(e) = \bigcup_{i\in I} A^i(e)$ for each $e \in E$ and denoted by $A_E = \widetilde{\bigcup}_{i\in I}A^i_E$.

Definition 2.2. ([17]) A subcollection $\mathcal{T}$ of $PS(\mathcal{U}, E)$ is called a soft topology on $\mathcal{U}$ if

(i) $\widetilde{\Phi}$ and $\widetilde{\mathcal{U}}$ belong to $\mathcal{T}$,
(ii) finite intersection of sets from $\mathcal{T}$ belongs to $\mathcal{T}$, and
(iii) any union of sets from $\mathcal{T}$ belongs to $\mathcal{T}$.

Terminologically, we call $(\mathcal{U}, \mathcal{T}, E)$ a soft topological space on $\mathcal{U}$. The elements of $\mathcal{T}$ are called soft $\mathcal{T}$-open sets (or simply soft open sets when no confusion arise), and their complements are called soft $\mathcal{T}$-closed sets (or soft closed sets).

In what follow, by $(\mathcal{U}, \mathcal{T}, E)$ we mean a soft topological space and by two distinct soft points $u(e), v(e')$, we mean either $u \neq v$ or $e \neq e'$.

Definition 2.3. ([8]) A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is called a soft base for the soft topology $\mathcal{T}$ if each element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Definition 2.4. ([1]) Let $\mathcal{F}$ be a collection soft sets over $\mathcal{U}$. The soft topology generated by $\mathcal{F}$ is the intersection of all soft topologies containing $\mathcal{F}$ and is denoted by $\mathcal{T}(\mathcal{F})$.

Definition 2.5. ([17]) Let A_E be a non-null soft subset of $(\mathcal{U}, \mathcal{T}, E)$. Then $\mathcal{T}_A := \{G_E\widetilde{\bigcap}A_E : G_E\widetilde{\in}\mathcal{T}\}$ is called a soft relative topology over A and $(A, \mathcal{T}_A, E)$ is a soft subspace of $(\mathcal{U}, \mathcal{T}, E)$.

Definition 2.6. ([17]) Let B_E be a soft subset of $(\mathcal{U}, \mathcal{T}, E)$. The soft interior of A_E, denoted by $\text{int}_{\mathcal{T}}(B_E)$, is the largest soft open set contained in B_E. The soft closure of B_E, denoted by $\text{cl}_{\mathcal{T}}(B_E)$, is the smallest soft closed set which contains B_E. The soft closure and interior of a soft subset B_E in the subspace $(A, \mathcal{T}_A, E)$ is respectively denoted by $\text{cl}_A(B_E)$ and $\text{int}_A(B_E)$.

Lemma 2.7. ([10]) *For a soft subset A_E of $(\mathcal{U}, \mathcal{T}, E)$,*

$$\text{int}(A^c_E) = (\text{cl}(A_E))^c \textit{ and } \text{cl}(A^c_E) = (\text{int}(A_E))^c.$$

Definition 2.8. ([7, 9]) A soft topological space $(\mathcal{U}, \mathcal{T}, E)$ is called

(i) soft T_0 if for each $u(e), v(e')$ over X with $u(e) \neq v(e')$, there exist soft open sets A_E, B_E such that $u(e)\widetilde{\notin}A_E$, $v(e')\widetilde{\notin}A_E$ or $v(e')\widetilde{\in}B_E$, $u(e)\widetilde{\notin}B_E$.

(ii) soft T_1 if for each $u(e), v(e')$ over X with $u(e) \neq v(e')$, there exist soft open sets A_E, B_E such that $u(e)\widetilde{\in}A_E$, $v(e')\widetilde{\notin}A_E$ and $v(e')\widetilde{\in}B_E$, $u(e)\widetilde{\notin}B_E$,

(iii) soft T_2 (soft Hausdorff) if for each $u(e), v(e')$ over X with $u(e) \neq v(e')$, there exist soft open sets A_E, B_E containing $u(e), v(e')$ respectively such that $A_E\widetilde{\bigcap}B_E = \widetilde{\Phi}$.

(iv) soft regular if for each soft closed set F_E and each soft point $u(e)$ with $u(e)\widetilde{\notin}F_E$, there exist soft open sets A_E, B_E such that $u(e)\widetilde{\in}A_E$, $F_E\widetilde{\subseteq}B_E$ and $A_E\widetilde{\bigcap}B_E = \widetilde{\Phi}$.

(v) soft normal if for each soft closed sets D_E, F_E with $D_E\widetilde{\bigcap}F_E = \widetilde{\Phi}$, there exist soft open sets A_E, B_E such that $D_E\widetilde{\subseteq}A_E$, $F_E\widetilde{\subseteq}B_E$ and $A_E\widetilde{\bigcap}B_E = \widetilde{\Phi}$.

The above soft separation axioms have been defined for the first time by Sabir and Naz [17] with respect to soft elements.

Definition 2.9. A space $(\mathcal{U}, \mathcal{T}, E)$ is called

(i) soft compact [5] if each soft open cover of $\widetilde{\mathcal{U}}$ has a finite subcover.

(ii) soft separable [16] if it has a countable soft set.

(iii) soft connected [12] if it cannot be written as a union of two disjoint soft open sets.

3. Extensions of soft topological spaces

Definition 3.1. Given a collection $\mathcal{F} = \{F_E^i : i \in I\}$ of non-open soft sets over $\mathcal{U}$, where I is any index set and a soft topological space $(\mathcal{U}, \mathcal{T}, E)$. The soft topology $\hat{\mathcal{T}}$ on $\mathcal{U}$ generated by $\mathcal{T}\widetilde{\bigcup}\mathcal{F}$ is called an extension of $\mathcal{T}$ with respect to $\mathcal{F}$. If $\mathcal{F}$ contains a single soft set F_E, say, then the generating soft topology $\hat{\mathcal{T}}$ is called a simple extension (for short s-extension) of $\mathcal{T}$. By the notation $\hat{\mathcal{T}} = \mathcal{T}[F_E]$ we mean $\hat{\mathcal{T}}$ is an s-extension of $\mathcal{T}$ with respect to the soft set F_E, and $(\mathcal{U}, \hat{\mathcal{T}}, E)$ is an s-extension soft topological space of $(\mathcal{U}, \mathcal{T}, E)$.

From the definition, we first give an easy remark.

Remark 3.2. If $\hat{\mathcal{T}} = \mathcal{T}[F_E]$ be an s-extension of $\mathcal{T}$ over $\mathcal{U}$, then

(i) $\hat{\mathcal{T}}$ is the smallest soft topology on $\mathcal{U}$ that contains $\mathcal{T}$ and $\{F_E\}$;

(ii) if $u(e)\widetilde{\notin}F_E$, the collection $\{O_E : O_E$ is soft $\mathcal{T}$-open around $u(e)\}$ forms a soft $\hat{\mathcal{T}}$-open base at $u(e)$, while if $u(e)\widetilde{\in}F_E$, the collection $\{G_E\widetilde{\bigcap}F_E : G_E$ is soft $\mathcal{T}$-open around $u(e)\}$ forms a soft $\hat{\mathcal{T}}$-open base at $u(e)$;

(iii) F_E is always soft $\hat{\mathcal{T}}$-open, but never soft $\mathcal{T}$-open;

(iv) soft $\hat{\mathcal{T}}$-open sets are of the form $G_E\widetilde{\bigcup}[O_E\widetilde{\bigcap}F_E]$, where G_E, O_E are soft $\mathcal{T}$-open;

(v) for a soft subset A_E over $\mathcal{U}$, $\operatorname{int}_{\hat{\mathcal{T}}}(A_E) = \operatorname{int}_{\mathcal{T}}(A_E)\widetilde{\bigcup}\operatorname{int}_F(A_E\widetilde{\bigcap}F_E)$;

(vi) for a soft subset B_E over $\mathcal{U}$, $\operatorname{cl}_{\hat{\mathcal{T}}}(B_E) = \operatorname{cl}_{\mathcal{T}}(B_E)\widetilde{\bigcap}\left[B_E^c\widetilde{\bigcup}\operatorname{cl}_F(B_E\widetilde{\bigcap}F_E)\right]$;

(vii) $(F, \mathcal{T}_F, E) = (F, \hat{\mathcal{T}}_F, E)$ and $(F^c, \mathcal{T}_{F^c}, E) = (F^c, \hat{\mathcal{T}}_{F^c}, E)$;

(viii) let $u(e)\widetilde{\notin}F_E$ and let B_E be soft set over $\mathcal{U}$, then $u(e)\widetilde{\in}\operatorname{cl}_{\hat{\mathcal{T}}}(B_E)$ if and only if $u(e)\widetilde{\in}\operatorname{cl}_{\mathcal{T}}(B_E)$.

Remark 3.3. At this place it is important to assert that if $\hat{\mathcal{T}}$ is an s-extension of a soft topology $\mathcal{T}$, then $\hat{\mathcal{T}}(e)$ need not be the extension of $\mathcal{T}(e)$ for all $e \in E$. Let $\mathcal{U} = \{u_1, u_2\}$, $E = \{e_1, e_2\}$, and $\mathcal{T} = \{\widetilde{\Phi}, \widetilde{\mathcal{U}}\}$ be the soft topology on $\mathcal{U}$. If $A_E = \{(e_1, \{u_1\}), (e_2, \mathcal{U})\}$, then $\hat{\mathcal{T}} = \mathcal{T}[A_E] = \{\widetilde{\Phi}, A_E, \widetilde{\mathcal{U}}\}$. Therefore $\hat{\mathcal{T}}(e_1) = \{\emptyset, \{u_1\}, \mathcal{U}\}$ is an s-extension of $\mathcal{T}(e_1) = \{\emptyset, \mathcal{U}\}$. On the other hand, $\hat{\mathcal{T}}(e_2) = \{\emptyset, \mathcal{U}\}$ is not an s-extension of $\mathcal{T}(e_2) = \{\emptyset, \mathcal{U}\}$.

Lemma 3.4. *Let $(\mathcal{U}, \mathcal{T}, E)$ be a soft topological space and let $\hat{\mathcal{T}} = \mathcal{T}[F_E]$. Then F_E is soft $\mathcal{T}$-closed if and only if F_E is soft $\hat{\mathcal{T}}$-closed.*

Proof. If F_E is soft $\mathcal{T}$-closed, then F_E^c is soft $\mathcal{T}$-open, by Remark 3.2 (i), $\mathcal{T}\widetilde{\subseteq}\hat{\mathcal{T}}$, so F_E^c is soft $\hat{\mathcal{T}}$-open and hence F_E is soft $\hat{\mathcal{T}}$-closed.

Conversely, suppose that F_E is soft $\hat{\mathcal{T}}$-closed. Then F_E^c is soft $\hat{\mathcal{T}}$-open. By Remark 3.2 (iv), $F_E^c = G_E\widetilde{\bigcup}[O_E\widetilde{\bigcap}F_E]$ for some soft $\mathcal{T}$-open sets G_E, O_E. But $F_E^c\widetilde{\bigcap}F_E = \widetilde{\Phi}$. Therefore $F_E^c = G_E\widetilde{\bigcup}\widetilde{\Phi}$ which is a soft $\mathcal{T}$-open set. Thus F_E is soft $\mathcal{T}$-closed. □

Lemma 3.5. *Let $(\mathcal{U},\mathcal{T},E)$ be a soft topological space and let $\hat{\mathcal{T}} = \mathcal{T}[F_E]$. If C_E is $\hat{\mathcal{T}}$-closed, then $C_E\widetilde{\bigcap}F_E$ is soft closed in $(F,\mathcal{T}_F,E) = (F,\hat{\mathcal{T}}_F,E)$ and $C_E\widetilde{\bigcap}F_E^c$ is soft closed in $(F^c,\mathcal{T}_{F^c},E) = (F^c,\hat{\mathcal{T}}_{F^c},E)$.*

Proof. Follows from Remark 3.2 (vii) and Lemma 3.4. □

Lemma 3.6. *Let $A_E, B_E\widetilde{\in}(\mathcal{U},\mathcal{T},E)$ with $A_E\widetilde{\subseteq}B_E$ and let G_E be soft open over $\mathcal{U}$. If $A_E\setminus \mathrm{int}(A_E) = B_E\setminus \mathrm{int}(B_E) = C_E$ and $\mathrm{cl}(B_E\setminus A_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}$, then*

(i) *$H_E = G_E\setminus \mathrm{cl}(G_E\widetilde{\bigcap}B_E\widetilde{\bigcap}A_E^c)$ is soft open;*

(ii) *$G_E\widetilde{\bigcap}\,\mathrm{int}(A_E) = H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)$; and*

(iii) *$G_E\widetilde{\bigcap}C_E = H_E\widetilde{\bigcap}C_E$.*

Proof. (i) Clear.

(ii) Let $u(e)\widetilde{\in}G_E\widetilde{\bigcap}\,\mathrm{int}(A_E)$. Then $u(e)\widetilde{\in}G_E$ and $u(e)\widetilde{\in}\,\mathrm{int}(A_E)\widetilde{\bigcap}\,\mathrm{int}(B_E)$. We claim that $u(e)\widetilde{\in}H_E$. If not, then $u(e)\widetilde{\in}\,\mathrm{cl}(G_E\widetilde{\bigcap}B_E\widetilde{\bigcap}A_E^c)\widetilde{\subseteq}\,\mathrm{cl}(G_E)\widetilde{\bigcap}\,\mathrm{cl}(B_E)\widetilde{\bigcap}\,\mathrm{cl}(A_E^c))$. This implies that $u(e)\widetilde{\in}\,\mathrm{cl}(A_E^c)) = (\mathrm{int}(A_E))^c$. Thus $u(e)\widetilde{\in}H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)$ and so $G_E\widetilde{\bigcap}\,\mathrm{int}(A_E)\widetilde{\subseteq}H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)$.

Conversely, since $H_E\widetilde{\subseteq}G_E$, we can only show that $H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)\widetilde{\subseteq}H_E\widetilde{\bigcap}\,\mathrm{int}(A_E)$. By (i), one can get the following simplifications

$$\begin{aligned} H_E\widetilde{\bigcap}\,\mathrm{int}(B_E) &= G_E\widetilde{\bigcap}\,\mathrm{int}(G_E^c\widetilde{\bigcup}B_E^c\widetilde{\bigcup}A_E)\widetilde{\bigcap}\,\mathrm{int}(B_E) \\ &= G_E\widetilde{\bigcap}\,\mathrm{int}((G_E^c\widetilde{\bigcap}B_E)\widetilde{\bigcup}A_E) \\ &= G_E\widetilde{\bigcap}\,\mathrm{int}(G_E^c\widetilde{\bigcup}A_E)\widetilde{\bigcap}\,\mathrm{int}(B_E), \end{aligned} \tag{1}$$

and correspondingly,

$$H_E\widetilde{\bigcap}\,\mathrm{int}(A_E) = G_E\widetilde{\bigcap}\,\mathrm{int}(G_E^c\widetilde{\bigcup}A_E)\widetilde{\bigcap}\,\mathrm{int}(A_E). \tag{2}$$

Let $u(e)\widetilde{\in}H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)$, then by statement (1), there exists a soft open O_E containing $u(e)$ such that $O_E\widetilde{\subseteq}G_E$, $O_E\widetilde{\subseteq}G_E^c\widetilde{\bigcup}A_E$, and $O_E\widetilde{\subseteq}B_E$. This means that $O_E\widetilde{\subseteq}\,\mathrm{int}(A_E)$. Therefore $u(e)\widetilde{\in}H_E\widetilde{\bigcap}\,\mathrm{int}(A_E)$ and hence $H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)$ $\widetilde{\subseteq}$ $H_E\widetilde{\bigcap}\,\mathrm{int}(A_E)$. Thus (ii) is proved.

(iii) $G_E\widetilde{\bigcap}C_E\widetilde{\subseteq}[G_E\setminus \mathrm{cl}(B_E\setminus A_E)]\widetilde{\bigcap}C_E\widetilde{\subseteq}[G_E\setminus \mathrm{cl}(A_E\widetilde{\bigcap}B_E\setminus A_E)]\widetilde{\bigcap}C_E = H_E\widetilde{\bigcap}C_E$ and $H_E\widetilde{\bigcap}C_E\widetilde{\subseteq}G_E\widetilde{\bigcap}C_E$ is always true, hence the result. □

Theorem 3.7. *Let $A_E, B_E\widetilde{\in}(\mathcal{U},\mathcal{T},E)$. Then $\mathcal{T}[A_E] = \mathcal{T}[B_E]$ if and only if A_E, B_E satisfies (i) $A_E\setminus \mathrm{int}(A_E) = B_E\setminus \mathrm{int}(B_E) = C_E$, and (ii) $\mathrm{cl}(B_E\setminus A_E)\widetilde{\bigcap}C_E = \mathrm{cl}(A_E\setminus B_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}$.*

Proof. Suppose that (i) and (ii) are true for the given subsets A_E, B_E over $\mathcal{U}$. First we consider the case if $A_E\widetilde{\subseteq}B_E$. By Remark 3.2 (i), the soft base $\mathcal{B}(B_E)$ of $\mathcal{T}[B_E]$ is equal to $\mathcal{T}\widetilde{\bigcup}\mathcal{T}_B$. We claim that $\mathcal{B}(B_E)\widetilde{\subseteq}\mathcal{T}[A_E]$. Take any $D_E\widetilde{\in}\mathcal{B}(B_E)$, if $D_E = G_E\widetilde{\bigcap}B_E = G_E\widetilde{\bigcap}(\mathrm{int}(B_E\widetilde{\bigcap}A_E)) = (G_E\widetilde{\bigcap}\,\mathrm{int}(B_E))\widetilde{\bigcup}(G_E\widetilde{\bigcap}A_E)$, then $D_E\widetilde{\in}\mathcal{T}[A_E]$, see Remark 3.2 (iv), and since $\mathcal{T}\widetilde{\subseteq}\mathcal{T}[A_E]$, therefore $\mathcal{T}[B_E]\widetilde{\subseteq}\mathcal{T}[A_E]$. We now show that $\mathcal{B}(A_E)\widetilde{\subseteq}\mathcal{B}(B_E)$. Let $F_E\widetilde{\in}\mathcal{B}(A_E)$. If $F_E = G_E\widetilde{\bigcap}A_E$, by Lemma 3.6, $F_E = G_E\widetilde{\bigcap}A_E = [G_E\widetilde{\bigcap}\,\mathrm{int}(A_E)]\widetilde{\bigcup}[G_E\widetilde{\bigcap}C_E] = [H_E\widetilde{\bigcap}\,\mathrm{int}(B_E)]\widetilde{\bigcup}[H_E\widetilde{\bigcap}C_E]\widetilde{\in}\mathcal{B}(B_E)$. Therefore $\mathcal{T}[A_E]\widetilde{\subseteq}\mathcal{T}[B_E]$.

Here, we consider a more general situation, if $A_E\widetilde{\bigcap}B_E = D_E$, then we start with

$$\begin{aligned} D_E\setminus \operatorname{int}(D_E) &= (A_E\widetilde{\bigcap}B_E)\widetilde{\bigcap}(\operatorname{int}(A_E)\widetilde{\bigcap}\operatorname{int}(B_E))^c \\ &= [A_E\widetilde{\bigcap}B_E\widetilde{\bigcap}(\operatorname{int}(A_E))^c]\widetilde{\bigcup}[A_E\widetilde{\bigcap}B_E\widetilde{\bigcap}(\operatorname{int}(B_E))^c] \\ &= [A_E\widetilde{\bigcap}C_E]\widetilde{\bigcup}[B_E\widetilde{\bigcap}C_E] = C_E. \end{aligned}$$

Now,

$$\begin{aligned} \operatorname{cl}(A_E\setminus D_E)\widetilde{\bigcap}C_E &= \operatorname{cl}(A_E\widetilde{\bigcap}[A_E^c\widetilde{\bigcup}B_E^c])\widetilde{\bigcap}C_E \\ &= \operatorname{cl}[(A_E\widetilde{\bigcap}A_E^c)\widetilde{\bigcup}(A_E\widetilde{\bigcap}B_E^c)]\widetilde{\bigcap}C_E \\ &= \operatorname{cl}(A_E\setminus B_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}. \end{aligned}$$

By the same way above, one can obtain $\operatorname{cl}(B_E\setminus D_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}$. In conclusion, from the first case, we get $\mathcal{T}[A_E] = \mathcal{T}[B_E] = \mathcal{T}[D_E]$.

Conversely, suppose that $\mathcal{T}[A_E] = \mathcal{T}[B_E]$. Let $u(e)\widetilde{\in}A_E\setminus \operatorname{int}(A_E)$. Assume that $u(e)\widetilde{\notin}B_E$ and $A_E = G_E\widetilde{\bigcup}(H_E\widetilde{\bigcap}B_E)$ for some soft $\mathcal{T}$-open sets G_E, H_E. Therefore $u(e)\widetilde{\in}G_E\widetilde{\subseteq}A_E$ and so $u(e)\widetilde{\in}\operatorname{int}(A_E)$, which is impossible.

Assume that $u(e)\widetilde{\in}\operatorname{int}(B_E)$. Then there exists a soft $\mathcal{T}$-open set O_E containing $u(e)$ such that $u(e)\widetilde{\in}O_E\widetilde{\subseteq}B_E$. Now, we have

$$\begin{aligned} u(e)\widetilde{\in}\operatorname{int}_{\mathcal{T}[B_E]}(O_E\widetilde{\bigcap}A_E) &= \operatorname{int}_{\mathcal{T}}(O_E\widetilde{\bigcap}A_E)\widetilde{\bigcup}\operatorname{int}_B(O_E\widetilde{\bigcap}A_E\widetilde{\bigcap}B_E) \\ &= [\operatorname{int}_{\mathcal{T}}(O_E)\widetilde{\bigcap}\operatorname{int}_{\mathcal{T}}(A_E)]\widetilde{\bigcup}\operatorname{int}_B(O_E\widetilde{\bigcap}A_E\widetilde{\bigcap}B_E). \end{aligned}$$

Since $u(e)\widetilde{\notin}\operatorname{int}(A_E)$, so $u(e)\widetilde{\in}\operatorname{int}_B(O_E\widetilde{\bigcap}A_E\widetilde{\bigcap}B_E) = \operatorname{int}_B(O_E)\widetilde{\bigcap}\operatorname{int}_B(A_E)\widetilde{\bigcap}\operatorname{int}_B(B_E)$. Therefore, there exists a soft $\mathcal{T}$-open set W_E containing $u(e)$ such that $u(e)\widetilde{\in}W_E\widetilde{\bigcap}B_E\widetilde{\subseteq}A_E$. On the other hand, $u(e)\widetilde{\in}W_E\widetilde{\bigcap}O_E\widetilde{\subseteq}W_E\widetilde{\bigcap}B_E\widetilde{\subseteq}A_E$, which implies that $u(e)\widetilde{\in}\operatorname{int}(A_E)$. This is a contradiction. Thus $A_E\setminus \operatorname{int}(A_E)\widetilde{\subseteq}B_E\setminus \operatorname{int}(B_E)$.

The other part can be proved by the same steps above.

We now show $\operatorname{cl}(B_E\setminus A_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}$. Suppose otherwise that $u(e)\widetilde{\in}\operatorname{cl}(B_E\setminus A_E)\widetilde{\bigcap}C_E$ and O_E is any soft $\mathcal{T}$-open containing $u(e)$. Then for each soft $\mathcal{T}[B_E]$-open set W_E of the form $G_E\widetilde{\bigcup}(H_E\widetilde{\bigcap}B_E)$ containing $u(e)$, $W_E\widetilde{\bigcap}(B_E\setminus A_E) \neq \widetilde{\Phi}$. Then again, $(O_E\widetilde{\bigcap}A_E)\widetilde{\bigcap}(B_E\setminus A_E) = \widetilde{\Phi}$ and so $O_E\widetilde{\bigcap}A_E \neq W_E$ for any soft $\mathcal{T}[B_E]$-open set W_E. This is impossible as $\mathcal{T}[A_E] = \mathcal{T}[B_E]$. The part $\operatorname{cl}(A_E\setminus B_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}$ can be followed in a similar way above. Hence the proof is finished. □

Corollary 3.8. *Let $A_E, B_E\widetilde{\in}(\mathcal{U},\mathcal{T},E)$. If $C_E = A_E\setminus \operatorname{int}(A_E)$ and $B_E = A_E\setminus F_E$, where $\operatorname{cl}_{\mathcal{T}}(F_E) = F_E$, $F_E\widetilde{\bigcap}C_E = \widetilde{\Phi}$, then $\mathcal{T}[A_E] = \mathcal{T}[B_E]$.*

Corollary 3.9. *Let $A_E, B_E\widetilde{\in}(\mathcal{U},\mathcal{T},E)$. If $C_E = A_E\setminus \operatorname{int}(A_E)$ and $B_E = A_E\widetilde{\bigcup}G_E$, where $\operatorname{int}_{\mathcal{T}}(G_E) = G_E$, $\operatorname{cl}_{\mathcal{T}}(G_E)\widetilde{\bigcap}C_E = \widetilde{\Phi}$, then $\mathcal{T}[A_E] = \mathcal{T}[B_E]$.*

4. Soft topological properties under s-extension

Lemma 4.1. ([9, Proposition 5.1]) *Let $(\mathcal{U},\mathcal{T},E)$ be a soft compact space and let $A_E\widetilde{\in}(\mathcal{U},\mathcal{T},E)$. If A_E is soft closed, then A_E is soft compact.*

Theorem 4.2. *Let $(\mathcal{U},\mathcal{T},E)$ be a soft compact space. Then $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft compact if and only if A_E^c is soft compact in $(\mathcal{U},\mathcal{T},E)$.*

Proof. Assume $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft compact. By Remark 3.2 (iii), A_E^c is soft $\mathcal{T}[A_E]$-closed and then A_E^c is soft compact in $(\mathcal{U}, \mathcal{T}[A_E], E)$. Since $\mathcal{T}\widetilde{\subseteq}\mathcal{T}[A_E]$, so A_E^c is soft compact in $(\mathcal{U}, \mathcal{T}, E)$.

Conversely, let $\mathcal{W} = \{W_E^i : i \in I\}$ be a soft open cover of $(\mathcal{U}, \mathcal{T}[A_E], E)$, where I is any index set. Then, for each i, $W_E^i = G_E^i \widetilde{\bigcup}(H_E^i \widetilde{\bigcap} A_E)$. Therefore $A_E^c \widetilde{\subseteq} \widetilde{\bigcup}_{i\in I} G_E^i$. But A_E^c is soft compact, so there exists a finite subset $M \subset I$ such that $A_E^c \widetilde{\subseteq} \widetilde{\bigcup}_{i=1}^{M} G_E^i$. On the other hand, $\widetilde{\mathcal{U}} = \widetilde{\bigcup}_{i\in I}(G_E^i \widetilde{\bigcup} H_E^i)$ and since $(\mathcal{U}, \mathcal{T}, E)$ is soft compact, then there exists another finite subset $N \subset I$ such that $\widetilde{\mathcal{U}} = \widetilde{\bigcup}_{i=1}^{N}(G_E^i \widetilde{\bigcup} H_E^i)$. Therefore $A_E = \widetilde{\bigcup}_{i=1}^{N}[G_E^i \widetilde{\bigcup}(H_E^i \widetilde{\bigcap} A_E)]$. In conclusion, we have

$$\widetilde{\mathcal{U}} = A_E^c \widetilde{\bigcup} A_E = \widetilde{\bigcup}_{i=1}^{M+N} [G_E^i \widetilde{\bigcup}(H_E^i \widetilde{\bigcap} A_E)].$$

Thus $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft compact. □

An s-extension of a soft compact space may not be soft compact.

Example 4.3. Let $\mathcal{T} = \{A_E \widetilde{\subseteq} \widetilde{\mathbb{R}} : 0(e_1) \widetilde{\notin} A_E\} \widetilde{\bigcup} \{\widetilde{\mathbb{R}}\}$ be a soft topology on the set of reals $\mathbb{R}$, where $E = \{e_1, e_2\}$. Evidently, $(\mathbb{R}, \mathcal{T}, E)$ is soft compact since the only soft open cover of $\mathbb{R}$ is $\{\widetilde{\mathbb{R}}\}$ which is finite. But $\mathcal{T}[\{(e_1, 0), (e_2, \emptyset)\}] = SP(\mathbb{R}, E)$ is the soft discrete topology and surely it cannot be soft compact as $\mathbb{R}$ is infinite.

Lemma 4.4. *Let $G_E, D_E \widetilde{\subseteq} (\mathcal{U}, \mathcal{T}, E)$. If G_E is soft open and D_E soft dense, then* $\mathrm{cl}(G_E) = \mathrm{cl}(G_E \widetilde{\bigcap} D_E)$.

Proof. Since $G_E \widetilde{\bigcap} D_E \widetilde{\subseteq} G_E$, then $\mathrm{cl}(G_E \widetilde{\bigcap} D_E) \widetilde{\subseteq} \mathrm{cl}(G_E)$. On the other hand, if $u(e) \widetilde{\in} \mathrm{cl}(G_E)$. Then for each soft open set O_E containing $u(e)$, $G_E \widetilde{\bigcap} O_E \neq \widetilde{\Phi}$. But D_E is soft dense, so it intersects each non-null soft open set. Therefore $(G_E \widetilde{\bigcap} D_E) \widetilde{\bigcap} O_E \neq \widetilde{\Phi}$. This implies that $u(e) \widetilde{\in} \mathrm{cl}(G_E \widetilde{\bigcap} D_E)$. Hence $\mathrm{cl}(G_E) \widetilde{\subseteq} \mathrm{cl}(G_E \widetilde{\bigcap} D_E)$. □

Theorem 4.5. *Let $(\mathcal{U}, \mathcal{T}, E)$ be a soft separable space. Then $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft separable if and only if $(A, \mathcal{T}_A, E)$ is soft separable.*

Proof. Assume $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft separable. Then there exists a conutable soft subset D_E of $(\mathcal{U}, \mathcal{T}[A_E], E)$ such that $\mathrm{cl}_{\mathcal{T}[A_E]}(D_E) = \widetilde{\mathcal{U}}$. Since A_E is soft $\mathcal{T}[A_E]$-open, by Lemma 4.4, $\mathrm{cl}_{\mathcal{T}[A_E]}(A_E) = \mathrm{cl}_{\mathcal{T}[A_E]}(A_E \widetilde{\bigcap} D_E)$. Therefore

$$A_E \widetilde{\subseteq} \mathrm{cl}_{\mathcal{T}[A_E]}(A_E) = \mathrm{cl}_{\mathcal{T}[A_E]}(A_E \widetilde{\bigcap} D_E) = \mathrm{cl}_{\mathcal{T}}(A_E \widetilde{\bigcap} D_E),$$

and so $A_E = A_E \widetilde{\bigcap} \mathrm{cl}_{\mathcal{T}}(A_E \widetilde{\bigcap} D_E) = \mathrm{cl}_A(A_E \widetilde{\bigcap} D_E)$. But $A_E \widetilde{\bigcap} D_E$ is soft countable, hence $(A, \mathcal{T}_A, E)$ is soft separable.

Conversely, let F_E be a countable soft dense in $(\mathcal{U}, \mathcal{T}, E)$ and let C_E be a countable soft dense in $(A, \mathcal{T}_A, E)$. Then $D_E = F_E \widetilde{\bigcup} C_E$ is countable. We now prove that D_E is soft dense in $(\mathcal{U}, \mathcal{T}[A_E], E)$. Let O_E be any non-null soft $\mathcal{T}[A_E]$-open. Then $O_E = G_E \widetilde{\bigcup}(H_E \widetilde{\bigcap} A_E)$ for some $\mathcal{T}$-open sets G_E, H_E. If $G_E \neq \widetilde{\Phi}$, then $G_E \widetilde{\bigcap} F_E \neq \widetilde{\Phi}$ and so $[G_E \widetilde{\bigcup}(H_E \widetilde{\bigcap} A_E)] \widetilde{\bigcap} D_E \neq \widetilde{\Phi}$. On the other hand, if $G_E = \widetilde{\Phi}$, $H_E \widetilde{\bigcap} A_E \neq \widetilde{\Phi}$. Therefore $(H_E \widetilde{\bigcap} A_E) \widetilde{\bigcap} C_E \neq \widetilde{\Phi}$ and $O_E \widetilde{\bigcap} D_E = [G_E \widetilde{\bigcup}(H_E \widetilde{\bigcap} A_E)] \widetilde{\bigcap} [F_E \widetilde{\bigcap} C_E] \neq \widetilde{\Phi}$. Thus D_E is soft dense in $(\mathcal{U}, \mathcal{T}[A_E], E)$, and hence $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft separable. □

Theorem 4.6. *If $(\mathcal{U}, \mathcal{T}, E)$ is a soft T_i-space, then $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft T_i-space, for $i = 0, 1, 2$.*

Proof. By Remark 3.2 (i), $\mathcal{T} \widetilde{\subseteq} \mathcal{T}[A_E]$. □

The converse is not valid in general.

Example 4.7. Let $\mathcal{U} = \{u\}$, $E = \{e_1, e_2\}$, and $\mathcal{T} = \{\widetilde{\Phi}, \widetilde{\mathcal{U}}\}$ be the soft topology on $\mathcal{U}$. If $A_E = \{(e_1, \{u\}), (e_2, \emptyset)\}$, then $\mathcal{T}[A_E] = \{\widetilde{\Phi}, A_E, \widetilde{\mathcal{U}}\}$ and so $(\mathcal{U}, \mathcal{T}[A_E], E)$ is a soft T_0-space, but $(\mathcal{U}, \mathcal{T}, E)$ is not.

Example 4.8. Consider the soft spaces $(\mathbb{R}, \mathcal{T}, E)$ and $(\mathbb{R}, \mathcal{T}[A_E], E)$ given in Example 4.3, there $A_E = \{(e_1, 0), (e_2, \emptyset)\}$. Then $(\mathbb{R}, \mathcal{T}[A_E], E)$ is soft T_2, while $(\mathbb{R}, \mathcal{T}, E)$ is not soft T_1.

Lemma 4.9. *Let $(\mathcal{U},\mathcal{T},E)$ be a soft regular space. Then $(\mathcal{U},\mathcal{T}[A_E],E)$ is not soft regular if and only if there exists $u(e)\widetilde{\in}A_E$ such that $G_E\widetilde{\bigcap}(\mathrm{cl}_{\mathcal{T}}(A_E)\backslash A_E)\neq\widetilde{\Phi}$ for each soft $\mathcal{T}$-open set G_E containing $u(e)$.*

Proof. Suppose, if possible, $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft regular. Then for each soft $\mathcal{T}[A_E]$-open set $G_E\widetilde{\subseteq}A_E$ containing $u(e)$, there is a soft $\mathcal{T}[A_E]$-closed set F_E such that $u(e)\widetilde{\subseteq}F_E\widetilde{\subseteq}G_E$. Again by soft regularity of $(\mathcal{U},\mathcal{T},E)$, there exists soft $\mathcal{T}$-open O_E such that $u(e)\widetilde{\in}O_E\widetilde{\bigcap}A_E\widetilde{\subseteq}F_E$. By assumption, there is $v(e)\widetilde{\in}O_E\widetilde{\bigcap}(\mathrm{cl}_{\mathcal{T}}(A_E)\backslash A_E)$. Therefore $v(e)\widetilde{\in}\,\mathrm{cl}_{\mathcal{T}}(O_E\widetilde{\bigcap}A_E)\widetilde{\subseteq}\,\mathrm{cl}_{\mathcal{T}}(F_E)=\mathrm{cl}_{\mathcal{T}[A_E]}(F_E)=F_E\widetilde{\subseteq}A_E$, which is impossible.

Conversely, suppose that $(\mathcal{U},\mathcal{T}[A_E],E)$ is not soft regular. There exists a soft $\mathcal{T}[A_E]$-open O_E containing $u(e)$ such that no soft $\mathcal{T}[A_E]$-closed set F_E containing $u(e)$ is a subset of O_E. On the other hand $u(e)$ must be in A_E. If not, then for the soft $\mathcal{T}$-open $D_E=G_E\widetilde{\bigcup}(\widetilde{\Phi}\widetilde{\bigcap}A_E)$ containing $u(e)$, there is a soft $\mathcal{T}$-closed set K_E such that $u(e)\widetilde{\in}K_E\widetilde{\subseteq}D_E$. By Lemma 3.4, K_E is soft $\mathcal{T}[A_E]$-closed, which shows that $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft regular, a contradiction.

Now, we assume that there is a soft $\mathcal{T}[A_E]$-open U_E containing $u(e)$ such that

$$U_E\widetilde{\bigcap}(\mathrm{cl}_{\mathcal{T}}(A_E)\backslash A_E)=\widetilde{\Phi}.$$

Let G_E be a soft $\mathcal{T}[A_E]$-open containing $u(e)$, where $G_E=\widetilde{\Phi}\widetilde{\bigcup}(H_E\widetilde{\bigcap}A_E)$. If F_E is any soft $\mathcal{T}$-closed set containing $u(e)$ with $F_E\widetilde{\subseteq}U_E\widetilde{\bigcap}H_E$, by Lemma 3.4,

$$\mathrm{cl}_{\mathcal{T}[A_E]}(F_E\widetilde{\bigcap}A_E)=\mathrm{cl}_{\mathcal{T}}(F_E\widetilde{\bigcap}A_E)\widetilde{\subseteq}\,\mathrm{cl}_{\mathcal{T}}(F_E)\widetilde{\bigcap}\,\mathrm{cl}_{\mathcal{T}}(A_E)=F_E\widetilde{\bigcap}\,\mathrm{cl}_{\mathcal{T}}(A_E).$$

Since $F_E\widetilde{\subseteq}(U_E\widetilde{\bigcap}H_E)$ and $U_E\widetilde{\bigcap}(\mathrm{cl}_{\mathcal{T}}(A_E)\backslash A_E)=\widetilde{\Phi}$, then $F_E\widetilde{\bigcap}\,\mathrm{cl}_{\mathcal{T}}(A_E)=F_E\widetilde{\bigcap}A_E$ and so $\mathrm{cl}_{\mathcal{T}[A_E]}(F_E\widetilde{\bigcap}A_E)\widetilde{\subseteq}F_E\widetilde{\bigcap}A_E\widetilde{\subseteq}(U_E\widetilde{\bigcap}H_E)\widetilde{\bigcap}A_E\widetilde{\subseteq}G_E$. This means that $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft regular, which contradicts the assumption. The proof is finished. □

From the above result, we have

Theorem 4.10. *Let $(\mathcal{U},\mathcal{T},E)$ be a soft regular space. Then $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft regular if and only $\mathrm{cl}_{\mathcal{T}}(A_E)\backslash A_E$ is soft $\mathcal{T}$-closed.*

Theorem 4.11. *Let $(\mathcal{U},\mathcal{T},E)$ be a soft regular space. If A_E is soft $\mathcal{T}$-closed, then $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft regular.*

Generally, the condition on the soft set A_E in the above results is essential.

Example 4.12. Let us use the details provided in Example 4.7. The space $(\mathcal{U},\mathcal{T},E)$ is soft regular. On the other hand, $(\mathcal{U},\mathcal{T}[A_E],E)$ is not soft regular as the set A_E is not soft closed.

Lemma 4.13. ([7, Theorem 5.7]) *Let $(\mathcal{U},\mathcal{T},E)$ be a soft normal space and A_E be a $\mathcal{T}$-closed set. Then $(A,\mathcal{T}_A,E)$ is soft normal.*

Lemma 4.14. ([17, Proposition 12, Theorem 2]) *Let $(Z,\mathcal{T}_Z,E)$ be a soft open (closed) subspace of $(\mathcal{U},\mathcal{T},E)$ and let $A_E\widetilde{\subseteq}Z_E$. Then A_E is soft $\mathcal{T}$-open ($\mathcal{T}$-closed) if and only if A_E is soft $\mathcal{T}_Z$-open ($\mathcal{T}_Z$-closed).*

Theorem 4.15. *Let $(\mathcal{U},\mathcal{T},E)$ be a normal space and let A_E be soft $\mathcal{T}$-closed. Then $(\mathcal{U},\hat{\mathcal{T}}=\mathcal{T}[A_E],E)$ is soft normal if and only $(A^c,\mathcal{T}_{A^c},E)$ is soft normal.*

Proof. Suppose $(\mathcal{U},\mathcal{T}[A_E],E)$ is soft normal. Since A_E is soft $\mathcal{T}$-closed, by Lemma 3.4, A_E is soft $\hat{\mathcal{T}}$-closed, by Lemma 4.13, $(A^c,\hat{\mathcal{T}}_{A^c},E)$ is soft normal, but $(A^c,\hat{\mathcal{T}}_{A^c},E)=(A^c,\mathcal{T}_{A^c},E)$ from Remark 3.2 (vii). This part is done.

Conversely, assume $(A^c,\mathcal{T}_{A^c},E)$ is soft normal. Let K_E,L_E be two soft $\hat{\mathcal{T}}$-closed sets with $K_E\widetilde{\bigcap}L_E=\widetilde{\Phi}$. Then $K_E\widetilde{\bigcap}A_E$ and $L_E\widetilde{\bigcap}A_E$ are soft closed in $(A,\hat{\mathcal{T}}_A,E)$, which imply, by Lemma 3.4, they are soft closed in

$(A, \mathcal{T}_A, E)$. By assumption and Lemma 4.14, $K_E \widetilde{\bigcap} A_E$ and $L_E \widetilde{\bigcap} A_E$ are soft closed in $(\mathcal{U}, \mathcal{T}, E)$. Since $(\mathcal{U}, \mathcal{T}, E)$ is soft normal, then there exist soft open sets G_E, H_E such that $K_E \widetilde{\bigcap} A_E \widetilde{\subseteq} G_E$, $L_E \widetilde{\bigcap} A_E \widetilde{\subseteq} H_E$, and $G_E \widetilde{\bigcap} H_E = \widetilde{\Phi}$. On the other hand, $K_E \widetilde{\bigcap} A_E^c$ and $L_E \widetilde{\bigcap} A_E^c$ are also soft closed in $(A^c, \hat{\mathcal{T}}_{A^c}, E) = (A^c, \mathcal{T}_{A^c}, E)$ from Lemma 3.5. By assumption, there are soft open sets U_E, V_E in $(A^c, \mathcal{T}_{A^c}, E) = (A^c, \hat{\mathcal{T}}_{A^c}, E)$ such that $K_E \widetilde{\bigcap} A_E^c \widetilde{\subseteq} U_E$, $K_E \widetilde{\bigcap} A_E^c \widetilde{\subseteq} V_E$, and $U_E \widetilde{\bigcap} V_E = \widetilde{\Phi}$. Since $\mathcal{T} \widetilde{\subseteq} \hat{\mathcal{T}}$ and A_E^c is $\hat{\mathcal{T}}$-open, then G_E, H_E, U_E, V_E are soft open in $(\mathcal{U}, \mathcal{T}[A_E], E)$. Therefore $K_E = (K_E \widetilde{\bigcap} A_E) \widetilde{\bigcup} (K_E \widetilde{\bigcap} A_E^c) \widetilde{\subseteq} (G_E \widetilde{\bigcap} A_E) \widetilde{\bigcup} U_E = O_E$, $L_E = (L_E \widetilde{\bigcap} A_E) \widetilde{\bigcup} (L_E \widetilde{\bigcap} A_E^c) \widetilde{\subseteq} (H_E \widetilde{\bigcap} A_E) \widetilde{\bigcup} V_E = W_E$, and $O_E \widetilde{\bigcap} W_E = \widetilde{\Phi}$. Thus $(\mathcal{U}, \hat{\mathcal{T}} = \mathcal{T}[A_E], E)$ is soft normal. □

Remark 4.16. If $(\mathcal{U}, \mathcal{T}[A_E], E)$ is an s-extension of $(\mathcal{U}, \mathcal{T}, E)$ and A_E is $\mathcal{T}$-closed, then A_E, A_E^c are soft closed and soft open. In this case, $(\mathcal{U}, \mathcal{T}[A_E], E)$ is always a soft disconnected space.

Theorem 4.17. *Let $(\mathcal{U}, \mathcal{T}, E)$ be any space. If either of the following statements is true, then $(\mathcal{U}, \mathcal{T}[A_E], E)$ is soft connected.*

(i) *if $(A, \mathcal{T}_A, E)$ is soft connected and A_E is soft dense in $(\mathcal{U}, \mathcal{T}, E)$; or*
(ii) *if $(A, \mathcal{T}_A, E)$ and $(A^c, \mathcal{T}_{A^c}, E)$ are soft connected but A_E is not soft $\mathcal{T}$-closed.*

Proof. (i) If $(\mathcal{U}, \mathcal{T}[A_E], E)$ is not soft connected, then there exists soft $\mathcal{T}[A_E]$-open sets O_E, W_E such that $\widetilde{\mathcal{U}} = O_E \widetilde{\bigcup} W_E$ and $O_E \widetilde{\bigcap} W_E = \widetilde{\Phi}$, where $O_E = G_E \widetilde{\bigcup} (H_E \widetilde{\bigcap} A_E)$ and $W_E = U_E \widetilde{\bigcup} (V_E \widetilde{\bigcap} A_E)$ for some $\mathcal{T}$-open sets G_E, H_E, U_E, V_E. Clearly $G_E \widetilde{\bigcup} H_E \neq \widetilde{\Phi}$ and $U_E \widetilde{\bigcup} V_E \neq \widetilde{\Phi}$. By soft density of A_E, $(G_E \widetilde{\bigcup} H_E) \widetilde{\bigcap} A_E \neq \widetilde{\Phi}$ and $(U_E \widetilde{\bigcup} V_E) \widetilde{\bigcap} A_E \neq \widetilde{\Phi}$. But $\widetilde{\mathcal{U}} = (G_E \widetilde{\bigcup} H_E) \widetilde{\bigcup} (U_E \widetilde{\bigcup} V_E)$ and so $A_E = [(G_E \widetilde{\bigcup} H_E) \widetilde{\bigcap} A_E] \widetilde{\bigcup} [(U_E \widetilde{\bigcup} V_E) \widetilde{\bigcap} A_E]$. This proves that $(A, \mathcal{T}_A, E)$ is not soft connected, a contradiction.

(ii) If $(\mathcal{U}, \mathcal{T}[A_E], E)$ is not soft connected, then there exists soft $\mathcal{T}[A_E]$-open sets O_E, W_E such that $\widetilde{\mathcal{U}} = O_E \widetilde{\bigcup} W_E$ and $O_E \widetilde{\bigcap} W_E = \widetilde{\Phi}$. By Remark 3.2 (iii), none of O_E and W_E is equal to A_E. Therefore either $O_E \widetilde{\bigcap} A_E \neq \widetilde{\Phi}$ and $W_E \widetilde{\bigcap} A_E \neq \widetilde{\Phi}$ or $O_E \backslash A_E \neq \widetilde{\Phi}$ and $W_E \backslash A_E \neq \widetilde{\Phi}$. Neither of two cases is possible as $(A, \mathcal{T}_A, E)$ and $(A^c, \mathcal{T}_{A^c}, E)$ are soft connected by assumption. □

5. Conclusion

The continuous supply of classes of topological spaces, examples, and their features and relations has aided the development of topology. It is, therefore, crucial to expand the area of soft topological spaces in the same manner. We have contributed this part to soft topology with some new classes of soft topological spaces by means of s-extension. We have shown that s-extension need not give a unique soft topology. By using the concept of soft point provided in [6], we have studied the preservation of some soft topological properties under s-extension. We have shown that the obtained results are similar to those found in classical topology, for instance: Theorems 4.15, 4.10, 4.6 are parallel to Theorems 1, 2, 5 in [11], but a soft topology with its simple extension does not get along well with its parametrized (crisp) topologies, see Remark 3.3. On the other hand, if we use the notion of soft elements, the latter soft topologies and their counterparts are nicely related, (see [2]).

In the future, this line of research could be developed by exploring other soft topological properties or by imposing some requirements on the $\mathcal{F}$ family in Definition 3.1.

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