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Baire Bijections of Extreme Points Generated by Small-Bound Isomorphisms of Simplex Spaces

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Abstract. We show that small-bound isomorphisms of spaces of affine continuous functions on Choquet simplices with Lindelöf boundaries induce Baire measurable bijections of their sets of extreme points.

1. Introduction

If K is a compact (Hausdorff) space then let $C(K)$ stand for the space of all real continuous functions on K, endowed with the supremum norm. Further, let $\mathcal{M}(K)$ stand for the space of all Radon measures on K endowed with the variation norm and the weak*-topology given by the duality $C(K)^* = \mathcal{M}(K)$, let $\mathcal{M}^+(K)$ stand for the set of positive elements from $\mathcal{M}(K)$, and let $\mathcal{M}^1(K)$ stand for the set of Radon probability measures on K. If E is a real Banach space then E^* stands for its dual space. We denote by B_E and S_E the unit ball and sphere in E, respectively, and we write $\langle \cdot, \cdot \rangle \colon E^* \times E \to \mathbb{R}$ for the duality mapping.

If X is a compact convex set in a locally convex (Hausdorff) space, then let $\mathfrak{A}(X)$ stand for the space of all continuous real affine functions on X, endowed with the supremum norm. Let us recall that if $\mu, \nu \in \mathcal{M}^+(X)$, then $\mu \prec \nu$ if $\mu(k) \leq \nu(k)$ for each convex continuous function k on X, and a measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is $\prec$-maximal. A measure $\mu \in \mathcal{M}(X)$ is *boundary* if its variation $|\mu|$ is maximal or if $\mu = 0$. Further, for any $\mu \in \mathcal{M}^1(X)$ there exists a unique point $r(\mu) \in X$, called the *barycenter* of μ, such that $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X)$, see [1, Propositions I.2.1 and I.2.2]. By the Choquet–Bishop–de-Leeuw representation theorem (see [1, Theorem I.4.8]), for each $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}^1(X)$ with $r(\mu) = x$. If this measure is uniquely determined for each $x \in X$, the set X is called a *simplex*, see [1, Theorem II.3.6]. When X is a simplex, the space $\mathfrak{A}(X)$ is sometimes termed a *simplex space*.

The well-known generalization of the Banach-Stone theorem given independently by Amir [2] and Cambern [5] states that compact spaces K_1 and K_2 are homeomorphic if there exists an isomorphism $T \colon C(K_1) \to C(K_2)$ with $\|T\| \left\|T^{-1}\right\| < 2$. This result has been successively extended in papers [7], [13], [8], and [18] to the context of spaces of affine continuous functions on compact convex sets, whose extreme points consist of weak peak points (we recall that a point $x \in \operatorname{ext} X$ is a *weak peak point* if given $\varepsilon \in (0,1)$ and an open set $U \subset X$ containing x, there exists a in $B_{\mathfrak{A}(X,\mathbb{R})}$ such that $|a| < \varepsilon$ on $\operatorname{ext} X \setminus U$ and $a(x) > 1 - \varepsilon$, see [6, p.

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73]). On the other hand, it is known that without the assumption of weak peak points the Amir-Cambern theorem for spaces of affine continuous functions in some sense fails completely even if the considered compact convex sets are simplices. Indeed, in [9], Hess shows that for each $\varepsilon > 0$, there exist simplices X_1, X_2 with nonhomeomorphic countable sets of extreme points and an isomorphism $T : \mathfrak{A}(X_1) \to \mathfrak{A}(X_2)$ with $\|T\| \left\|T^{-1}\right\| < 1 + \varepsilon$. Thus it seems natural to ask, given two simplex spaces that are isomorphic by an isomorphism with a small-bound, how different the sets of extreme points of these simplices can be. We show that, under some topological assumptions, the sets of extreme points are in this case in a certain sense quite similar. More precisely, we prove the following result saying that there exists a bijection of the sets of extreme points that is measurable in both directions with respect to hierarchies of Baire sets, with a small complexity (for the definition of the unexplained notions see the next section).

Theorem 1.1. *Let for $i = 1, 2$, X_i be a simplex with* $\operatorname{ext} X_i$ *being Lindelöf, and assume that there exists an isomorphism $T : \mathfrak{A}(X_1) \to \mathfrak{A}(X_2)$ with $\|T\| \left\|T^{-1}\right\| < \frac{3}{2}$. Then there exist mutually bijective mappings $\rho_1 : \operatorname{ext} X_1 \to \operatorname{ext} X_2$ and $\rho_2 : \operatorname{ext} X_2 \to \operatorname{ext} X_1$ with the following properties.*

(i) For $i \neq j \in \{1, 2\}$, and for each zero set $F \subseteq \operatorname{ext} X_j$,
$\rho_i^{-1}(F) \in \Delta_3(\operatorname{Bas}(\operatorname{ext} X_i))$.

(ii) For $i \neq j \in \{1, 2\}$, if $\operatorname{ext} X_j$ is moreover resolvable, then for each zero set $F \subseteq \operatorname{ext} X_j$, $\rho_i^{-1}(F) \in \Delta_2(\operatorname{Bas}(\operatorname{ext} X_i))$.

In case where the considered simplices are metrizable, the sets of extreme points are G_δ, see [1, Corollary I.4.4], and hence Polish spaces. The hierarchies of Baire and Borel sets in this case coincide, since any closed subset of a metrizable space is G_δ (see e.g. [14, Proposition A.43]). Moreover, the sets of extreme points are resolvable if and only if they are F_σ. Thus in this case, using the standard notation (see e.g. [11, Definition (24.2)]), the above result can be stated in the following simpler way.

Corollary 1.2. *Let for $i = 1, 2$, X_i be a metrizable simplex, and assume that there exists an isomorphism $T : \mathfrak{A}(X_1) \to \mathfrak{A}(X_2)$ with $\|T\| \left\|T^{-1}\right\| < \frac{3}{2}$. Then there exist mutually bijective mappings $\rho_1 : \operatorname{ext} X_1 \to \operatorname{ext} X_2$ and $\rho_2 : \operatorname{ext} X_2 \to \operatorname{ext} X_1$ with the following properties.*

(i) For $i \in \{1, 2\}$, the mapping ρ_i is Δ^0_3-measurable.

(ii) For $i \neq j \in \{1, 2\}$, if $\operatorname{ext} X_j$ is F_σ, then the mapping ρ_i is Δ^0_2-measurable.

We recall that two uncountable Polish spaces are always Borel isomorphic (see [11, Theorem 15.6]). Further, by [17], the sets of extreme points of two simplices have the same cardinality if the corresponding simplex spaces are isomorphic. Thus the new information that we obtain here is that under the assumptions of Corollary 1.2, the Borel isomorphism has a very low complexity.

The paper is organized as follows. In sections 2 and 3, we collect preliminaries from topology and the theory of affine functions on compact convex sets, respectively. Section 4 is devoted to the proofs of the results, and in Section 5 we present several examples concerning the question of sharpness of the results. In particular, we prove that Theorem 1.1 does not hold without the assumption that the sets of extreme points are Lindelöf, see Example 5.1.

2. Topological preliminaries

Let X be a topological space. We recall that X is *Lindelöf*, if any open covering of X has a countable subcover. A subset A of X is G_δ if it is a countable intersection of open sets, and the complement of a G_δ set is called an F_σ set. The set A is *resolvable* if for any nonempty $B \subset X$ (equivalently, for any nonempty closed $B \subset X$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. By [14, Proposition A.117], a subset of a completely metrizable space is resolvable if and only if it is of type F_σ and G_δ.

Next we recall that a *zero set* in X is the inverse image of a closed set in $\mathbb{R}$ under a continuous function $f : X \to \mathbb{R}$. The complement of a zero set is a *cozero set*. If X is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set. We recall that *Borel sets* are members of the σ-algebra generated by the family of all open subsets of X and *Baire sets* are members of the σ-algebra generated by the family of all cozero sets in X.

Next we recall the hierarchies of Baire and Borel sets and mappings. For details we refer the reader to [20]. The hierarchy of Baire sets in X is defined as follows. Let $\mathrm{Bas}(X)$ stand for the algebra generated by cozero sets in X and $\Sigma_2(\mathrm{Bas}(X))$ for countable unions of sets from $\mathrm{Bas}(X)$ (we recall that a family $\mathcal{F}$ of subsets of X is an *algebra* if $\emptyset, X \in \mathcal{F}$ and $\mathcal{F}$ is closed with respect to complements and finite unions). Further, let $\Pi_2(\mathrm{Bas}(X))$ be made of all countable intersections of sets from $\mathrm{Bas}(X)$, and let $\Delta_2(\mathrm{Bas}(X)) = \Sigma_2(\mathrm{Bas}(X)) \cap \Pi_2(\mathrm{Bas}(X))$. Proceeding inductively, for any $\alpha \in (2, \omega_1)$ we let $\Sigma_\alpha(\mathrm{Bas}(X))$ be made of all countable unions of sets from $\bigcup_{1\le\beta<\alpha} \Pi_\beta(\mathrm{Bas}(X))$, $\Pi_\alpha(\mathrm{Bas}(X))$ is made of all countable intersections of sets from $\bigcup_{1\le\beta<\alpha} \Sigma_\beta(\mathrm{Bas}(X))$, and let $\Delta_\alpha(\mathrm{Bas}(X)) = \Sigma_\alpha(\mathrm{Bas}(X)) \cap \Pi_\alpha(\mathrm{Bas}(X))$. Further, let $\mathrm{Bos}(X)$ stand for the algebra generated by closed sets in X. Proceeding analogically as above, we obtain the hierarchy of Borel sets in X.

Further, let F be an another topological space. The *Baire measurable mappings* are the mappings measurable with respect to Baire sets (we recall that, given a family $\mathcal{F}$ of sets in a set X, a mapping $f\colon X \to F$ is called *$\mathcal{F}$-measurable* if $f^{-1}(U) \in \mathcal{F}$ for every $U \subset F$ open). The hierarchy of Baire measurable mappings is defined as follows. Let

$$\mathrm{Baf}_1(X, F) = \{f\colon X \to F;\ f^{-1}(U) \in \Sigma_2(\mathrm{Bas}(X)), U \subset F \text{ open}\}.$$

Given an ordinal $\alpha \in (1, \omega_1)$, assuming that $\mathrm{Baf}_\beta(X, F)$ has been already defined for each $\beta < \alpha$, let $\mathrm{Baf}_\alpha(X, F)$ stand for the mappings that are pointwise limits of sequences contained in $\bigcup_{1\le\beta<\alpha} \mathrm{Baf}_\beta(X, F)$. The hierarchy of Borel measurable mappings is defined analogously, starting the procedure with mappings *of the first Borel class*, i.e., with

$$\mathrm{Bof}_1(X, F) = \{f\colon X \to F;\ f^{-1}(U) \in \Sigma_2(\mathrm{Bos}(X)), U \subset F \text{ open}\}.$$

Further, let $C_0(X, F) = C(X, F)$ be the family of all continuous mappings from X to F, and for an ordinal $\alpha \in (0, \omega_1)$, assuming that $C_\beta(X, F)$ has been already defined for each $\beta < \alpha$, we define $C_\alpha(X, F)$ as pointwise limits of sequences contained in $\bigcup_{0\le\beta<\alpha} C_\beta(X, F)$.

Then we have the following results (see [20, Theorem 5.2] and [21, Theorem 3.7(i)]).

Proposition 2.1. *Let $f\colon X \to \mathbb{R}$ be a function on a Tychonoff space X and $\alpha \in [1, \omega_1)$. Then $f \in \mathrm{Baf}_\alpha(X, \mathbb{R})$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bas}(X))$-measurable.*

Proposition 2.2. *If X is a normal topological space, then $C_\alpha(X, \mathbb{R}) = \mathrm{Baf}_\alpha(X, \mathbb{R})$ for each $\alpha \in [1, \omega_1)$.*

3. Affine functions on compact convex sets

Let X be a compact convex set in a locally convex (Hausdorff) space. We consider the evaluation mapping

$$\phi : X \to \mathfrak{A}(X)^*, \quad \phi(x)(f) = f(x), \quad x \in X, f \in \mathfrak{A}(X).$$

It is a standard part of the theory of compact convex sets (see e.g [14, Section 4.3]) that the mapping ϕ is a homeomorphic embedding of X into the unit ball $B_{\mathfrak{A}(X)^*}$ equipped with the weak* topology, and moreover,

$$\mathrm{ext}\, B_{\mathfrak{A}(X)^*} = \phi(\mathrm{ext}\, X) \cup -\phi(\mathrm{ext}\, X), \quad \overline{\mathrm{ext}\, B_{\mathfrak{A}(X)^*}} \subseteq \phi(X) \cup -\phi(X)$$

and

$$B_{\mathfrak{A}(X)^*} = \mathrm{co}(\phi(X) \cup -\phi(X)).$$

Further, if $\mu \in \mathcal{M}(X)$ and $f\colon X \to \mathbb{R}$ is a μ-measurable function, then we write $\mu(f)$ for $\int_X f \,\mathrm{d}\mu$. There exists a natural restriction $R : \mathcal{M}(X) \to \mathfrak{A}(X)^*$ defined by

$$R(\mu)(f) = \mu(f), \quad \mu \in \mathcal{M}(X), f \in \mathfrak{A}(X).$$

Further, the symbol ε_x stands for the Dirac measure at a point $x \in X$. Clearly, $R(\varepsilon_x) = \phi(x)$.

Next we recall that a function $f\colon X \to \mathbb{R}$ is called *strongly affine* if, for any measure $\mu \in \mathcal{M}^1(X)$, f is μ-integrable and $\int_X f \,\mathrm{d}\mu = f(r(\mu))$. Each strongly affine function is affine, but the converse is not true in general. Every strongly affine function is bounded, see [14, Lemma 4.5]. The function f is said to satisfy the *maximum principle* if

$$\sup_{x\in X} \left|f(x)\right| = \sup_{x\in \operatorname{ext} X} \left|f(x)\right|.$$

Consequently, if an element $a^{**} \in \mathfrak{A}(X)^{**}$ satisfies the maximum principle on the compact convex set $B_{\mathfrak{A}(X)^*}$, then a^{**} is determined by its values on $\operatorname{ext} X$ in the sense that

$$\begin{aligned} \|a^{**}\| &= \sup_{s^*\in B_{\mathfrak{A}(X)^*}} |\langle a^{**}, s^*\rangle| = \sup_{s^*\in \operatorname{ext} B_{\mathfrak{A}(X)^*}} |\langle a^{**}, s^*\rangle| = \\ &= \sup_{s^*\in \phi(\operatorname{ext} X)\cup -\phi(\operatorname{ext} X)} |\langle a^{**}, s^*\rangle| = \sup_{x\in \operatorname{ext} X} \left|\langle a^{**}, \phi(x)\rangle\right|. \end{aligned} \tag{1}$$

Important examples of functions satisfying the maximum principle on a compact convex set are Baire strongly affine functions (see e.g. [14, Theorem 3.86]), and affine functions of the first Borel class (see [8] and [12, Theorem 2.3]). Further, by [13, Lemma 2.1] if X is a compact convex set with the set $\operatorname{ext} X$ being Lindelöf, each strongly affine function $f : X \to \mathbb{R}$ satisfies the maximum principle. In this case, also the set $\operatorname{ext} B_{\mathfrak{A}(X)^*} = \phi(\operatorname{ext} X) \cup -\phi(\operatorname{ext} X)$ is Lindelöf, and hence for each element $a^{**} \in \mathfrak{A}(X)^{**}$ that is strongly affine on $B_{\mathfrak{A}(X)^*}$,

$$\|a^{**}\| \overset{(1)}{=} \sup_{x\in \operatorname{ext} X} \left|\langle a^{**}, \phi(x)\rangle\right|. \tag{2}$$

4. Results

In [17] it was showed how to extend the characteristic function $\chi_{\{x\}}$ of an extreme point x of a simplex X to an element $a_x^{**} \in \mathfrak{A}(X)^{**}$ satisfying the maximum principle on each ball $rB_{\mathfrak{A}(X)^*}$, $r > 0$. We briefly recall the way this was done.

The characteristic function $\chi_{\{x\}}$ is convex and upper semicontinuous. Moreover, since X is a simplex, its upper envelope $\chi^*_{\{x\}}$ defined for $z \in X$ as

$$\chi^*_{\{x\}}(z) = \inf\{a(z)\colon a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\} = \lim\{a(z)\colon a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\},$$

(where the system $\{a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\}$ is considered as a downward directed net) is upper semicontinuous and affine, see [14, Theorem 6.5], and coincides with $\chi_{\{x\}}$ on the set of extreme points of X, see [14, Theorem 3.24]. Thus by [18, Lemma 2.4], the formula

$$\langle a_x^{**}, s^*\rangle = \lim\{\langle s^*, a\rangle\colon a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\}, \quad s^* \in \mathfrak{A}(X)^*,$$

defines an element $a_x^{**} \in \mathfrak{A}(X)^{**}$, satisfying that $\langle a_x^{**}, \phi(z)\rangle = \chi^*_{\{x\}}(z)$ for $z \in X$. Moreover, a_x^{**} is of the first Borel class on each ball $rB_{\mathfrak{A}(X)^*}$, $r > 0$, and hence satisfies the maximum principle on each such ball.

Next we need to show that for each boundary measure $\mu \in \mathcal{M}(X)$ it holds that

$$\langle a_x^{**}, R(\mu)\rangle = \mu(\{x\}). \tag{3}$$

To see this, we write $\mu = \alpha_1\mu_1 - \alpha_2\mu_2$, where $\alpha_1, \alpha_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}^1(X)$. Since the function $\chi^*_{\{x\}}$ is affine and upper semicontinuous, it is strongly affine, see [14, Proposition 4.7]. Moreover, since the measure μ is boundary, it holds that $\mu(\chi^*_{\{x\}}) = \mu(\chi_{\{x\}})$, see [14, Theorem 3.68]. Thus

$$\begin{aligned} \langle a_x^{**}, R(\mu)\rangle &= \lim\{\langle R(\mu), a\rangle\colon a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\} = \\ &= \lim\{\alpha_1\mu_1(a) - \alpha_2\mu_2(a)\colon a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\} = \\ &= \lim\{\alpha_1 a(r(\mu_1)) - \alpha_2 a(r(\mu_2))\colon a \in \mathfrak{A}(X,\mathbb{R}), a > \chi_{\{x\}}\} = \\ &= \alpha_1\chi^*_{\{x\}}(r(\mu_1)) - \alpha_2\chi^*_{\{x\}}(r(\mu_2)) = \\ &= \alpha_1\mu_1(\chi^*_{\{x\}}) - \alpha_2\mu_2(\chi^*_{\{x\}}) = \mu(\chi^*_{\{x\}}) = \mu(\chi_{\{x\}}), \end{aligned}$$

and (3) holds.

Now we aim to show that also Baire subsets of extreme points of a simplex X can be extended to elements of $\mathfrak{A}(X)^{**}$, although the demonstration of this fact differs substantially from the above arguments for a single extreme point. Here the construction uses the solution of the abstract Dirichlet problem on X, i.e., the possibility of extending a Baire function defined on $\operatorname{ext} X$ to a strongly affine Baire function on X.

Lemma 4.1. *Let X be a compact convex set, and let $f \in \mathrm{Baf}_\alpha(X, \mathbb{R})$ be a strongly affine function for some ordinal $\alpha \in [0, \omega_1)$. Then the formula*

$$\tilde{f}(s^*) = \mu(f), \text{ where } R(\mu) = s^*, \quad s^* \in \mathfrak{A}(X)^*,$$

and where $R : \mathcal{M}(X) \to \mathfrak{A}(X)^*$ is the restriction mapping, defines an extension of f to an element in $\mathfrak{A}(X)^{**}$ satisfying that for each $r > 0$, $\tilde{f} \in \mathrm{Baf}_\alpha(rB_{\mathfrak{A}(X)^*}, \mathbb{R})$ is strongly affine.

Proof. First we show that the function $\tilde{f}$ is defined correctly. To this end, let measures $\mu, \nu \in \mathcal{M}(X)$ satisfy $R(\mu) = R(\nu)$. We write

$$\mu = \alpha_1\mu_1 - \alpha_2\mu_2, \quad \nu = \beta_1\nu_1 - \beta_2\nu_2,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are nonnegative real numbers and $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}^1(X)$. By the assumption, we have $\mu(1) = \nu(1)$, hence

$$\alpha_1 + \beta_2 = \alpha_2 + \beta_1.$$

If $\alpha_1 + \beta_2 = 0$, then $\mu = \nu = 0$. Thus we assume that $\alpha_1 + \beta_2 \neq 0$, and we pick an arbitrary function $h \in \mathfrak{A}(X)$. Then, since $\mu(h) = \nu(h)$, we obtain that

$$\frac{\alpha_1\mu_1(h) + \beta_2\nu_2(h)}{\alpha_1 + \beta_2} = \frac{\alpha_2\mu_2(h) + \beta_1\nu_1(h)}{\alpha_2 + \beta_1},$$

in others words, the probabilities $\frac{\alpha_1\mu_1+\beta_2\nu_2}{\alpha_1+\beta_2}$ and $\frac{\alpha_2\mu_2+\beta_1\nu_1}{\alpha_2+\beta_1}$ share the same barycenter. Thus, since the function f is strongly affine, we have

$$\frac{\alpha_1\mu_1(f) + \beta_2\nu_2(f)}{\alpha_1 + \beta_2} = \frac{\alpha_2\mu_2(f) + \beta_1\nu_1(f)}{\alpha_2 + \beta_1},$$

and it follows that $\mu(f) = \nu(f)$. Hence the function $\tilde{f}$ is a correctly defined linear functional on $\mathfrak{A}(X)^*$ and it is an extension of f in the sense that $\tilde{f}(\phi(x)) = \varepsilon_x(f) = f(x)$ for each $x \in X$. Hence $\tilde{f}(-\phi(x)) = -f(x)$ for $x \in X$. Moreover, since f is bounded by [14, Lemma 4.5], it is clear that also $\tilde{f}$ is bounded on $B_{\mathfrak{A}(X)^*}$, and hence it belongs to the space $\mathfrak{A}(X)^{**}$.

Next we show that $\tilde{f}$ is a strongly affine function in $\mathrm{Baf}_\alpha(B_{\mathfrak{A}(X)^*}, \mathbb{R})$. We recall that $B_{\mathfrak{A}(X)^*} = \operatorname{co}(\phi(X) \cup -\phi(X))$ with $\overline{\operatorname{ext} B_{\mathfrak{A}(X)^*}} \subseteq \phi(X) \cup -\phi(X)$. By the assumption, $f \in \mathrm{Baf}_\alpha(X, \mathbb{R})$. Thus $\tilde{f} \in \mathrm{Baf}_\alpha(\phi(X) \cup -\phi(X), \mathbb{R})$, and hence also $\tilde{f} \in \mathrm{Baf}_\alpha(\overline{\operatorname{ext} B_{\mathfrak{A}(X)^*}}, \mathbb{R})$.

Moreover, since the function f is strongly affine on X, it follows that for each measure $\mu \in \mathcal{M}^1(\overline{\operatorname{ext} B_{\mathfrak{A}(X)^*}})$, $\mu(\tilde{f}) = \tilde{f}(r(\mu))$. To see this, given a measure $\mu \in \mathcal{M}^1(\overline{\operatorname{ext} B_{\mathfrak{A}(X)^*}})$, we write $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$, where $\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$, $\mu_1 \in \mathcal{M}^1(\phi(X))$ and $\mu_2 \in \mathcal{M}^1(-\phi(X))$. Then $r(\mu_1) \in \phi(X)$, $r(\mu_2) \in -\phi(X)$, and $r(\mu) = \alpha_1 r(\mu_1) + \alpha_2 r(\mu_2)$. Also, since the function f is strongly affine on X, it follows that the function $f \circ \phi^{-1}$ is strongly affine on $\phi(X)$, see [19, Proposition 3.2]. Thus we have

$$\begin{aligned}\mu(\tilde{f}) &= \alpha_1\mu_1(\tilde{f}) + \alpha_2\mu_2(\tilde{f}) = \alpha_1\mu_1(f \circ \phi^{-1}) - \alpha_2\mu_2(f \circ \phi^{-1}) = \\ &= \alpha_1(f \circ \phi^{-1})(r(\mu_1)) - \alpha_2(f \circ \phi^{-1})(r(\mu_2)) = \\ &= \alpha_1\tilde{f}(r(\mu_1)) + \alpha_2\tilde{f}(r(\mu_2)) = \tilde{f}(\alpha_1 r(\mu_1) + \alpha_2 r(\mu_2)) = \tilde{f}(r(\mu)).\end{aligned}$$

Thus it follows by [19, Theorem 3.3] that $\tilde{f}$ is a strongly affine function and $\tilde{f} \in \mathrm{Baf}_\alpha(B_{\mathfrak{A}(X)^*}, \mathbb{R})$.

Finally, since for any $r > 0$, the ball $rB_{\mathfrak{A}(X)^*}$ is affinely homeomorphic to $B_{\mathfrak{A}(X)^*}$, it easily follows that $\tilde{f}$ is a strongly affine function on $rB_{\mathfrak{A}(X)^*}$ and $\tilde{f} \in \mathrm{Baf}_\alpha(rB_{\mathfrak{A}(X)^*}, \mathbb{R})$. The proof is finished. □

Lemma 4.2. *Let X be a simplex with $\operatorname{ext} X$ Lindelöf, and let $F \subseteq \operatorname{ext} X$ be a zero set (in $\operatorname{ext} X$). Then the characteristic function χ_F may be uniquely extended to a function $a_F^{**} \in \mathfrak{A}(X)^{**}$ satisfying the maximum principle on $B_{\mathfrak{A}(X)^*}$. Moreover, for each $r > 0$, a_F^{**} is a strongly affine function in $\operatorname{Baf}_2(rB_{\mathfrak{A}(X)^*}, \mathbb{R})$. If, in addition, $\operatorname{ext} X$ is resolvable, then $a_F^{**} \in \operatorname{Baf}_1(rB_{\mathfrak{A}(X)^*}, \mathbb{R})$.*

Proof. The uniqueness part of the statement follows from (1). To prove the existence, we note that the function χ_F clearly belongs to $\operatorname{Baf}_1(\operatorname{ext} X, \mathbb{R})$. Since $\operatorname{ext} X$ is normal, being Lindelöf and regular, we have $\chi_F \in C_1(\operatorname{ext} X, \mathbb{R})$ by Proposition 2.2. Thus by [16, Theorem 2.1], χ_F can be extended to a strongly affine function $h_F \in C_2(X, \mathbb{R}) = \operatorname{Baf}_2(X, \mathbb{R})$. By Lemma 4.1, the function h_F may be extended to a function $a_F^{**} \in \mathfrak{A}(X)^{**}$ such that for each $r > 0$, a_F^{**} is strongly affine function in $\operatorname{Baf}_2(rB_{\mathfrak{A}(X)^*}, \mathbb{R})$.

In the case when $\operatorname{ext} X$ is moreover resolvable, $h_F \in \operatorname{Baf}_1(X, \mathbb{R})$ by [16, Theorem 2.3], and hence $a_F^{**} \in \operatorname{Baf}_1(rB_{\mathfrak{A}(X)^*}, \mathbb{R})$. Finally, since a_F^{**} is a strongly affine Baire function on $B_{\mathfrak{A}(X)^*}$, it satisfies the maximum principle. $\square$

We note that from the uniqueness part of the previous lemma it follows that the notation a_x^{**} is not ambiguous, meaning that in the case when $\operatorname{ext} X$ is Lindelöf and $x \in \operatorname{ext} X$ is such that $\{x\}$ is a zero set, the element a_x^{**} defined at the beginning of this section coincides with the element $a_{\{x\}}^{**}$ given by Lemma 4.2.

Now we prove our main result.

Proof. [Proof of Theorem 1.1]

For $i = 1, 2$ we consider the evaluation mapping $\phi_i : X_i \to \mathfrak{A}(X_i)^*$, and we note that since $\operatorname{ext} X_i$ is Lindelöf by the assumption, so is $\operatorname{ext} B_{\mathfrak{A}(X_i)^*} = \phi_i(\operatorname{ext} X_i) \cup -\phi_i(\operatorname{ext} X_i)$. Further, for each $x \in \operatorname{ext} X_1$ we consider the element $a_x^{**} \in \mathfrak{A}(X_1)^{**}$, and analogously, for each $y \in \operatorname{ext} X_2$ we consider the element $b_y^{**} \in \mathfrak{A}(X_2)^{**}$.

Next we define the desired mappings ρ_1, ρ_2 in the following way. We find $\varepsilon \in (0, \frac{1}{4})$ such that $\|T\| \left\|T^{-1}\right\| < \frac{3}{2} - 2\varepsilon$, and let

$$\rho_1(x) = \{y \in \operatorname{ext} X_2 : \left|\langle (T^{**})^{-1} b_y^{**}, \phi_1(x)\rangle\right| > \frac{1-\varepsilon}{\|T\|}\}, \quad x \in \operatorname{ext} X_1,$$

$$\rho_2(y) = \{x \in \operatorname{ext} X_1 : \left|\langle T^{**} a_x^{**}, \phi_2(y)\rangle\right| > \frac{1-\varepsilon}{\left\|T^{-1}\right\|}\}, \quad y \in \operatorname{ext} X_2.$$

We need to show that the relations ρ_1 and ρ_2 are mutual bijections. This is done very similarly as in, e.g., [18]. Let L_1 and L_2 denote the domain of ρ_1 and ρ_2, respectively.

First we show that $\rho_2 : L_2 \to \operatorname{ext} X_1$ is a mapping. To this end, assume that for some $y \in \operatorname{ext} X_2$ there exist distinct points $x_1, x_2 \in \operatorname{ext} X_1$ such that

$$\left|\langle T^{**} a_{x_i}^{**}, \phi_2(y)\rangle\right| > \frac{1-\varepsilon}{\left\|T^{-1}\right\|}, \quad i = 1, 2.$$

Then we find $\alpha_1, \alpha_2 \in S_{\mathbb{R}}$ satisfying

$$\alpha_i \langle T^{**} a_{x_i}^{**}, \phi_2(y)\rangle > \frac{1-\varepsilon}{\left\|T^{-1}\right\|}, \quad i = 1, 2,$$

and so we have

$$\langle T^{**}(\alpha_1 a_{x_1}^{**} + \alpha_2 a_{x_2}^{**}), \phi_2(y)\rangle > 2\frac{1-\varepsilon}{\left\|T^{-1}\right\|}.$$

Consequently, $\left\|T^{**}(\alpha_1 a_{x_1}^{**} + \alpha_2 a_{x_2}^{**})\right\| > 2\frac{1-\varepsilon}{\|T^{-1}\|}$.

On the other hand, since the function $\alpha_1 a_{x_1}^{**} + \alpha_2 a_{x_2}^{**}$ satisfies the maximum principle on $B_{\mathfrak{A}(X_1)^*}$, by (1) we obtain that

$$\left\|\alpha_1 a_{x_1}^{**} + \alpha_2 a_{x_2}^{**}\right\| = \sup_{x \in \operatorname{ext} X_1} \left|\alpha_1 \chi_{\{x_1\}}(x) + \alpha_2 \chi_{\{x_2\}}(x)\right| = 1,$$

and hence $\left\|T^{**}(\alpha_1 a^{**}_{x_1} + \alpha_2 a^{**}_{x_2})\right\| \le \|T^{**}\| = \|T\|$. Thus it follows that

$$\|T\| > 2\frac{1-\varepsilon}{\left\|T^{-1}\right\|} > \frac{3}{2}\frac{1}{\left\|T^{-1}\right\|}.$$

This contradicts the assumption that $\|T\|\left\|T^{-1}\right\| < \frac{3}{2}$, and hence ρ_2 is a mapping.

Next we show that the mapping $\rho_2 : L_2 \to \operatorname{ext} X_1$ is surjective. To this end, let $x \in \operatorname{ext} X_1$ be given. We know that a^{**}_x is a strongly affine function on any ball $rB_{\mathfrak{A}(X_1)^*}$, $r > 0$. Thus, since $T^{**}a^{**}_x = a^{**}_x \circ T^*$ and T^* is an affine weak*-weak* homeomorphism, it follows that $T^{**}a^{**}_x$ is a strongly affine function on the compact convex set $B_{\mathfrak{A}(X_2)^*}$, see [19, Proposition 3.2]. Consequently, by (2) and since $\left\|a^{**}_x\right\| = 1$, we have

$$\frac{1-\varepsilon}{\left\|T^{-1}\right\|} < \frac{1}{\left\|T^{-1}\right\|} = \frac{1}{\left\|(T^{**})^{-1}\right\|} \le \left\|T^{**}a^{**}_x\right\| = \sup_{y\in\operatorname{ext} X_2}\left|\langle T^{**}a^{**}_x, \phi_2(y)\rangle\right|.$$

Thus there exists $y \in \operatorname{ext} X_2$ such that $\frac{1-\varepsilon}{\|T^{-1}\|} < \left|\langle T^{**}a^{**}_x, \phi_2(y)\rangle\right|$, that is, $\rho_2(y) = x$. Similarly we would show that $\rho_1 : L_1 \to \operatorname{ext} X_2$ is a surjective mapping.

Further, we show that $L_2 = \operatorname{ext} X_2$ and for each $y \in \operatorname{ext} X_2$, $\rho_2(y) \in L_1$ and $\rho_1(\rho_2(y)) = y$. To this end, for a fixed $y \in \operatorname{ext} X_2$, since the mapping ρ_1 is surjective, we find $x \in \operatorname{ext} X_1$ such that $\rho_1(x) = y$, and, since the mapping ρ_2 is surjective, we find $\tilde{y} \in \operatorname{ext} X_2$ satisfying that $\rho_2(\tilde{y}) = x$. We want to show that $\tilde{y} = y$. Thus we assume that $\tilde{y} \ne y$, and we find a boundary measure $\mu \in \mathcal{M}(X_1)$ satisfying that $R(\mu) = T^*(\phi_2(\tilde{y}))$ and $\left\|\mu\right\| = \left\|T^*(\phi_2(\tilde{y}))\right\| \le \|T\|$, see e.g. [10]. We write $\mu = \lambda\varepsilon_x + \nu$, where $\lambda \in \mathbb{R}$ and $\nu \in \mathcal{M}(X_1)$ satisfies $\nu(\{x\}) = 0$. Further, since $\rho_2(\tilde{y}) = x$, we have

$$\frac{1-\varepsilon}{\left\|T^{-1}\right\|} < \left|\langle T^{**}a^{**}_x, \phi_2(\tilde{y})\rangle\right| = \left|\langle a^{**}_x, T^*(\phi_2(\tilde{y}))\rangle\right| = \left|\langle a^{**}_x, R(\mu)\rangle\right| =^{(3)} \left|\mu(\{x\})\right| = |\lambda|.$$

Consequently,

$$\|\nu\| = \left\|\mu\right\| - |\lambda| < \|T\| - \frac{1-\varepsilon}{\left\|T^{-1}\right\|}.$$

Further, we have

$$\begin{aligned} 0 &= \chi_{\{y\}}(\tilde{y}) = \chi^*_{\{y\}}(\tilde{y}) = \langle b^{**}_y, \phi_2(\tilde{y})\rangle = \langle (T^{**})^{-1}b^{**}_y, T^*\phi_2(\tilde{y})\rangle = \\ &= \langle (T^{**})^{-1}b^{**}_y, R(\mu)\rangle = \langle (T^{**})^{-1}b^{**}_y, R(\lambda\varepsilon_x + \nu)\rangle = \langle (T^{**})^{-1}b^{**}_y, \lambda\phi_1(x) + R(\nu)\rangle, \end{aligned}$$

hence

$$\left|\langle (T^{**})^{-1}b^{**}_y, \lambda\phi_1(x)\rangle\right| = \left|\langle (T^{**})^{-1}b^{**}_y, R(\nu)\rangle\right|.$$

Thus, recalling that $\rho_1(x) = y$ we finally obtain that

$$\begin{aligned} &\frac{2}{3}(1-\varepsilon)^2 < \frac{1-\varepsilon}{\left\|T^{-1}\right\|}\cdot\frac{1-\varepsilon}{\|T\|} < |\lambda|\left|\langle (T^{**})^{-1}b^{**}_y, \phi_1(x)\rangle\right| = \\ &= \left|\langle (T^{**})^{-1}b^{**}_y, R(\nu)\rangle\right| \le \|\nu\|\left\|(T^{**})^{-1}b^{**}_y\right\| < (\|T\| - \frac{1-\varepsilon}{\left\|T^{-1}\right\|})\cdot\left\|T^{-1}\right\| = \\ &= \|T\|\left\|T^{-1}\right\| - (1-\varepsilon) < \frac{3}{2} - 2\varepsilon - (1-\varepsilon) = \frac{1}{2} - \varepsilon. \end{aligned}$$

However, it is easy to check that the inequality

$$\frac{2}{3}(1-\varepsilon)^2 < \frac{1}{2} - \varepsilon$$

does not hold for any $\varepsilon > 0$. This contradiction shows that $L_2 = \operatorname{ext} X_2$, and for each $y \in \operatorname{ext} X_2$, $\rho_2(y) \in L_1$ and $\rho_1(\rho_2(y)) = y$.

Moreover, since ρ_2 is surjective, it follows that $L_1 = \operatorname{ext} X_1$. Further, for a given $x \in \operatorname{ext} X_1$ we find $y \in \operatorname{ext} X_2$ satisfying that $\rho_2(y) = x$, and then

$$\rho_2(\rho_1(x)) = \rho_2(\rho_1(\rho_2(y))) = \rho_2(y) = x.$$

Thus the mappings ρ_1 and ρ_2 are mutual bijections.

It remains to show that the mappings ρ_1 and ρ_2 are measurable in accordance with the statements (i) and (ii). We prove the measurability of the mapping ρ_1, the proof for ρ_2 would be analogous. Thus we pick a zero set $F \subseteq \operatorname{ext} X_2$ and we consider the element b_F^{**} given by Lemma 4.2. We want to show that

$$\begin{aligned} \rho_1^{-1}(F) &= \{x \in \operatorname{ext} X_1 : \left|\langle (T^{**})^{-1} b_F^{**}, \phi_1(x)\rangle\right| > \frac{1}{2\,\|T\|}\} = \\ &= \{x \in \operatorname{ext} X_1 : \left|\langle (T^{**})^{-1} b_F^{**}, \phi_1(x)\rangle\right| \geq \frac{1-2\varepsilon}{2\,\|T\|}\}. \end{aligned} \tag{4}$$

We consider a fixed $x \in \rho_1^{-1}(F)$. Thus $y = \rho_1(x) \in F$. First we show that

$$\left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle\right| < \frac{1-2\varepsilon}{2\,\|T\|}.$$

To this end, we find $\alpha, \beta \in S_{\mathbb{R}}$ satisfying that

$$\left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle\right| = \alpha\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle$$

and

$$\left|\langle (T^{**})^{-1} b_y^{**}, \phi_1(x)\rangle\right| = \beta\langle (T^{**})^{-1} b_y^{**}, \phi_1(x)\rangle.$$

Then, since both the functions b_F^{**} and b_y^{**} are strongly affine on the compact convex set $B_{\mathfrak{A}(X_2)^*}$, so is the function $\alpha(b_F^{**} - b_y^{**}) + \beta b_y^{**}$. Thus, since $\operatorname{ext} B_{\mathfrak{A}(X_2)^*}$ is Lindelöf, by (2) we have

$$\left\|\alpha(b_F^{**} - b_y^{**}) + \beta b_y^{**}\right\| = \sup_{\tilde{y} \in \operatorname{ext} X_2} \left|\alpha \chi_{F\setminus\{y\}}(\tilde{y}) + \beta \chi_{\{y\}}(\tilde{y})\right| = 1.$$

Hence

$$\left|\langle (T^{**})^{-1}(\alpha(b_F^{**} - b_y^{**}) + \beta b_y^{**}), \phi_1(x)\rangle\right| \leq \left\|T^{-1}\right\|.$$

Thus if we assume that $\left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle\right| \geq \frac{1-2\varepsilon}{2\|T\|}$, we would have

$$\begin{aligned} &\langle (T^{**})^{-1}(\alpha(b_F^{**} - b_y^{**}) + \beta b_y^{**}), \phi_1(x)\rangle = \\ &= \alpha\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle + \beta\langle (T^{**})^{-1} b_y^{**}, \phi_1(x)\rangle = \\ &= \left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle\right| + \left|\langle (T^{**})^{-1} b_y^{**}, \phi_1(x)\rangle\right| > \frac{1-2\varepsilon}{2\,\|T\|} + \frac{1-\varepsilon}{\|T\|}. \end{aligned}$$

But this would yield

$$\left\|T^{-1}\right\| > \frac{1-2\varepsilon}{2\,\|T\|} + \frac{1-\varepsilon}{\|T\|},$$

that is, $\|T\| \left\|T^{-1}\right\| > \frac{3}{2} - 2\varepsilon$. This contradiction with the choice of ε shows that

$$\left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle\right| < \frac{1-2\varepsilon}{2\,\|T\|}.$$

Consequently, we have

$$\begin{aligned}&\left|\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle\right| = \left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}) + (T^{**})^{-1}b_y^{**}, \phi_1(x)\rangle\right| \geq \\ &\geq \left|\langle (T^{**})^{-1}b_y^{**}, \phi_1(x)\rangle\right| - \left|\langle (T^{**})^{-1}(b_F^{**} - b_y^{**}), \phi_1(x)\rangle\right| > \frac{1-\varepsilon}{\|T\|} - \frac{1-2\varepsilon}{2\,\|T\|} = \frac{1}{2\,\|T\|}.\end{aligned}$$

On the other hand, let $x \in \operatorname{ext} X_1 \setminus \rho_1^{-1}(F)$ be given. Thus $y = \rho_1(x) \notin F$. We want to show that

$$\left|\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle\right| < \frac{1-2\varepsilon}{2\,\|T\|}.$$

This we show similarly as above. So we find $\alpha, \beta \in S_{\mathbb{R}}$ satisfying that

$$\left|\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle\right| = \alpha\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle$$

and

$$\left|\langle (T^{**})^{-1}b_y^{**}, \phi_1(x)\rangle\right| = \beta\langle (T^{**})^{-1}b_y^{**}, \phi_1(x)\rangle.$$

Then as above, since the function $\alpha b_F^{**} + \beta b_y^{**}$ is strongly affine, by (2) we have

$$\left\|\alpha b_F^{**} + \beta b_y^{**}\right\| = \sup_{\tilde{y} \in \operatorname{ext} X_2} \left|\alpha \chi_F(\tilde{y}) + \beta \chi_{\{y\}}(\tilde{y})\right| = 1,$$

and thus

$$\left|\langle (T^{**})^{-1}(\alpha b_F^{**} + \beta b_y^{**}), \phi_1(x)\rangle\right| \leq \left\|T^{-1}\right\|.$$

Thus, assuming that $\left|\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle\right| \geq \frac{1-2\varepsilon}{2\|T\|}$, we would have

$$\begin{aligned}&\langle (T^{**})^{-1}(\alpha b_F^{**} + \beta b_y^{**}), \phi_1(x)\rangle = \\ &= \alpha\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle + \beta\langle (T^{**})^{-1}b_y^{**}, \phi_1(x)\rangle = \\ &= \left|\langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle\right| + \left|\langle (T^{**})^{-1}b_y^{**}, \phi_1(x)\rangle\right| \geq \frac{1-2\varepsilon}{2\,\|T\|} + \frac{1-\varepsilon}{\|T\|},\end{aligned}$$

which in the same way as above gives a contradiction with $\|T\|\left\|T^{-1}\right\| < \frac{3}{2} - 2\varepsilon$, and hence (4) holds.

Further, since $b_F^{**} \in \operatorname{Baf}_2(rB_{\mathfrak{A}(X_2)^*}, \mathbb{R})$ for each $r > 0$, it follows that the function $(T^{**})^{-1}b_F^{**} = b_F^{**} \circ T^*$ belongs to $\operatorname{Baf}_2(B_{\mathfrak{A}(X_1)^*}, \mathbb{R})$. Hence $(T^{**})^{-1}b_F^{**}$ is $\Sigma_3(\operatorname{Bas}(B_{\mathfrak{A}(X_1)^*}))$-measurable by Proposition 2.1. Thus, since the mapping ϕ_1 is a homeomorphic embedding and the system $\Sigma_3(\operatorname{Bas}(\operatorname{ext} X_1))$ is closed with respect to finite unions, the set

$$\begin{aligned}\rho_1^{-1}(F) = &\{x \in \operatorname{ext} X_1 : \langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle > \frac{1}{2\,\|T\|}\} \\ &\cup \{x \in \operatorname{ext} X_1 : \langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle < -\frac{1}{2\,\|T\|}\}\end{aligned}$$

belongs to $\Sigma_3(\operatorname{Bas}(\operatorname{ext} X_1))$. Similarly from the equality

$$\begin{aligned}\rho_1^{-1}(F) = &\{x \in \operatorname{ext} X_1 : \langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle \geq \frac{1-2\varepsilon}{2\,\|T\|}\} \\ &\cup \{x \in \operatorname{ext} X_1 : \langle (T^{**})^{-1}b_F^{**}, \phi_1(x)\rangle \leq -\frac{1-2\varepsilon}{2\,\|T\|}\}\end{aligned}$$

we obtain that $\rho_1^{-1}(F) \in \Pi_3(\operatorname{Bas}(\operatorname{ext} X_1))$, and hence $\rho_1^{-1}(F) \in \Delta_3(\operatorname{Bas}(\operatorname{ext} X_1))$.

If, moreover, the set $\operatorname{ext} X_2$ is resolvable, then $(T^{**})^{-1}b_F^{**} \in \operatorname{Baf}_1(B_{\mathfrak{A}(X_1)^*}, \mathbb{R})$ by Lemma 4.2, and hence as above we see that $\rho_1^{-1}(F) \in \Delta_2(\operatorname{Bas}(\operatorname{ext} X_1))$. This finishes the proof. □

5. Examples

In this final section we discuss the optimality of our results. To start with, we do not know whether the constant $\frac{3}{2}$, appearing in Theorem 1.1, can be replaced by 2, which could be expected in view of the Amir-Cambern theorem and related results for spaces of affine functions. We note that in our proof, this better constant is needed only for the measurability of the constructed mappings ρ_1, ρ_2, the mutual bijectivity of these mappings could be proven also with the constant 2.

On the other hand, it turns out that the topological assumption of Lindelöf sets of extreme points is necessary. To prove this, we modify the example of Hess [9].

Example 5.1. *For each $\varepsilon > 0$ there exist simplices X_1, X_2 such that there exists an isomorphism $T : \mathfrak{A}(X_1) \to \mathfrak{A}(X_2)$ with $\|T\| \left\|T^{-1}\right\| \leq 1 + \varepsilon$, but there is no Baire bimeasurable bijection of the sets $\operatorname{ext} X_1$ and $\operatorname{ext} X_2$.*

Proof. Let $\varepsilon > 0$ be given. We consider the function space (see [14, Definition 3.1])

$$\mathcal{H}_1 = \{f \in C([0, \omega_1]) : f(\omega_1) = \frac{2}{2+\varepsilon} f(0) + \frac{\varepsilon}{2+\varepsilon} f(1)\},$$

and let $\mathcal{H}_2 = C([0, \omega_1])$. Then $\mathcal{M}^1([0, \omega_1])$ is a simplex with $\operatorname{ext} \mathcal{M}^1([0, \omega_1])$ homeomorphic to $[0, \omega_1]$, see e.g. [1, Corollary II.4.2], and moreover, $\mathcal{H}_2$ is isometric to the space $\mathfrak{A}(\mathcal{M}^1([0, \omega_1]))$, see e.g. [1, Corollary II.3.13]. Further, we show that the state space

$$S(\mathcal{H}_1) = \{s^* \in \mathcal{H}_1^* : s^* \geq 0, s^*(1) = 1\}$$

is a simplex. To start with, it is standard to check that $\operatorname{ext} S(\mathcal{H}_1)$ is homeomorphic to the interval $[0, \omega_1)$.

Moreover, it is easy to see that if a nonzero measure $\mu \in \mathcal{M}([0, \omega_1])$ annihilates $\mathcal{H}_1$ then it is a nonzero multiple of

$$\varepsilon_{\omega_1} - \frac{2}{2+\varepsilon} \varepsilon_0 - \frac{\varepsilon}{2+\varepsilon} \varepsilon_1,$$

so it is not carried by $[1, \omega_1)$. Thus $S(\mathcal{H}_1)$ is a simplex (see [14, Propositions 6.9. and 3.66.]). Moreover, the space $\mathcal{H}_1$ can be identified in a standard way with $\mathfrak{A}(S(\mathcal{H}_1))$, see [14, Proposition 4.26.].

To proceed further, we define $T : \mathcal{H}_1 \to \mathcal{H}_2$ for $\alpha \in [0, \omega_1]$ as

$$Tf(\alpha) = \begin{cases} f(\alpha + 1), & \alpha \in [0, \omega), \\ f(\alpha), & \text{otherwise.} \end{cases}$$

Then T is injective, since if $Tf = 0$ then $f(\alpha) = 0$ for each $\alpha \geq 1$, and $f(0) = \frac{2+\varepsilon}{2}(f(\omega_1) - \frac{\varepsilon}{2+\varepsilon} f(1)) = 0$. Moreover, T is surjective, since T^{-1} can be written for $g \in \mathcal{H}_2$ as

$$T^{-1}g(\alpha) = \begin{cases} \frac{2+\varepsilon}{2}(g(\omega_1) - \frac{\varepsilon}{2+\varepsilon} g(0)), & \alpha = 0, \\ g(\alpha - 1), & \alpha \in [1, \omega), \\ g(\alpha), & \text{otherwise.} \end{cases}$$

Further, it is clear that $\|T\| \leq 1$, and for $g \in \mathcal{H}_2$,

$$\left\|T^{-1}g\right\| \leq \max\{\frac{2+\varepsilon}{2}(g(\omega_1) - \frac{\varepsilon}{2+\varepsilon} g(0)), \left\|g\right\|\} \leq \left\|g\right\| (\frac{2+\varepsilon}{2})(1 + \frac{\varepsilon}{2+\varepsilon}) = \left\|g\right\| (1 + \varepsilon).$$

Finally, it is easy to see that each singleton in $[0, \omega_1)$ is a zero set, but it is known that $\{\omega_1\}$ is not a Baire set in $[0, \omega_1]$. (To see this, assume for a contradiction that $\{\omega_1\}$, and hence also $[0, \omega_1)$, is a Baire set in $[0, \omega_1]$. But then $[0, \omega_1)$ would be K-analytic, see [14, Definition A.110 and Theorem A.111(i)], and hence Lindelöf, see [14, Theorem A.111(d)], which is clearly not true.)

Thus it follows that there exists no Baire bimeasurable bijection $\rho : [0, \omega_1) \to [0, \omega_1]$, because otherwise the singleton $\rho^{-1}(\omega_1)$ would not be a Baire set, which is not possible. The proof is finished. □

Finally, it is not clear whether in Theorem 1.1, it is possible to prove that the constructed mappings ρ_i are measurable also with respect to the Borel hierarchy of sets instead of the Baire hierarchy. While we do not know the solution to this problem, we show that it cannot be answered in positive using our methods, which rely on the solution of the abstract Dirichlet problem. To this end, imitating standard constructions, see e.g. [4, Section VII], [1, Proposition I.4.15], [3, Theorem 3.2.4] or [13], we present an example of the characteristic function of a closed subset of a simplex with Lindelöf boundary for which the solution of the abstract Dirichlet problem does not exist.

Example 5.2. *There exists a simplex X with Lindelöf set of extreme points and a closed set $F \subseteq \operatorname{ext} X$ such that χ_F cannot be extended to an affine Borel function on X.*

Proof. Let $B \subseteq [0,1]$ be a Bernstein set, i.e., a set which intersects each nonempty perfect subset of $[0,1]$ but contains no such set (see [15, Theorem 5.3]), and let

$$K = (B \times \{-1,1\}) \cup ([0,1] \times \{0\})$$

be endowed with the "porcupine" topology, see [4, Section VII]. Precisely, let $B\times\{-1\}$ and $B\times\{1\}$ be endowed with the discrete topology, and let a point $x \in [0,1] \times \{0\}$ has a basis of neighbourhoods consisting of the sets of the form

$$(U \times \{0\}) \cup (((B \cap U) \setminus \bigcup_{i=1}^{n} \{x_i\}) \times \{-1\}) \cup (((B \cap U) \setminus \bigcup_{i=1}^{k} \{y_i\}) \times \{1\}),$$

where $k, n \in \mathbb{N}$, $U \subseteq [0,1]$ is an euclidean open neighbourhood of x, and
$x_1, \dots x_n, y_1, \dots y_k \in U$. Then it is easy to check that K is a compact Hausdorff space.

Further, we consider the function space

$$\mathcal{H} = \{f \in C(K) : f(x,0) = \frac{1}{2} f(x,-1) + \frac{1}{2} f(x,1), x \in B\}.$$

Then $\mathcal{H}$ is an example of the so-called Stacey function space, see [14, Definition 6.13]. Thus it follows from [14, Lemma 6.14, Theorem 6.54 and Proposition 4.26] that its state space

$$X = \{s^* \in \mathcal{H}^* : s^* \geq 0, s^*(1) = 1\}$$

is a simplex, $\mathcal{H}$ is isometric to $\mathfrak{A}(X)$, and $\operatorname{ext} X$ is homeomorphic to

$$(B \times \{-1,1\}) \cup ([0,1] \setminus B) \times \{0\}.$$

We check that this set is Lindelöf. Thus we consider an arbitrary open covering $\mathcal{U}$ of this set. Then, since $([0,1] \setminus B) \times \{0\}$ is a separable metric space, it is Lindelöf, hence we may extract from $\mathcal{U}$ a countable family $\{U_n\}_{n\in\mathbb{N}}$ covering the set $([0,1] \setminus B) \times \{0\}$. Then the set

$$(B \times \{0\}) \setminus \bigcup_{n=1}^{\infty} U_n = ([0,1] \times \{0\}) \setminus \bigcup_{n=1}^{\infty} U_n$$

is at most countable. Indeed, if this set were uncountable, then, since it is closed (in the euclidean topology on $[0,1] \times \{0\}$), it would contain a perfect set, which is not possible due to the fact that B is a Bernstein set. Thus it follows from the definition of the topology of K that also the set $(B \times \{-1\}) \setminus \bigcup_{n\in\mathbb{N}} U_n$ is at most countable, and hence we may extract an another countable subfamily $\{V_n\}_{n\in\mathbb{N}}$ from $\mathcal{U}$ such that $\bigcup_{n\in\mathbb{N}}(U_n \cup V_n)$ covers $(([0,1] \setminus B) \times \{0\}) \cup (B \times \{-1\})$. Similarly we can cover the set $B \times \{1\}$, and it follows that $\operatorname{ext} X$ is Lindelöf.

Further, the set $F = ([0,1] \setminus B) \times \{0\}$ is closed in $\operatorname{ext} X$. Finally, if $f : X \to \mathbb{R}$ is an affine function extending χ_F, then $f(x,0) = 1$ for $x \in [0,1] \setminus B$, and, since f is affine, $f(x,0) = 0$ for $x \in B$. Thus, since B is a Bernstein set, f is not Borel, which finishes the proof. $\square$

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