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Lupaş Bernstein-Kantorovich Operators Using Jackson and Riemann Type (p, q)-Integrals

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Abstract. In this paper, Lupaş Bernstein-Kantorovich operators have been studied using Jackson and Riemann type (p, q)-integrals. It has been shown that (p, q)-integrals as well as Riemann type (p, q)-integrals are not well defined for $0 < q < p < 1$ and thus further analysis is needed. Throughout the paper, the case $1 \leq q < p < \infty$ has been used. Advantages of using Riemann type (p, q)-integrals are discussed over general (p, q)-integrals. Lupaş Bernstein-Kantorovich operators constructed via Jackson integral need not be positive for every $f \geq 0$. So to make these operators based on general (p, q)-integral positive, one need to consider strictly monotonically increasing functions, and to handle this situation Lupaş Bernstein-Kantorovich operators are constructed using Riemann type (p, q)-integrals. However Lupaş (p, q)-Bernstein-Kantorovich operators based on Riemann type (p, q)-integrals are always positive linear operators. Approximation properties for these operators based on Korovkin's type approximation theorem are investigated. The rate of convergence via modulus of continuity and function f belonging to the Lipschitz class is computed.

1. Introduction and preliminaries

In 1912, S.N. Bernstein [4] introduced the famous Bernstein polynomial(operator) for any bounded function $f : [0, 1] \to R$ as follows

$$B_r(f; z) = \sum_{j=0}^{r} \left(\begin{array}{c} r \\ j \end{array} \right) z^j (1 - z)^{r-j} f\left(\frac{j}{r}\right), \quad z \in [0, 1] \tag{1}$$

and proved the sequence of operators $B_r : C[0, 1] \to C[0, 1]$ for any $r \in N$ and $f \in C[0, 1]$ converges uniformely to f on [0,1] [8].

Further, based on q-Calculus, Lupaş [14] in 1987 proposed the first q-Bernstein operators (rational) [4]. After that, in 1996, Phillips introduced another q-operator (polynomials) [25] to study approximation properties via positive linear operators.

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Recently, in the field of Approximation Theory [23] and Computer Aided Geometric Design (CAGD) [13] the applications of (p,q)-calculus emerged as a new area . To mimic the shape of curves and surfaces better one needs parameters which can provide flexibility. In this sequel, applications of post quantum calculus plays an important role in CAGD, see [13]. The (p,q)-calculus development has led to the discovery of various generalizations of Bernstein polynomials based on (p,q)-integers.

It all started when Mursaleen et al [23] introduced (p,q)-calculus in approximation theory and constructed the (p,q)-analogue of Bernstein operators (extension of Bernstein Phillips polynomials) for $0 < q < p \leq 1$.

Khalid and Lobiyal [13] recently defined post quantum analogue of Lupaş Bernstein operators (an extension of q-analogue of Lupaş Bernstein operators (rational) [14]) as follows:

For any $p > 0$ and $q > 0$, the linear operators $L^r_{p,q} : C[0,1] \to C[0,1]$

$$L^r_{p,q}(f;z) = \sum_{j=0}^{r} \frac{f\left(\frac{p^{r-j}\,[j]_{p,q}}{[r]_{p,q}}\right) \left[\begin{array}{c} r \\ j \end{array}\right]_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} z^j \,(1-z)^{r-j}}{\prod_{j=1}^{r} \{p^{j-1}(1-z) + q^{j-1}z\}}, \tag{2}$$

is (p,q)-analogue of Lupaş Bernstein operators.

Lupaş q-Bernstein operators [14] can be obtained from Lupaş (p,q)-Bernstein operators on substituting $p = 1$. Similarly, classical Bernstein operators [4] can be deduced from Lupaş (p,q)-Bernstein operators by substituting $p = q = 1$.

Some other advantages of using the extra parameter p have been discussed in the field of approximations on compact disk [20] and in CAGD [13].

For classical approximation theory related to positive linear operators, see [4, 8] and for quantum calculus, refer [18, 25, 29]. Acar et.al. investigated approximation properties by constructing some operators via post quantum calculus [2]. Cai et. al. investigated approximation properties by constructing post quantum analogue of Lambda-Bernstein operators [6] and Kantorovich type Bernstein-Stancu-Schurer operators [5]. Kadak et al studied (p,q)-Szasz operators involving Brenke type polynomials [12]. Wafi and Rao studied approximation properties by (p,q)-Bivariate-Bernstein-Chlowdosky operators and (p,q) variants of stancu schurer operators [26, 28]. Also Mishra and Pandey investigated properties of Chlowdosky variant of (p,q) Kantrovich-Stancu-Schurer operators [17] and Milovanovic et. al. Dunkl generalization of Szász-Kantorovich operators [16]. Kantrovich variants of several operators in q-calculus are studied in [19], [21] and [22].

Let us recall some definitions and notations of (p,q)-calculus:

(p,q)-integers $[r]_{p,q}$ for any $p > 0$ and $q > 0$, are defined by

$$[r]_{p,q} = p^{r-1} + p^{r-2}q + p^{r-3}q^2 + ... + pq^{r-2} + q^{r-1} = \begin{cases} \frac{p^r - q^r}{p-q}, & \text{when } p \neq q \neq 1 \\ r\,p^{r-1}, & \text{when } p = q \neq 1 \\ q, & \text{when } p = 1 \\ r, & \text{when } p = q = 1 \end{cases}$$

where $[r]_q$ is q-integers for $r = 0, 1, 2, \cdots$. The (p,q)-Binomial expansion is

$$(z+w)^r_{p,q} := (z+w)(pz+qw)(p^2z+q^2w)\cdots(p^{r-1}z+q^{r-1}w)$$

and the (p,q)-analogue of Binomial coefficients are defined as

$$\left[\begin{array}{c} r \\ j \end{array}\right]_{p,q} := \frac{[r]_{p,q}!}{[j]_{p,q}![r-j]_{p,q}!}.$$

The first general (p,q)-definite integrals in [10] of the function f are defined as

$$\int_0^c f(z)d_{p,q}z = (p-q)c\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}c\right), \quad \text{when} \quad \left|\frac{q}{p}\right| < 1. \tag{3}$$

It is easy to note that for range of p and q satisfying $0 < q < p < 1$, integration (3) is not well defined. When $0 < p < 1$, then $\frac{c}{p} > c$ for $c > 0$. For $j = 0$ in (3), input of f is $\frac{c}{p}$, but $\frac{c}{p} \notin [0,c]$. But function may not be defined outside the interval $[0,c]$, so integration (3) is not well defined.

Example 1.1. *Let us consider function*

$$f(z) = \frac{1}{z-3c}, \quad z \in [0,c].$$

Take $p = \frac{1}{3}$ and $q = \frac{1}{4}$. This implies that $\frac{c}{p} = 3c$. While opening the series for $j = 0$, term $f(3c)$ appears. But given function is not defined at $z = 3c$. So integration defined by 3 is not well defined here.

Therefore we consider the case $1 \leq q < p < \infty$ throughout the paper. Generally accepted definition for (p,q)-integral (4) over $[c,d]$ is defined as

$$\int_c^d f(z)d_{p,q}z = \int_0^d f(z)d_{p,q}z - \int_0^c f(z)d_{p,q}z. \tag{4}$$

For (p,q)-calculus details, one can refer [10, 11].
All the notions of q-calculus can be re-obtained from (p,q)-calculus on putting $p = 1$ [31].

Bernstein-Kantorovich operators using q-calculus are introduced by Dalmanoglu [7] as follows:

$$K_{r,q}(f;z) = [r+1]_q\sum_{j=0}^{r}p_{r,j}(q;z)\int_{[j]_q/\ [r+1]_q}^{[j+1]_q/[r+1]_q} f(t)d_qt, \quad z \in [0,1], \tag{5}$$

$$p_{r,j}(q;z) := \begin{bmatrix} r \\ j \end{bmatrix}_q z^j\prod_{s=0}^{r-j-1}(1-q^sz).$$

where $K_{r,q} : C[0,1] \to C[0,1]$ are defined for any function $f \in C[0,1]$ and for any $r \in N$.

For details about classical Kantorovich operators and solutions, one can refer [30, 32].
Motivated by above mentioned work, in section 2, we recall Lupaş (p,q)-Bernstein-Kantorovich operators based on Jackson integral and Riemann type (p,q)-integral from [24] and will study its approximation properties over $[0,1]$. Advantages of using Riemann type (p,q)-integrals are discussed over general (p,q)-integrals. Approximation properties for these operators are studied via Korovkin's type approximation theorem. The order of approximation using usual modulus of continuity and also the rate of convergence for the function f belonging to the class $\mathrm{Lip}_M(\alpha)$ are computed.

2. Construction of Operators

Lupaş (p,q)-Bernstein-Kantorovich operators were first constructed in 2017 in [24]. We recall these operators here as follows.

$$A_r^{(p,q)}(f;z) = [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}} f(t)d_{p,q}t, \quad z \in [0,1]. \tag{6}$$

where

$$B_{p,q}^{j,r}(z) = \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} z^j (1-z)^{r-j}}{\prod_{j=1}^{r} \{p^{j-1}(1-z) + q^{j-1}z\}}.$$

Here $B_{p,q}^{0,r}(z), B_{p,q}^{1,r}(z), \cdots, B_{p,q}^{r,r}(z)$ are the (p,q)-analogue of the Lupaş q-Bernstein rational functions [14] of degree r on the interval $[0,1]$.

We further analyze these operators for approximation. Before this, we would like to recall some basics regarding general (p,q)-integral (3) for $f(z) = 1$, $f(z) = z$, $f(z) = z^2$. These calculations, we are going to use in proving Lemma (2.1).

1. $f(z) = 1$,

$$\begin{aligned}
\int_c^d 1 d_{p,q}z &= \int_0^d 1 d_{p,q}z - \int_0^c 1 d_{p,q}z \\
&= (q-p).d \sum_{j=0}^{\infty} \frac{p^j}{q^{j+1}} - (q-p)c \sum_{j=0}^{\infty} \frac{p^j}{q^{j+1}} \\
&= (q-p).\frac{d}{q} \sum_{j=0}^{\infty} \left(\frac{p}{q}\right)^j - (q-p).\frac{c}{q} \sum_{j=0}^{\infty} \left(\frac{p}{q}\right)^j \\
&= (q-p).\frac{d}{q}\left(\frac{q}{q-p}\right) - (q-p).\frac{c}{q}\left(\frac{q}{q-p}\right) \\
&= d - c
\end{aligned}$$

$$\int_c^d 1 d_{p,q}z = d - c.$$

2. $f(z) = z$

$$\begin{aligned}
\int_c^d z d_{p,q}z &= \int_0^d z d_{p,q}z - \int_0^c z d_{p,q}z \\
&= (q-p).d \sum_{j=0}^{\infty} \frac{p^j}{q^{j+1}}\left(\frac{p^j}{q^{j+1}}.d\right) - (q-p)c \sum_{j=0}^{\infty} \frac{p^j}{q^{j+1}}\left(\frac{p^j}{q^{j+1}}.c\right) \\
&= (q-p)(d^2 - c^2) \sum_{j=0}^{\infty} \frac{p^2}{q^{2(j+1)}} \\
&= (q-p)\frac{(d^2 - c^2)}{q^2} \sum_{j=0}^{\infty} \left(\frac{p^2}{q^2}\right)^j
\end{aligned}$$

$$\begin{aligned}
\int_c^d z d_{p,q}z &= (q-p)\frac{(d^2 - c^2)}{q^2} \frac{q^2}{q^2 - p^2} \\
&= \frac{(d^2 - c^2)}{q+p}
\end{aligned}$$

$$\int_c^d z d_{p,q}z = \frac{(d^2 - c^2)}{q+p}.$$

3. $f(z) = z^2$

$$\begin{aligned}
\int_c^d z^2 d_{p,q}z &= \int_0^d z^2 d_{p,q}z - \int_0^c z^2 d_{p,q}z \\
&= (q-p).d\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}\Big(\frac{p^j}{q^{j+1}}.d\Big)^2 - (q-p)c\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}\Big(\frac{p^j}{q^{j+1}}.c\Big)^2 \\
&= (q-p).d^3\sum_{j=0}^{\infty}\frac{p^{3j}}{q^{3(j+1)}} - (q-p).c^3\sum_{j=0}^{\infty}\frac{p^{3j}}{q^{3(j+1)}} \\
&= (q-p)(d^3-c^3)\sum_{j=0}^{\infty}\frac{p^3}{q^{3(j+1)}} \\
&= (q-p)\frac{(d^3-c^3)}{q^3}\sum_{j=0}^{\infty}\Big(\frac{p^3}{q^3}\Big)^j \\
&= (q-p)\frac{(d^3-c^3)}{q^3}\frac{q^3}{q^3-p^3} \\
&= \frac{(d^3-c^3)}{q^2+qp+p^2} \\
\int_c^d z^2 d_{p,q}z &= \frac{(d^3-c^3)}{q^2+qp+p^2}.
\end{aligned}$$

By simple computation, we have the following basic lemmas based on integral given by (3) for the operator (6).

Lemma 2.1. *For $1 \leq q < p < \infty,\ z \in [0,1]$*

(i) $A_r^{(p,q)}(1;z) = 1$,

(ii) $A_r^{(p,q)}(t;z) = z + \frac{p^r}{[2]_{p,q}[r]_{p,q}}$,

(iii) $A_r^{(p,q)}(t^2;z) = \frac{q^2}{[r]_{p,q}}\frac{[r-1]_{p,q}}{\{p(1-z)+qz\}}z^2 + \Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}} + \frac{p^{r-1}}{[r]_{p,q}}\Big)z + \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}}$,

(iv) $A_r^{(p,q)}\Big((t-z)^2;z\Big) = \Big(\frac{q^2}{[r]_{p,q}}\frac{[r-1]_{p,q}}{\{p(1-z)+qz\}} - 1\Big)z^2 + \Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}} + \frac{p^{r-1}}{[r]_{p,q}} - \frac{2p^r}{[2]_{p,q}[r]_{p,q}}\Big)z + \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}}$.

Proof. (i)

$$A_r^{(p,q)}(1;z) = [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} 1\ d_{p,q}t \ = 1.$$

(ii)

$$\begin{aligned}
A_r^{(p,q)}(t;z) &= [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} t\ d_{p,q}t \\
&= \frac{1}{[2]_{p,q}[r]_{p,q}}\sum_{j=0}^{r}\frac{B_{r,j}^{(p,q)}(z)}{p^{r-j}q^j}\Big(\frac{[j+1]^2_{p,q}-p^2[j]^2_{p,q}}{p^{2j-2r}}\Big).
\end{aligned}$$

On substituting $[j+1]_{p,q} = p^j + q[j]_{p,q}$, we have

$$\begin{aligned}
A_r^{(p,q)}(t;z) &= \frac{1}{[2]_{p,q}[r]_{p,q}} \sum_{j=0}^{r} \frac{B_{r,j}^{(p,q)}(z)(p^j + [2]_{p,q}[j]_{p,q})}{p^{j-r}} \\
&= \frac{p^r}{[2]_{p,q}[r]_{p,q}} \Big(\sum_{j=0}^{r} B_{r,j}^{(p,q)}(z) + [2]_{p,q} \sum_{j=0}^{r} B_{r,j}^{(p,q)}(z) \frac{[j]_{p,q}}{p^j} \Big) \\
&= \frac{p^r}{[2]_{p,q}[r]_{p,q}} \Big(1 + [2]_{p,q} \sum_{j=0}^{r} \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} z^j (1-z)^{r-j}}{\prod_{j=1}^{r} \{p^{j-1}(1-z) + q^{j-1}z\}} \frac{[j]_{p,q}}{p^j} \Big) \\
&= \frac{p^r}{[2]_{p,q}[r]_{p,q}} + \sum_{j=0}^{r-1} p^{r-j-1} \frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j+1)}{2}} z^{j+1} (1-z)^{r-j-1}}{\prod_{j=1}^{n} \{p^{j-1}(1-z) + q^{j-1}z\}} \\
&= \frac{p^r}{[2]_{p,q}[r]_{p,q}} + \frac{z}{1-z} \sum_{j=0}^{r-1} p^{r-j-1} \frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j-1)}{2}} \left(\frac{qz}{1-z}\right)^j}{\prod_{j=1}^{r} \{p^{j-1} + q^{j-1}\frac{z}{1-z}\}} \\
&= \frac{p^r}{[2]_{p,q}[r]_{p,q}} + \frac{u}{1+u} \sum_{j=0}^{r-1} \frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j-1)}{2}} \left(\frac{qu}{p}\right)^j}{\prod_{j=0}^{r-2} \{p^j + q^j\left(\frac{qu}{p}\right)\}}, \ where\ \ u = \frac{z}{1-z} \\
&= \frac{p^r}{[2]_{p,q}[r]_{p,q}} + z.
\end{aligned}$$

(iii)

$$\begin{aligned}
A_r^{(p,q)}(t^2;z) &= [r]_{p,q} \sum_{j=0}^{r} \frac{B_{r,j}^{(p,q)}(z)}{p^{r-j}q^j} \int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}} t^2 \, d_{p,q}t \\
&= \frac{1}{[3]_{p,q}[r]_{p,q}^2} \sum_{j=0}^{r} \frac{B_{r,j}^{(p,q)}(z)}{p^{r-j}q^j} \Big(\frac{[j+1]_{p,q}^3 - p^3[j]_{p,q}^3}{p^{3j-3r}} \Big) \\
&= \frac{p^{2r}}{[3]_{p,q}[r]_{p,q}^2} \sum_{j=0}^{r} B_{r,j}^{(p,q)}(z) \Big(1 + (2q+p)\frac{[j]_{p,q}}{p^j} + \frac{[3]_{p,q}[j]_{p,q}^2}{p^{2j}} \Big) \\
&= \frac{p^{2r}}{[3]_{p,q}[r]_{p,q}^2} \Big(1 + (2q+p) \sum_{j=0}^{r} B_{r,j}^{(p,q)}(z) \frac{[j]_{p,q}}{p^j} + \ [3]_{p,q} \sum_{j=0}^{r} B_{r,j}^{(p,q)}(z) \frac{[j]_{p,q}^2}{p^{2j}} \Big).
\end{aligned}$$

With the help of the previous calculations, we have

$$\frac{p^r}{[r]_{p,q}} \sum_{j=0}^{r} B_{r,j}^{(p,q)}(z) \frac{[j]_{p,q}}{p^j} = z,$$

and

$$\frac{p^{2r}}{[r]_{p,q}^2}\sum_{j=0}^{r}B_{r,j}^{(p,q)}(z)\frac{[j]_{p,q}^2}{p^{2j}} = \frac{p^{2r}}{[r]_{p,q}^2}\sum_{j=0}^{r}\frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}}\, z^j\,(1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}\frac{[j]_{p,q}^2}{p^{2j}}$$

$$= \frac{p^{2r}}{[r]_{p,q}}\sum_{j=0}^{r-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j+1)}{2}}\, z^{j+1}\,(1-z)^{r-j-1}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}\times\frac{[j+1]_{p,q}}{p^{2j+2}}.$$

On substituting $[j+1]_{p,q}=p^j+q[j]_{p,q}$, we get

$$\frac{p^{2r}}{[r]_{p,q}^2}\sum_{j=0}^{r}B_{r,j}^{(p,q)}(z)\frac{[j]_{p,q}^2}{p^{2j}}$$

$$= \frac{p^{2r}}{[r]_{p,q}}\sum_{j=0}^{r-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j+1)}{2}}\, z^{j+1}\,(1-z)^{r-j-1}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}\Big(\frac{1}{p^{j+2}}+\frac{q[j]_{p,q}}{p^{2j+2}}\Big)$$

$$= \frac{p^{2r-2}}{[r]_{p,q}}\frac{z}{1-z}\sum_{j=0}^{r-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j-1)}{2}}\Big(\frac{qz}{p(1-z)}\Big)^j}{\prod_{j=0}^{r-1}\{p^j+q^j\Big(\frac{z}{1-z}\Big)\}}$$

$$+\frac{q^2p^{2r-4}[r-1]_{p,q}}{[r]_{p,q}}\Big(\frac{z}{1-z}\Big)^2\sum_{j=0}^{r-2}\frac{\begin{bmatrix} r-2 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-2)(r-j-3)}{2}} q^{\frac{j(j-1)}{2}}\Big(\frac{q^2z}{p^2(1-z)}\Big)^j}{\prod_{j=0}^{r-1}\{p^j+q^j\Big(\frac{z}{1-z}\Big)\}}$$

$$= \frac{p^{r-1}}{[r]_{p,q}}\frac{u}{1+u}\sum_{j=0}^{r-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j-1)}{2}}\Big(\frac{qu}{p}\Big)^j}{\prod_{j=0}^{r-2}\{p^j+q^j\Big(\frac{qu}{p}\Big)\}}$$

$$+\frac{q^2[r-1]_{p,q}}{[r]_{p,q}}\frac{u^2}{(1+u)(p+qu)}\sum_{j=0}^{r-2}\frac{\begin{bmatrix} n-2 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-2)(r-j-3)}{2}} q^{\frac{j(j-1)}{2}}\Big(\frac{q^2u}{p^2}\Big)^j}{\prod_{j=0}^{r-3}\{p^j+q^j\Big(\frac{q^2u}{p^2}\Big)\}}\qquad \textit{where } u=\frac{z}{1-z}$$

$$= \frac{p^{r-1}}{[r]_{p,q}}\frac{u}{1+u}+\frac{q^2[r-1]_{p,q}}{[r]_{p,q}}\Big(\frac{u^2}{(1+u)(p+qu)}\Big)$$

$$= \frac{p^{r-1}}{[r]_{p,q}}z+\frac{q^2[r-1]_{p,q}}{[r]_{p,q}}\Big(\frac{z^2}{(1-z)p+qz}\Big).$$

Using the above equalities, we have

$$A_r^{(p,q)}(t^2;z)=q^2\frac{[r-1]_{p,q}}{[r]_{p,q}}\Big(\frac{z^2}{p(1-z)+qz}\Big)+\Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}}+\frac{p^{r-1}}{[r]_{p,q}}\Big)z+\frac{p^{2r}}{[3]_{p,q}[r]_{p,q}^2}.$$

(iv) As the operators $A_r^{(p,q)}$ are linear, we have

$$\begin{aligned} &A_r^{(p,q)}\Big((t-z)^2;z\Big)\\ &= A_r^{(p,q)}(t^2;z) - 2zA_r^{(p,q)}(t;z) + z^2A_r^{(p,q)}(1;z)\\ &= \ q^2\frac{[r-1]_{p,q}}{[r]_{p,q}}\Big(\frac{z^2}{p(1-z)+qz}\Big) + \Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}} + \frac{p^{r-1}}{[r]_{p,q}}\Big)z + \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}} - 2z\Big(z + \frac{p^r}{[2]_{p,q}[r]_{p,q}}\Big)\\ &+ z^2\\ &= \Big(\ q^2\frac{[r-1]_{p,q}}{[r]_{p,q}(p(1-z)+qz)} - 1\Big)z^2 + \Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}} + \frac{p^{r-1}}{[r]_{p,q}} - \frac{2p^r}{[2]_{p,q}[r]_{p,q}}\Big)z + \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}}. \end{aligned}$$

Remark 2.2. *As in usual definite integration, if f is defined and integrable on $[c,d]$ and $f \geq 0$ then $\int_c^d f(z)dz \geq 0$. But integration defined by (3) and (4) may not carry this property.*

Example 2.3. *Consider a function $f(z) = z-1,\ \ z \in [1,2]$. f is monotonically increasing on $[1,2]$ and $f \geq 0$ on $[1,2]$. Take $q = 10$ and $p = 11$. Then $\int_1^2 f(z)d_{p,q}z = \int_1^2 zd_{p,q}z - \int_1^2 1d_{p,q}z$. After little bit calculation using definition 3 and 4, we have $\int_c^d 1d_{p,q}z = (d-c)$ and $\int_c^d zd_{p,q}z = \frac{(d^2-c^2)}{q+p}$*
Hence $\int_1^2 f(z)d_{p,q}z = \frac{3}{21} - 1 = -\frac{6}{7}$.

In above example f is not non-negative on $[0,2]$. Authors in [1] mentioned that if we take f to be non decreasing then integration is positive on $[c,d]$. Here we give example which claims that by taking $f \geq 0$ and monotonic increasing on interval $[0,d]$, still integration need not be positive on $[0,d]$ and its subinterval.

Example 2.4.

$$f(z) = \begin{cases} 0, & \text{if } z \in [0,1/2] \\ z-1/2, & \text{if } z \in [1/2,1] \end{cases}$$

Take $q = 2$ and $p = 3$. Clearly $f \geq 0$ and nonotonically increasing on $[0,1]$. After some simple calculation, we have $\int_0^1 f(z)d_{p,q}z = \int_{\frac{1}{2}}^1 (z-\frac{1}{2})d_{p,q}z = \int_{\frac{1}{2}}^1 zd_{p,q}z - \int_{\frac{1}{2}}^1 \frac{1}{2}d_{p,q}z = \frac{3}{20} - \frac{1}{4} = -\frac{1}{10}$.

This shortcoming can be removed by taking f to be positive and strictly monotonically increasing on interval $[0,d]$ otherwise tail part of the difference of two integrals in (4) can exceed.

Lemma 2.5. *Let $f \geq 0$ on $[0,d]$, where $\ 0 < c < d\ $ and $\ 1 \leq q < p < \infty$. If f is strictly monotonic increasing on $[0,d]$, then $\ \int_c^d f(z)d_{p,q}z \geq\ 0$.*

Proof: Consider $\int_c^d f(z)d_{p,q}z = \int_0^d f(z)d_{p,q}z - \int_0^c f(z)d_{p,q}z$.
$\int_c^d f(z)d_{p,q}z = (p-q)d\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}d\right) - (p-q)c\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}c\right)$
$\int_c^d f(z)d_{p,q}z \geq (p-q)c\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}\Big(f\left(\frac{q^j}{p^{j+1}}d\right) - f\left(\frac{q^j}{p^{j+1}}a\right)\Big)$. As f is strictly monotonically increasing on $[0,d]$. Thus $f\left(\frac{q^j}{p^{j+1}}d\right) - f\left(\frac{q^j}{p^{j+1}}c\right) \geq 0$. This implies that $\int_c^d f(z)d_{p,q}z \geq 0$.

Remark 2.6. *Observe that operator $A_r^{(p,q)}(f;z)$ satisfies $A_r^{(p,q)}(\alpha f + \beta g;z) = \alpha A_r^{(p,q)}(f;z) + \beta A_r^{(p,q)}(g;z)$ for all $f,g \in C[0,1]$ and $\alpha,\beta \in R$. Means $A_r^{(p,q)}(f;z)$ is a linear operator. But $A_r^{(p,q)}(f;z)$ need not be positive for every $f \geq 0$ defined on $[0,1]$. The reason behind this integration defined by (4) need not be positive for every $f \geq 0$ on subinterval $[c,d]$ of $[0,1]$. If we take f to be positive and strictly monotonically increasing on $[0,1]$, then $A_r^{(p,q)}(f;z) \geq 0$.*

Recall Classical Korovkin approximation theorem [8] is as follows:

Let $T_r : C[c,d] \to C[c,d]$ be the sequence of positive linear operators. Then $\lim_r \|T_r(f_s,z) - f_s(z)\|_{C[c,d]} = 0$, for $s = 0,1,2$, where $f_0(z) = 1$, $f_1(z) = z$ and $f_2(z) = z^2$ if and only if $\lim_r \|T_r(f,z) - f(z)\|_{C[c,d]} = 0$, for all $f \in C[c,d]$.

Remark 2.7. *For $1 \leq q < p < \infty$ and it is easy to see that $\lim\limits_{r\to\infty}[r]_{p,q} = \infty$ and $\lim\limits_{r\to\infty} \frac{[r-1]_{p,q}}{[r]_{p,q}} = \frac{1}{p}$. In order to obtain the convergence results of the operator $A_r^{(p,q)}(f;z)$, let us choose a sequence $1 \leq q_r < p_r < \infty$ such that $\lim\limits_{r\to\infty} p_r = 1$, and $\lim\limits_{r\to\infty} p_r^r = 1$,. By using squeeze theorem, we get $\lim\limits_{r\to\infty} q_r = 1$, and $\lim\limits_{r\to\infty} q_r^r = 1$, $\lim\limits_{r\to\infty}[r]_{p_r,q_r} = \infty$ and $\lim\limits_{r\to\infty} \frac{[r-1]_{p_r,q_r}}{[r]_{p_r,q_r}} = 1$.*

Theorem 2.8. *Let $1 \leq q_r < p_r < \infty$ such that $\lim\limits_{r\to\infty} p_r = 1$ and $\lim\limits_{r\to\infty} p_r^r = 1$ satisfying Remark 2.7. Then for each strictly monotonic increasing positive function $f \in C[0,1]$, $A_r^{(p_r,q_r)}(f;z)$ converges uniformly to f on $[0,1]$.*

Proof. It is sufficient to show using Korovkin Theorem that

$$\lim_{r\to\infty} \|A_r^{(p_r,q_r)}(t^m;z) - z^m\|_{C[0,1]} = 0, \quad m = 0,1,2.$$

It is clear from Lemma 2.1 (i) that

$$\lim_{r\to\infty} \|A_r^{(p_r,q_r)}(1;z) - 1\|_{C[0,1]} = 0.$$

Now, by Lemma 2.1 (ii)

$$|A_r^{(p_r,q_r)}(t;z) - z| = \frac{p_r^r}{[2]_{p_r,q_r}[r]_{p_r,q_r}}$$

which yields

$$\lim_{r\to\infty} \|A_r^{(p_r,q_r)}(t;z) - z\|_{C[0,1]} = 0.$$

Similarly,

$$\begin{aligned}
&|A_r^{(p_r,q_r)}(t^2;z) - z^2| \\
&= \left|\left(q_r^2 \frac{[r-1]_{p_r,q_r}}{[r]_{p_r,q_r}(p_r(1-z) + q_r z)} - 1\right) z^2 + \left(\frac{p_r^r(2q_r + p_r)}{[3]_{p_r,q_r}[r]_{p_r,q_r}} + \frac{p_r^{r-1}}{[r]_{p_r,q_r}}\right) z + \frac{p_r^{2r}}{[3]_{p_r,q_r}[r]_{p_r,q_r}^2}\right| \\
&\leq \left(q_r^2 \frac{[r-1]_{p_r,q_r}}{[r]_{p_r,q_r}(p_r(1-z) + q_r z)} - 1\right) z^2 + \left(\frac{p_r^r(2q_r + p_r)}{[3]_{p_r,q_r}[r]_{p_r,q_r}} + \frac{p_r^{r-1}}{[r]_{p_r,q_r}}\right) z + \frac{p_r^{2r}}{[3]_{p_r,q_r}[r]_{p_r,q_r}^2}.
\end{aligned}$$

In above inequality, if we take maximum on both sides then we get

$$\|A_r^{(p_r,q_r)}(t^2;z) - z^2\| \leq \frac{q_r\, [r-1]_{p_r,q_r}}{[r]_{p_r,q_r}} - 1 + \frac{p_r^r(2q_r + p_r)}{[3]_{p_r,q_r}[r]_{p_r,q_r}} + \frac{p_r^{r-1}}{[r]_{p_r,q_r}} + \frac{p_r^{2r}}{[3]_{p_r,q_r}[r]_{p_r,q_r}^2}$$

which concludes

$$\lim_{r\to\infty} \|A_r^{(p_r,q_r)}(t^2;z) - z^2\|_{C[0,1]} = 0.$$

Remark 2.9. *As we observed that for positivity of $A_r^{(p,q)}(f;z)$, f must be a strictly monotonic increasing function on* $[0,1]$. *But to estimate the order of approximation, this condition is not sufficient enough to assess the rate of convergence. As operator $A_r^{(p,q)}(f;z)$ does not approximate every continuous function on* $[0,1]$. *Therefore to overcome such shortcomings discussed above, special type of* (p,q)*-integrals which are the restricted* (p,q)*-integrals and the Riemann type* (p,q)*-integrals [1] which are used to construct new operators $\hat{A}_r^{(p,q)}(f;z)$. We obtain the rate of convergence and approximation of continuous function f defined on* $[0,1]$ *by $\hat{A}_r^{(p,q)}(f;z)$.*

In [1], for the development of post quantum integral, authors proposed the definition of Riemann type (p,q)-integral as follows:

$$\int_c^d f(z)\, d_{p,q}^R z = (p-q)(d-c)\sum_{j=0}^{\infty} f\Big(c+(d-c)\frac{q^j}{p^{j+1}}\Big)\frac{q^j}{p^{j+1}} \quad \text{when } |\frac{q}{p}|<1$$

Again, same problem appears here as discussed earlier, it will not be well defined for $0<q<p<1$. As for $j=0$, in the right hand side f takes input $c+(d-c)\frac{1}{p}$. Notice that $c+(d-c)\frac{1}{p}>d$. But function may not be defined outside the interval $[0,d]$. However, it is well defined for $1\leq q<p<\infty$. Thus one needs to re think over some other possible extension of Riemann type q-integrals.
Therefore an open question will arise:
How to re-define general (p,q)-integral (3) and Riemann type (p,q)-integral for $0<q<p<1$?
Therefore we consider the case $1\leq q<p<\infty$ for further analysis. Using definition (3), (4) and the idea of restricted q-integral in [9], the restricted (p,q)-integral can be re-shaped as follows.

Definition 2.10. *Let $1\leq q<p<\infty$ and r be a positive integer. The restricted* (p,q)*-integral is defined as*

$$\int_c^d f(z)\, d_{p,q}z = \int_{\frac{dq^r}{p^{r+1}}}^d f(z)\, d_{p,q}z = (p-q)(d-c)\sum_{j=0}^{r-1} f\Big(c+(d-c)\frac{q^j}{p^{j+1}}\Big)\frac{q^j}{p^{j+1}}. \tag{7}$$

Taking limit $r\to\infty$ in (7) gives the following definition of Riemann type (p,q)-integral.

Definition 2.11. *Let $1\leq q<p<\infty$ and $0<c<d$. The Riemann type* (p,q)*-integral is defined as*

$$\int_c^d f(z)\, d_{p,q}^R z = (p-q)(d-c)\sum_{j=0}^{\infty} f\Big(c+(d-c)\frac{q^j}{p^{j+1}}\Big)\frac{q^j}{p^{j+1}}. \tag{8}$$

For details on Riemann type q-integral, one can see [3, 9, 15].
Now we prove Hölder's type inequality for Riemann type (p,q)-integral and calculate Riemann type (p,q)-integral for $f(z)=1,\ z$ and z^2, which is used in proving Lemma 2.14.

Lemma 2.12. *Let $\alpha,\beta>0$ satisfying $\frac{1}{\alpha}+\frac{1}{\beta}=1$. For $1\leq q<p<\infty$ and $0<c<d$, $R_{p,q}(f;c,d)$ satisfies the following inequality*

$$R_{p,q}(|fg|;c,d)\leq\Big(R_{p,q}(|f|^{\alpha};c,d)\Big)^{\frac{1}{\alpha}}\Big(R_{p,q}(|f|^{\beta};c,d)\Big)^{\frac{1}{\beta}}$$

where $R_{p,q}(f;c,d)=\int_c^d f(z)\, d_{p,q}^R z=(p-q)(d-c)\sum\limits_{j=0}^{\infty} f\Big(c+(d-c)\frac{q^j}{p^{j+1}}\Big)\frac{q^j}{p^{j+1}}$.

Proof:

$$R_{p,q}(|fg|;c,d)=\int_c^d |f(z)g(z)|\, d_{p,q}^R z=(p-q)(d-c)\sum_{j=0}^{\infty}|f(c+(d-c)\frac{q^j}{p^{j+1}})g(c+(d-c)\frac{q^j}{p^{j+1}})|\frac{q^j}{p^{j+1}}$$

$$= (p-q)(d-c)\sum_{j=0}^{\infty}|\Big(f(c+(d-c)\frac{q^j}{p^{j+1}})(\frac{q^j}{p^{j+1}})^{\frac{1}{\alpha}}\Big)\Big(g(c+(d-c)\frac{q^j}{p^{j+1}}\Big)(\frac{q^j}{p^{j+1}})^{\frac{1}{\beta}}\Big)|.$$

Now using Holder's inequality, we have

$$R_{p,q}(|fg|;c,d) \leq (p-q)(d-c)\Big(\sum_{j=0}^{\infty}|\Big(f(c+(d-c)\frac{q^j}{p^{j+1}})|^{\alpha}(\frac{q^j}{p^{j+1}})\Big)\Big)^{\frac{1}{\alpha}}\Big(\sum_{j=0}^{\infty}|\Big(g(c+(d-c)\frac{q^j}{p^{j+1}})|^{\beta}(\frac{q^j}{p^{j+1}})\Big)\Big)^{\frac{1}{\beta}}$$

$$\leq (p-q)(d-c)\Big(\sum_{j=0}^{\infty}|\Big(f(c+(d-c)\frac{q^j}{p^{j+1}})|^{\alpha}(\frac{q^j}{p^{j+1}})\Big)\Big)^{\frac{1}{\alpha}}(p-q)(d-c)\Big(\sum_{j=0}^{\infty}|\Big(g(c+(d-c)\frac{q^j}{p^{j+1}})|^{\beta}(\frac{q^j}{p^{j+1}})\Big)\Big)^{\frac{1}{\beta}}$$

$$= \Big(R_{p,q}(|f|^{\alpha};c,d)\Big)^{\frac{1}{\alpha}}\Big(R_{p,q}(|f|^{\beta};c,d)\Big)^{\frac{1}{\beta}}.$$

Hence the proof is completed.
Again, we have calculated integration for $f(z)=1$, $f(z)=z$, $f(z)=z^2$ based on definition 2.11.
Case 1: For $f(z)=1$

$$\begin{aligned}
\int_c^d 1d^R_{p,q}z =& (q-p)(d-c)\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}.1\\
&= (q-p)(d-c)\frac{1}{q}\sum_{j=0}^{\infty}\frac{p^j}{q^j}\\
&= (q-p)(d-c)\frac{1}{q}.\frac{q}{q-p}\\
&= d-c
\end{aligned}$$

$$\int_c^d 1d^R_{p,q}z = d-c.$$

Case 2. $f(z)=z$

$$\begin{aligned}
\int_c^d zd^R_{p,q}z =& (q-p)(d-c)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}(c+(d-c)\frac{p^j}{q^{j+1}})\\
&= (q-p)(d-c)\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}(c+\frac{p^j}{q^{j+1}}+(d-c)\frac{p^2j}{q^{2(j+1)}})\\
&= (q-p)(d-c)\Big[\frac{c}{q}\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}+\frac{(d-c)}{q^2}\sum_{j=0}^{\infty}(\frac{p^2}{q^2})^j\Big]\\
&== (q-p)(d-c)\Big[\frac{c}{q}\frac{q}{q-p}+\frac{(d-c)}{q^2}\frac{q^2}{q^2-p^2}\Big]\\
&= (d-c)\Big[c+\frac{d-c}{q+p}\Big]\\
&= (d-c)\Big[\frac{cq+cp+d-c}{q+p}\Big]
\end{aligned}$$

$$\int_c^d zd^R_{p,q}z = (d-c)\Big[\frac{cq+cp+d-c}{q+p}\Big].$$

Case 3. $f(z) = z^2$

$$\int_c^d z^2 d_{p,q}^R z = (q-p)(d-c)\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}\Big((c+(d-c)\frac{p^j}{q^{j+1}}\Big)^2$$

$$= (q-p)(d-c)\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}\Big[c^2+\Big(\frac{(d-c).p^j}{q^{j+1}}\Big)^2+2c(d-c).\frac{p^j}{q^{j+1}}\Big]$$

$$= (q-p)(d-c)\sum_{j=0}^{\infty}\frac{p^j}{q^{j+1}}\Big[c^2+\frac{(d-c)^2}{q^2}.\Big(\frac{p^2}{q^2}\Big)^j+2c\frac{(d-c)}{q}.\Big(\frac{p}{q}\Big)^j\Big]$$

$$= (q-p)(d-c)\Big[\frac{c^2}{q}\sum_{j=0}^{\infty}\Big(\frac{p}{q}\Big)^j+\frac{(d-c)^2}{q^3}\sum_{j=0}^{\infty}\Big(\frac{p^3}{q^3}\Big)^j+\frac{2c(d-c)}{q^2}\sum_{j=0}^{\infty}\Big(\frac{p^2}{q^2}\Big)^j\Big]$$

$$\int_c^d z^2 d_{p,q}^R z = (d-c)\Big[c^2+\frac{(d-c)^2}{q^2+qp-p^2}+\frac{2c(d-c)}{q+p}\Big].$$

In 2017, authors proposed Lupaş Bernstein-Kantorovich Operators using Riemann type (p,q)-integral in [24] as follows:

$$\hat{A}_r^{(p,q)}(f;z) = [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}} f(t)d_{p,q}^R t,\ \ z\in[0,1] \tag{9}$$

where

$$B_{p,q}^{j,r}(z) = \frac{\left[\begin{array}{c} r \\ j \end{array}\right]_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}}\ z^j\ (1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}. \tag{10}$$

$B_{p,q}^{0,r}(z), B_{p,q}^{1,r}(z),\cdots, B_{p,q}^{r,r}(z)$ are the (p,q)-analogue of the Lupaş q-Bernstein rational functions [14] of degree r on the interval $[0,1]$.

Here we study approximation properties for operators (9) for $1\le q<p<\infty$.

Remark 2.13. *Let $f\ge 0$ on $[0,1]$. Then $\hat{A}_r^{(p,q)}(f;z)$ is a linear and positive operator.*

Lemma 2.14. *The following equalities hold for $1\le q<p<\infty$.*

(i) $\hat{A}_r^{(p,q)}(1;z) = 1$,

(ii) $\hat{A}_r^{(p,q)}(t;z) = z + \frac{p^r(1-z)+q^r z}{[2]_{p,q}[r]_{p,q}}$,

(iii) $\hat{A}_r^{(p,q)}(t^2;z) = \Big(1+\frac{2(q-p)}{p[2]_{p,q}}\Big)\frac{q^2p^2\ [r-1]_{p,q}}{[r]_{p,q}\ \ \{p(1-z)+qz\}}z^2+\Big(1+\frac{2(q-p)}{p\ [2]_{p,q}}+\frac{2}{[2]_{p,q}}\Big)\frac{p^{r+1}}{[r]_{p,q}}z+\frac{\big[p^r(1-z)+q^r z\big]\big[p^{r+1}(1-z)+q^{r+1}z\big]}{[3]_{p,q}[r]_{p,q}^2\ \{p(1-z)+qz\}}$.

Proof. In the sequel of proof, following results are used:

(a) $\int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}} 1\ d_{p,q}^R t = \frac{q^j}{[r]_{p,q}\ p^{(}j-r)}$

(b) $\int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}} t\ d_{p,q}^R t = \frac{q^j}{p^{2j-2r}[r]_{p,q}^2}\Big([j]_{p,q}+\frac{q^j}{[2]_{p,q}}\Big)$

(c) $\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} t^2\, d^R_{p,q}t = \frac{q^j}{p^{3j-3r}[r]^3_{p,q}}\left(p^2[j]^2_{p,q} + \frac{q^{2j}}{[3]_{p,q}} + \frac{2p[j]_{p,q}\, q^j}{[2]_{p,q}}\right)$

(i)

$$\hat{A}_r^{(p,q)}(1;z) = [r]_{p,q}\sum_{j=0}^{r}\frac{B^{j,r}_{p,q}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} 1\;\, d^R_{p,q}t = 1.$$

(ii)

$$\begin{aligned}
\hat{A}_r^{(p,q)}(t;z) &= [r]_{p,q}\sum_{j=0}^{r}\frac{B^{j,r}_{p,q}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} t\, d^R_{p,q}t \\
&= [r]_{p,q}\sum_{j=0}^{r}\frac{B^{j,r}_{p,q}(z)}{p^{r-j}q^j}\,\frac{q^j}{p^(2j-2r)[r]^2_{p,q}}\left([j]_{p,q} + \frac{q^j}{[2]_{p,q}}\right) \\
&= \sum_{j=0}^{r-1} p^{r-j-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j+1)}{2}}\; z^{j+1}\,(1-z)^{r-j-1}}{\prod_{j=1}^{r}\{p^{j-1}(1-z) + q^{j-1}z\}} \\
&+ \sum_{j=0}^{r}\frac{q^j}{p^{j-r}[r]_{p,q}[2]_{p,q}}\,\frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}}\; z^j\,(1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z) + q^{j-1}z\}} \\
&= \frac{z}{1-z}\sum_{j=0}^{r-1} p^{r-j-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j-1)}{2}}\left(\frac{qz}{1-z}\right)^j}{\prod_{j=1}^{r}\{p^{j-1} + q^{j-1}\frac{z}{1-z}\}} \\
&+ \frac{p^r}{[r]_{p,q}[2]_{p,q}}\sum_{j=0}^{r}\frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}}\left(\frac{qz}{p(1-z)}\right)^j}{\prod_{j=1}^{r}\{p^{j-1} + q^{j-1}\frac{z}{1-z}\}} \\
&= \frac{u}{1+u}\sum_{j=0}^{r-1}\frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j-1)}{2}}\left(\frac{qu}{p}\right)^j}{\prod_{j=0}^{r-2}\{p^j + q^j\left(\frac{qu}{p}\right)\}} \\
&+ \frac{p^r + q^r u}{(1+u)[r]_{p,q}[2]_{p,q}}\sum_{j=0}^{r}\frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}}\left(\frac{qu}{p}\right)^j}{\prod_{j=0}^{r-1}\{p^j + q^j\left(\frac{qu}{p}\right)\}} \qquad \textit{where}\;\; u = \frac{z}{1-z} \\
&= \frac{u}{1+u} + \frac{p^r + q^r u}{(1+u)[r]_{p,q}[2]_{p,q}} \\
&= z + \frac{p^r(1-z) + q^r z}{[r]_{p,q}[2]_{p,q}}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\hat{A}_r^{(p,q)}(t^2;z) &= [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}[r]_{p,q}}} t^2\, d_{p,q}^R t\\
&= [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\frac{q^j}{p^{3j-3r}[r]_{p,q}^3}\Big(p^2[j]_{p,q}^2+\frac{q^{2j}}{[3]_{p,q}}+\frac{2p[j]_{p,q}\,q^j}{[2]_{p,q}}\Big)\\
&= \frac{p^{2r+2}}{[r]_{p,q}^2}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{2j}}[j]_{p,q}^2+\frac{2p^{2r+1}}{[2]_{p,q}[r]_{p,q}^2}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{2j}}[j]_{p,q}q^j+\frac{p^{2r}}{[3]_{p,q}[r]_{p,q}^2}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{2j}}q^{2j}
\end{aligned}$$

Now we compute value of each term in the above sum.

$$\begin{aligned}
\frac{p^{2r+2}}{[r]_{p,q}^2}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{2j}}[j]_{p,q}^2 &= \frac{p^{2r+2}}{[r]_{p,q}^2}\sum_{j=0}^{r}\frac{[j]_{p,q}^2}{p^{2j}}\frac{\begin{bmatrix} r\\ j\end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}}q^{\frac{j(j-1)}{2}}\,z^j\,(1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}\\
&= \frac{p^{2r+2}}{[r]_{p,q}}\sum_{j=0}^{r-1}\frac{[j+1]_{p,q}}{p^{2j+2}}\frac{\begin{bmatrix} r-1\\ j\end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}}q^{\frac{j(j+1)}{2}}\,z^j\,(1-z)^{r-j-1}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}\\
&\quad \text{using } [j+1]=p^j+q[j]_{p,q} \text{ and previous calculations, we have}\\
&= \frac{p^{2r+2}}{[r]_{p,q}}\sum_{j=0}^{r-1}\Big(\frac{p^j+q[j]_{p,q}}{p^{2j+2}}\Big)\frac{\begin{bmatrix} r-1\\ j\end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}}q^{\frac{j(j+1)}{2}}\,z^j\,(1-z)^{r-j-1}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}\\
&= \frac{p^{r+1}}{[r]_{p,q}}\frac{u}{1+u}\sum_{j=0}^{r-1}\frac{\begin{bmatrix} r-1\\ j\end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}}q^{\frac{j(j-1)}{2}}\left(\frac{qu}{p}\right)^j}{\prod_{j=0}^{r-2}\{p^j+q^j\left(\frac{qu}{p}\right)\}}\\
&+\frac{q^2p^2u^2[r-1]_{p,q}}{[r]_{p,q}(1+u)(p+qu)}\sum_{j=0}^{r-2}\frac{\begin{bmatrix} r-2\\ j\end{bmatrix}_{p,q} p^{\frac{(r-j-2)(r-j-3)}{2}}q^{\frac{j(j-1)}{2}}\left(\frac{q^2u}{p^2}\right)^j}{\prod_{j=0}^{r-3}\{p^j+q^j\left(\frac{q^2u}{p^2}\right)\}}\quad \textit{where}\ \ u=\frac{z}{1-z}\\
&= \frac{p^{r+1}}{[r]_{p,q}}\frac{u}{1+u}+\frac{q^2p^2u^2[r-1]_{p,q}}{[r]_{p,q}(1+u)(p+qu)}\\
&= \frac{p^{r+1}}{[r]_{p,q}}z+\frac{[r-1]_{p,q}q^2p^2z^2}{[r]_{p,q}\{p(1-z)+qz\}}.
\end{aligned}$$

$$\frac{2p^{2r+1}}{[2]_{p,q}[r]_{p,q}^2}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{2j}}[j]_{p,q}q^j = \frac{2p^{2r+1}}{[2]_{p,q}[r]_{p,q}^2}\sum_{j=0}^{r}\frac{[j]_{p,q}q^j}{p^{2j}}\frac{\begin{bmatrix} r\\ j\end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}}q^{\frac{j(j-1)}{2}}\,z^j\,(1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}$$

using $q^j=(q-p)[j]_{p,q}+p^j$ and previous calculations

$$= \frac{2p^{2r+1}}{[2]_{p,q}[r]^2_{p,q}} \sum_{j=0}^{r} \frac{[j]_{p,q}\{(q-p)[j]_{p,q}+p^j\}}{p^{2j}} \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} z^j (1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}$$

$$= \frac{2p^{2r+1}}{[2]_{p,q}[r]^2_{p,q}}(q-p) \sum_{j=0}^{r} \frac{[j]^2_{p,q}}{p^{2j}} \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} z^j (1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}$$

$$+ \frac{2p^{2r+1}}{[2]_{p,q}[r]_{p,q}} \sum_{j=0}^{r-1} \frac{1}{p^{j+1}} \frac{\begin{bmatrix} r-1 \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j-1)(r-j-2)}{2}} q^{\frac{j(j+1)}{2}} z^{j+1} (1-z)^{-j-1}}{\prod_{j=1}^{r}\{p^{j-1}+q^{j-1}\frac{z}{1-z}\}}$$

$$= \frac{2(q-p)}{p[2]_{p,q}}\Big(\frac{p^{r+1}}{[r]_{p,q}}z + \frac{[r-1]_{p,q}q^2p^2z^2}{[r]_{p,q}\{p(1-z)+qz\}}\Big) + \frac{2p^{r+1}}{[2]_{p,q}[r]_{p,q}}z$$

$$\frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}} \sum_{j=0}^{r} \frac{B^{j,r}_{p,q}(z)}{p^{2j}} q^{2j} = \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}} \sum_{j=0}^{r} \frac{q^{2j}}{p^{2j}} \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} z^j (1-z)^{r-j}}{\prod_{j=1}^{r}\{p^{j-1}(1-z)+q^{j-1}z\}}$$

$$= \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}} \sum_{j=0}^{r} \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} \left(\frac{q^2 z}{p^2(1-z)}\right)^j}{\prod_{j=1}^{r}\{p^{j-1}+q^{j-1}\frac{z}{1-z}\}}.$$

$$\frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}} \sum_{j=0}^{r} \frac{B^{j,r}_{p,q}(z)}{p^{2j}} q^{2j} = \frac{p^{2r}}{[3]_{p,q}[r]^2_{p,q}(1+u)(p+qu)p^{2r-4}} \sum_{j=0}^{r} \frac{\begin{bmatrix} r \\ j \end{bmatrix}_{p,q} p^{\frac{(r-j)(r-j-1)}{2}} q^{\frac{j(j-1)}{2}} \left(\frac{q^2 u}{p^2}\right)^j}{\prod_{j=0}^{r-3}\{p^j+q^j\frac{q^2}{p^2}\}}$$

$$\text{where } u = \frac{z}{1-z}$$

$$= \frac{[p^r(1-z)+q^r z]\left[p^{r+1}(1-z)+q^{r+1}z\right]}{[3]_{p,q}[r]^2_{p,q}\{p(1-z)+qz\}}.$$

Using these values, we have

$$\hat{A}^{(p,q)}_r(t^2;z) = \frac{p^{r+1}}{[r]_{p,q}}z + \frac{[r-1]_{p,q}q^2p^2z^2}{[r]_{p,q}\{p(1-z)+qz\}} + \frac{2(q-p)}{p[2]_{p,q}}\Big(\frac{p^{r+1}}{[r]_{p,q}}z + \frac{[r-1]_{p,q}q^2p^2z^2}{[r]_{p,q}\{p(1-z)+qz\}}\Big) + \frac{2p^{r+1}}{[2]_{p,q}[r]_{p,q}}z$$

$$+ \frac{[p^r(1-z)+q^r z]\left[p^{r+1}(1-z)+q^{r+1}z\right]}{[3]_{p,q}[r]^2_{p,q}\{p(1-z)+qz\}}$$

$$= \frac{[r-1]_{p,q}q^2p^2z^2}{[r]_{p,q}\{p(1-z)+qz\}}\Big(1+\frac{2(q-p)}{p[2]_{p,q}}\Big) + \frac{p^{r+1}}{[r]_{p,q}}z\Big(1+\frac{2(q-p)}{p[2]_{p,q}}+\frac{2}{[2]_{p,q}}\Big)$$

$$+ \frac{[p^r(1-z)+q^r z]\left[p^{r+1}(1-z)+q^{r+1}z\right]}{[3]_{p,q}[r]^2_{p,q}\{p(1-z)+qz\}}.$$

.

Theorem 2.15. *Let $p = p_r$ and $q = q_r$ be sequence of real number satisfying $1 \leq q_r < p_r < \infty$ such that $\lim_{r\to\infty} p_r = 1$ and $\lim_{r\to\infty} p_r^r = 1$. Then for each $f \in C[0,1]$, $\hat{A}_r^{(p_r,q_r)}(f;z)$ converges uniformly to f on $[0,1]$.*

Proof. It is sufficient to show using Korovkin's Theorem that

$$\lim_{r\to\infty} \|\hat{A}_r^{(p_r,q_r)}(t^m;z) - z^m\|_{C[0,1]} = 0, \quad m = 0,1,2.$$

By Lemma 2.14, it is clear that

$$\lim_{r\to\infty} \|\hat{A}_r^{(p_r,q_r)}(1;z) - 1\|_{C[0,1]} = 0.$$

Now, by Lemma 2.14 (ii)

$$|\hat{A}_r^{(p_r,q_r)}(t;z) - z| = \frac{p_r^r(1-z) + q_r^r z}{[2]_{p_r,q_r}[r]_{p_r,q_r}} \leq \frac{p^r + q^r}{[2]_{p_r,q_r}[r]_{p_r,q_r}}$$

Using remark 2.7, we get

$$\lim_{r\to\infty} \|\hat{A}_r^{(p_r,q_r)}(t;z) - z\|_{C[0,1]} = 0.$$

$$|\hat{A}_r^{(p,q)}(t^2;z) - z^2| = \Bigg|\bigg(\Big(1 + \frac{2(q_r - p_r)}{p_r[2]_{p_r,q_r}}\Big)\frac{q_r^2 p_r^2\ [r-1]_{p_r,q_r}}{[r]_{p_r,q_r}\ \{p_r(1-z) + q_r z\}} - 1\bigg)z^2 + \Big(1 + \frac{2(q_r - p_r)}{p_r\ [2]_{p_r,q_r}} + \frac{2}{[2]_{p_r,q_r}}\Big) \frac{p^{r+1}}{[r]_{p_r,q_r}} z + \frac{[p_r^r(1-z) + q_r^r z]\left[p_r^{r+1}(1-z) + q_r^{r+1} z\right]}{[3]_{p_r,q_r}[r]^2_{p_r,q_r}\ \{p_r(1-z) + q_r z\}}\Bigg|$$

$$\leq \Bigg|\bigg(\Big(1 + \frac{2(q_r - p_r)}{p_r[2]_{p_r,q_r}}\Big)\frac{q_r^2 p_r^2\ [r-1]_{p_r,q_r}}{[r]_{p_r,q_r}\ \{p_r(1-z) + q_r z\}} - 1\bigg)\Bigg| + \Bigg|\Big(1 + \frac{2(q_r - p_r)}{p_r\ [2]_{p_r,q_r}} + \frac{2}{[2]_{p_r,q_r}}\Big)\frac{p^{r+1}}{[r]_{p_r,q_r}}\Bigg| + \Bigg|\frac{[p_r^r(1-z) + q_r^r z]\left[p_r^{r+1}(1-z) + q_r^{r+1} z\right]}{[3]_{p_r,q_r}[r]^2_{p_r,q_r}\ \{p_r(1-z) + q_r z\}}\Bigg|$$

Again using remark 2.7, finally we have the conclusion

$$\lim_{r\to\infty} \|\hat{A}_r^{(p_r,q_r)}(t^2;z) - z^2\|_{C[0,1]} = 0.$$

Hence, the proof is completed.

3. Rate of convergence

Here, with the help of modulus of continuity and functions of the lipschitz class, the approximation order for the operators $A_r^{(p,q)}(f;z)$ and $\hat{A}_r^{(p,q)}(f;z)$ are being studied.

Let $f \in C[0,1]$ and consider notation $\omega(f,\delta)$ to denote the modulus of continuity of f where $\delta > 0$, defined as

$$\omega(f,\delta) = \sup_{|z-w|\leq\delta} |f(z) - f(w)|, \ \ z,w \in [0,1].$$

It is well known that $\lim_{\delta\to 0^+} \omega(f,\delta) = 0$ for $f \in C[0,1]$ and for any $\delta > 0$ one has

$$|f(w) - f(z)| \leq \omega(f,\delta)\Big(\frac{|w-z|}{\delta} + 1\Big).$$

First, we evaluate the rates of convergence by means of modulus of continuity for both $A_r^{(p,q)}(f;z)$ and $\hat{A}_r^{(p,q)}(f;z)$.

Theorem 3.1. *Let $q = (q_r)$ ard $p = (p_r)$ with $1 \leq q_r < p_r < \infty$ be the sequences and if f is any positive strictly monotonic increasing continuous function defined on* $[0,1]$, *then*

$$\Big|A_r^{(p,q)}(f;z) - f(z)\Big| \leq 2\omega\Big(f, \delta_r(z)\Big)$$

where

$$\begin{aligned}\delta_r(z) &= \sqrt{A_r^{(p,q)}\Big((t-z)^2;z\Big)}\\ &= \sqrt{\Big(\frac{q^2\ [r-1]_{p,q}}{[r]_{p,q}\ \{p(1-z)+qz\}} - 1\Big)z^2 + \Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}} + \frac{p^{r-1}}{[r]_{p,q}} - \frac{2p^r}{[2]_{p,q}[r]_{p,q}}\Big)z + \frac{p^{2r}}{[3]_{p,q}[r]_{p,q}^2}}\end{aligned}$$

Proof. Since $A_r^{(p,q)}(1;z) = 1$, we have

$$\begin{aligned}\Big|A_r^{(p,q)}(f;z) - f(z)\Big| &\leq A_r^{(p,q)}\Big(|f(t) - f(z)|;x\Big)\\ &\leq [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}} |f(t) - f(z)| d_{p,q}t.\end{aligned}$$

In view of (3.1), we get

$$\begin{aligned}\Big|A_r^{(p,q)}(f;z) - f(z)\Big| &\leq \Big\{[r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\ [r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\ [r]_{p,q}}}\Big(\frac{|t-z|}{\delta} + 1\Big)d_{p,q}t\Big\}\omega(f,\delta)\\ &= \Big\{\frac{1}{\delta^2}A_r^{(p,q)}\Big((t-z)^2;z\Big) + 1\Big\}\omega(f,\delta).\end{aligned}$$

Choosing $\delta = \delta_r(z) = \sqrt{\Big(\frac{q^2\ [r-1]_{p,q}}{[r]_{p,q}\ \{p(1-z)+qz\}} - 1\Big)z^2 + \Big(\frac{p^r(2q+p)}{[3]_{p,q}[r]_{p,q}} + \frac{p^{r-1}}{[r]_{p,q}} - \frac{2p^r}{[2]_{p,q}[r]_{p,q}}\Big)z + \frac{p^{2r}}{[3]_{p,q}[r]_{p,q}^2}}$, we have

$$\Big|A_r^{(p,q)}(f;z) - f(z)\Big| \leq 2\omega\Big(f, \delta_r(z)\Big).$$

Theorem 3.2. *Let $q = (q_r)$ and $p = (p_r)$ with $1 \leq q_r < p_r < \infty$ be the sequences and if f is any continuous function defined on* $[0,1]$, *then*

$$\Big|\hat{A}_r^{(p,q)}(f;z) - f(z)\Big| \leq 2\omega\Big(f, \delta_r(z)\Big)$$

where

$$\delta_r(z) = \Big(\hat{A}_r^{(p,q)}\Big((t-z)^2;z\Big)\Big)^{\frac{1}{2}}$$

with

$$\begin{aligned}\hat{A}_r^{(p,q)}\Big((t-z)^2;z\Big) &= \left\{\Big(1 + \frac{2(q_r - p_r)}{p_r[2]_{p_r,q_r}}\Big)\frac{q_r^2 p_r^2\ [r-1]_{p_r,q_r}}{[r]_{p_r,q_r}\ \{p_n(1-z) + q_r z\}} - 1\right\}z^2\\ &+ \left\{\Big(1 + \frac{2(q_r - p_r)}{p\ [2]_{p_r,q_r}} + \frac{2}{[2]_{p_r,q_r}}\Big)\frac{p_r^{r+1}}{[r]_{p_r,q_r}} - 2\frac{p^r(1-z) + q^r z}{[2]_{p,q}[r]_{p,q}}\right\}z\\ &+ \frac{[p_r^r(1-z) + q_r^r z]\Big[p_r^{r+1}(1-z) + q_r^{r+1}z\Big]}{[3]_{p_r,q_r}[r]_{p_r,q_r}^2\ \{p_r(1-z) + q_r z\}}\end{aligned}$$

Proof. Since $\hat{A}_r^{(p,q)}(1;z) = 1$, we have

$$\begin{aligned}\left|\hat{A}_r^{(p,q)}(f;z) - f(z)\right| &\leq \hat{A}_r^{(p,q)}\Big(|f(t) - f(z)|;z\Big)\\ &\leq [r]_{p,q} \sum_{j=0}^{r} \frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j} \int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} |f(t) - f(z)| d_{p,q}^R t.\end{aligned}$$

In view of (3.1), we get

$$\begin{aligned}\left|\hat{A}_r^{(p,q)}(f;z) - f(z)\right| &\leq \left\{[r]_{p,q} \sum_{j=0}^{r} \frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j} \int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} \left(\frac{|t-z|}{\delta} + 1\right) d_{p,q}^R t\right\}\omega(f,\delta)\\ &= \left\{\frac{1}{\delta^2}\hat{A}_r^{(p,q)}\Big((t-z)^2;z\Big) + 1\right\}\omega(f,\delta).\end{aligned}$$

Choosing $\delta = \delta_r(z) = \left(\hat{A}_r^{(p,q)}\Big((t-z)^2;z)\right)^{\frac{1}{2}}$ with

$$\begin{aligned}\hat{A}_r^{(p,q)}\Big((t-z)^2;z\Big) &= \left\{\left(1 + \frac{2(q_r - p_r)}{p_r[2]_{p_r,q_r}}\right)\frac{q_r^2 p_r^2\,[r-1]_{p_r,q_r}}{[r]_{p_r,q_r}\,\{p_r(1-z) + q_r z\}} - 1\right\} z^2\\ &+ \left\{\left(1 + \frac{2(q_r - p_r)}{p\,[2]_{p_r,q_r}} + \frac{2}{[2]_{p_r,q_r}}\right)\frac{p_r^{r+1}}{[r]_{p_r,q_r}} - 2\frac{p^r(1-z) + q^r z}{[2]_{p,q}[r]_{p,q}}\right\} z\\ &+ \frac{[p_r^r(1-z) + q_r^r z]\left[p_r^{r+1}(1-z) + q_r^{r+1} z\right]}{[3]_{p_r,q_r}[r]_{p_r,q_r}^2\,\{p_r(1-z) + q_r z\}}\end{aligned}$$

we have

$$\left|A_r^{(p,q)}(f;z) - f(z)\right| \leq 2\omega\Big(f, \delta_r(z)\Big).$$

With the help of usual Lipschitz class $\mathrm{Lip}_M(\alpha)$, convergence rate for the operators $A_r^{(p,q)}(f;z)$ and $\hat{A}_r^{(p,q)}(f;z)$ will be studied.

If f satisfies the inequality

$$|f(w) - f(z)| \leq M|w - z|^\alpha, \quad (w, z \in [0,1])$$

for $f \in C[0,1]$, $M > 0$ and $0 < \alpha \leq 1$, then f is said to be member of the class $\mathrm{Lip}_M(\alpha)$.

Theorem 3.3. *Let $q = (q_r)$ and $p = (p_r)$ with $1 \leq q_r < p_r < \infty$ be the sequences. Then for each $f \in Lip_M(\alpha)$, we have*

$$|A_r^{(p_r,q_r)}(f;z) - f(z)| \leq M\delta_r^\alpha(z),$$

where

$$\delta_r(z) = \sqrt{A_r^{(p_r,q_r)}\Big((t-z)^2;z\Big)}$$

with $A_r^{(p_r,q_r)}\Big((t-z)^2;z\Big)$ as in remark (2.3).

Proof. As the operators $A_r^{(p,q)}$ are monotone, we can write

$$\begin{aligned}\left|A_r^{(p,q)}(f;z) - f(z)\right| &\leq A_r^{(p,q)}\Big(|f(t) - f(z)|;z\Big)\\ &\leq [r]_{p,q} \sum_{j=0}^{r} \frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j} \int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} |f(t) - f(z)| d_{p,q} t\\ &\leq M[r]_{p,q} \sum_{j=0}^{r} \frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j} \int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}} |t - z|^\alpha\, d_{p,q} t.\end{aligned}$$

For $p_1 = \frac{2}{\alpha}$ and $p_2 = \frac{2}{2-\alpha}$ and applying the Hölder's inequality for the sum, we have

$$\begin{aligned} |A_r^{(p,q)}(f;z) - f(z)| &\leq M\sum_{j=0}^{r}\left\{[r]_{p,q}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}(t-z)^2 d_{p,q}t\right\}^{\frac{\alpha}{2}}\left\{[r]_{p,q}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}1 d_{p,q}t\right\}^{\frac{2-\alpha}{2}} \\ &\leq M\left\{[r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}(t-z)^2 d_{p,q}t\right\}^{\frac{\alpha}{2}}\left\{[r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}1 d_{p,q}t\right\}^{\frac{2-\alpha}{2}} \\ &= M\left\{A_r^{(p,q)}\left((t-z)^2;z\right)\right\}^{\frac{\alpha}{2}}. \end{aligned}$$

Theorem 3.4. *Let $q = (q_r)$ and $p = (p_r)$ with $1 \leq q_r < p_r < \infty$ be the sequences. Then for each $f \in Lip_M(\alpha)$, we have*

$$|\hat{A}_r^{(p_r,q_r)}(f;z) - f(z)| \leq M\delta_r^{\alpha}(z),$$

where

$$\delta_r(z) = \sqrt{\hat{A}_r^{(p_r,q_r)}\left((t-z)^2;z\right)}$$

Proof. As the operators $\hat{A}_r^{(p,q)}$ are monotone, one can write

$$\begin{aligned} \left|\hat{A}_r^{(p,q)}(f;z) - f(z)\right| &\leq \hat{A}_r^{(p,q)}\left(|f(t) - f(z)|;z\right) \\ &\leq [r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}|f(t) - f(z)| d_{p,q}^R t \\ &\leq M[r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}|t-z|^{\alpha}\, d_{p,q}^R t. \end{aligned}$$

Again for $p_1 = \frac{2}{\alpha}$ and $p_2 = \frac{2}{2-\alpha}$, and applying the Hölder's inequality for the sum, we have

$$\begin{aligned} |\hat{A}_r^{(p,q)}(f;z) - f(z)| &\leq M\sum_{j=0}^{r}\left\{[r]_{p,q}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}(t-z)^2 d_{p,q}^R t\right\}^{\frac{\alpha}{2}}\left\{[r]_{p,q}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}1 d_{p,q}^R t\right\}^{\frac{2-\alpha}{2}} \\ &\leq M\left\{[r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}(t-z)^2 d_{p,q}^R t\right\}^{\frac{\alpha}{2}}\left\{[r]_{p,q}\sum_{j=0}^{r}\frac{B_{p,q}^{j,r}(z)}{p^{r-j}q^j}\int_{\frac{[j]_{p,q}}{p^{j-r-1}\,[r]_{p,q}}}^{\frac{[j+1]_{p,q}}{p^{j-r}\,[r]_{p,q}}}1 d_{p,q}^R t\right\}^{\frac{2-\alpha}{2}} \\ &= M\left\{A_r^{(p,q)}\left((t-z)^2;z\right)\right\}^{\frac{\alpha}{2}}. \end{aligned}$$

If we take sequences $q = q_r$ and $p = p_r$ with $1 \leq q_r < p_r < \infty$ with $\delta^2(z) = \delta_r^2(z) = \hat{A}_r^{(p,q)}\left((t-z)^2;z\right)$, we arrive at our desired result.

4. Conclusion

As discussed in Section 2, we can say general (p,q)-integral (3) and Riemann type (p,q)-integral are not exact extension of general q-integral and Riemann type q-integral. Therefore an open question is: Can we redefine the general (p,q) integral and the Riemann type (p,q)-integral such that it becomes well defined also for $0 < q < p < 1$?

Competing interests
The authors declare that they have no competing interests.
Authors' contributions
All the authors contributed equally and significantly in writing this paper.

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