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Certain Weighted Young and Pólya-Szegö-type Inequalities Involving Marichev-Saigo-Maeda Fractional Integral Operators with Applications

Wengui Yang[a,b]

[a]*Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472000, China*
[b]*College of Applied Engineering, Henan University of Science and Technology, Sanmenxia 472000, China*

Abstract. In this paper, we establishes certain new weighted Young and Pólya-Szegö-type inequalities involving Marichev-Saigo-Maeda fractional integral operators. Meanwhile, corresponding weighted Cauchy-Schwarz type inequalities, Shisha-Mond type inequalities and Diaz-Metcalf type inequalities for Marichev-Saigo-Maeda fractional integral operators are also obtained. As applications, some estimates for weighted Chebyshev-type inequalities with two unknown functions for Marichev-Saigo-Maeda fractional integral operators are presented. The main results of this paper are more general and include a great number of existing classical inequalities.

1. Introduction

The classical Young inequality says that

$$\mathbb{x}^{\mathbb{p}}/\mathbb{p}+\mathbb{y}^{\mathbb{q}}/\mathbb{q}\geq \mathbb{x}\mathbb{y} \ \text{ for } \ \mathbb{x},\mathbb{y}\geq 0,\ \ 1/\mathbb{p}+1/\mathbb{q}=1 \ \text{ with } \ \mathbb{p},\mathbb{q}>1, \tag{1.1}$$

with equality if and only if $\mathbb{x}^{\mathbb{p}}=\mathbb{y}^{\mathbb{q}}$. If $\theta\in[0,1]$, the inequality (1.1) can be reexpressed as the following weighted arithmetic-geometric mean inequality

$$\theta\mathbb{x}+(1-\theta)\mathbb{y}\geq \mathbb{x}^{\theta}\mathbb{y}^{1-\theta} \ \text{ with equality if and only if } \ \mathbb{x}=\mathbb{y}. \tag{1.2}$$

By using classification and analysis, Kittaneh and Manasrah [7] presented a refinement of inequality of weighted arithmetic-geometric mean inequality (1.2) in the following form

$$\theta\mathbb{x}+(1-\theta)\mathbb{y}\geq \mathbb{x}^{\theta}\mathbb{y}^{1-\theta}+r_0(\sqrt{\mathbb{x}}-\sqrt{\mathbb{y}})^2 \ \text{ for } \ r_0=\min\{\theta,1-\theta\}. \tag{1.3}$$

Pólya and Szegö [9] established the following inequality

$$1\leq \Big(\int_a^b f^2(x)dx\int_a^b g^2(x)dx\Big)/\Big(\int_a^b f(x)g(x)dx\Big)^2\leq (\Phi_1\Psi_1+\Phi_2\Psi_2)^2/(4\Phi_1\Psi_1\Phi_2\Psi_2), \tag{1.4}$$

where f and g are two integral functions defined on $[a,b]$ satisfying the following condition

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Email address: wgyang0617@yahoo.com (Wengui Yang)

$$0 < \Phi_1 \le f(x) \le \Phi_2 \text{ and } 0 < \Psi_1 \le g(x) \le \Psi_2 \text{ for constants } \Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R} \text{ and } \forall x \in [a, b]. \tag{1.5}$$

Later, inequality (1.4) was called as the Pólya-Szegö inequality. Pólya-Szegö inequality has received widespread attention from many scholars since it can be applied to some fields, for example, mathematical analysis, linear algebra, probability and statistical problems, etc. Decades later, Greub and Rheinboldt [4] obtained the weighted discrete Pólya-Szegö inequality, which was called as the Greub-Rheinboldt inequality. Assume $0 < \Phi_1 \le f_j \le \Phi_j$ and $0 < \Psi_1 \le g_j \le \Psi_j$ for $j = 1, 2, \dots, n$, $n \ge 1$, $\Phi_1\Psi_1 < \Phi_2\Psi_2$, and $\xi_1, \xi_2, \dots, \xi_n$ are nonzero real numbers. Shisha and Mond [9] obtained the following inequality

$$(\textstyle\sum_{j=1}^n f_j^2\xi_j^2)/(\sum_{j=1}^n f_jg_j\xi_j^2) - (\sum_{j=1}^n f_jg_j\xi_j^2)/(\sum_{j=1}^n g_j^2\xi_j^2) \le (\sqrt{\Phi_2/\Psi_1} - \sqrt{\Phi_1/\Psi_2})^2, \tag{1.6}$$

which was known as the Shisha and Mond inequality. Employing the same method, Shisha and Mond type integral inequality can be easily acquired. Suppose $g(x)$ and $f(x)$ are nonzero continuous functions defined on $[a, b]$ satisfying the condition $\mathbb{m} \le f(x)/g(x) \le \mathbb{M}$ for all almost $x \in [a, b]$. Diaz and Metcalf [2] established the following inequality

$$\int_a^b f^2(x)dx + \mathbb{m}\mathbb{M} \int_a^b g^2(x)dx \le (\mathbb{m} + \mathbb{M}) \int_a^b f(x)dx \int_a^b g(x)dx. \tag{1.7}$$

In 2003, Dragomir and Diamond [3] drew support from Pólya-Szegö inequality (1.2) and Shisha-Mond integral inequality to give the following result: let f and g be two integral functions defined on $[a, b]$ satisfying the condition (1.5), and $T(f, g, a, b) = \int_a^b f(x)g(x)dx/(b-a) - \int_a^b f(x)dx \int_a^b g(x)dx/(b-a)^2$, then

$$|T(f, g, a, b)| \le \tfrac{(\Phi_2-\Phi_1)(\Psi_2-\Psi_1)}{4(b-a)^2\sqrt{\Phi_1\Psi_1\Phi_2\Psi_2}} \int_a^b f(x)dx \int_a^b g(x)dx \quad (\text{or } \tfrac{(\sqrt{\Phi_2}-\sqrt{\Phi_1})(\sqrt{\Psi_2}-\sqrt{\Psi_1})}{b-a}(\int_a^b f(x)dx \int_a^b g(x))^{1/2}). \tag{1.8}$$

During the past decade, a large number of scholars have extensively studied fractional integral inequalities based on the different types of known fractional integral operators. Therefore, there exist many results on fractional integral inequalities since they have been proved to be one of the most effective and significant tools for the development of fractional calculus systems. The interested readers can refer to the literatures [12, 24, 25] and the references quoted therein. For example, Set *et al.* [15] investigated some Pólya-Szegö type inequalities for the generalized proportional Hadamard fractional integrals. From the relevant review papers[16, 17], the introductory overview of the theory of fractional-calculus operators based upon the Fox-Wright function and related Mittag-Leffler type functions as well as recent developments of ordinary and partial fractional differintegral equations were presented, respectively. Srivastava [18] investigated a great deal of fractional calculus operators and integral transformations introduced the general non-trivial family of the Riemann-Liouville type fractional integrals and derivatives.

On the other hand, the Marichev-Saigo-Maeda fractional integral operators involving Appell's function appeared in the literatures [8, 14]. Based on the fractional calculus operators with Gaussian hypergeometric function, Srivastava and Saigo [20] investigated the solutions of various boundary value problems involving the celebrated Euler-Darboux equation. Joshi *et al.* [5], Tassaddiq *et al.* [23] and Nale *et al.* [10] considered the Grüss inequalities, reverse Minkowski inequalities and some related inequalities with monotone functions for the generalized Marichev-Saigo-Maeda fractional integral operators, respectively. Srivastava *et al.* [19, 21] introduced certain formulas and integral transforms related to the Marichev-Saigo-Maeda fractional calculus operators with some applications to (p, q)-extended Bessel functions and Fox-Wright generalized hypergeometric functions, respectively.

In this paper, motivated by the above previously mentioned references, we will investigate the weighted Young and Pólya-Szegö-type inequalities for Marichev-Saigo-Maeda fractional integral operators. To the best knowledge of the author, there does not exist any literature dealing with the Young and Pólya-Szegö-type inequalities involving Marichev-Saigo-Maeda fractional integral operators. Therefore, it is necessary and important to study the weighted Young and Pólya-Szegö-type inequalities involving Marichev-Saigo-Maeda fractional integral operators. At the same time, some new related weighted Cauchy-Schwarz type inequalities, weighted Shisha-Mond type inequalities and weighted Diaz-Metcalf type inequalities for Marichev-Saigo-Maeda fractional integral operators are also are introduced. As applications, several estimates of Chebyshev-type weighted Marichev-Saigo-Maeda fractional integral inequalities with two unknown functions are presented based on the Heaviside unit step function and Pólya-Szegö-type inequalities. The main results of this paper are more general and extend some existing classical inequalities.

2. Preliminaries

In this section, we firstly introduce the definitions of the Marichev-Saigo-Maeda fractional integral operators involving Appell's functions or Horn's function as follows.

Definition 2.1 (See [8, 14]). Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$ and $x > 0$. Then the left and right-sided Marichev-Saigo-Maeda fractional integral operators involving Appell's function or Horn's function are given as follows

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}t^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tfrac{t}{x},1-\tfrac{x}{t})f(t)dt \ \text{ for } \gamma > 0, \tag{2.1}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',x,\infty}f)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tfrac{x}{t},1-\tfrac{t}{x})f(t)dt \ \text{ for } \gamma > 0, \tag{2.2}$$

where F_3 denotes the known Appell's function or Horn's function with two variables defined by

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \ \text{ for } \ \max\{|x|,|y|\} < 1, \tag{2.3}$$

and $(\alpha)_m$ represents the Pochhammer symbol defined by $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$.

Definition 2.2 (See [8, 14]). Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{R}$ and $x > 0$. Then the left and right-sided Marichev-Saigo-Maeda fractional derivatives involving Appell's function or Horn's function are given as follows

$$(\mathscr{D}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}f)(x) = (\mathscr{I}^{-\beta',-\beta,-\gamma}_{-\alpha',-\alpha,0,x}f)(x) = (\tfrac{d}{dx})^{\kappa}(\mathscr{I}^{-\beta'+\kappa,-\beta,-\gamma+\kappa}_{-\alpha',-\alpha,0,x}f)(x) \ \text{ for } \ \gamma > 0 \ \text{ and } \ \kappa = [\gamma+1], \tag{2.4}$$

$$(\mathscr{D}^{\beta,\beta',\gamma}_{\alpha,\alpha',x,\infty}f)(x) = (\mathscr{I}^{-\beta',-\beta,-\gamma}_{-\alpha',-\alpha,x,\infty}f)(x) = (-\tfrac{d}{dx})^{\kappa}(\mathscr{I}^{-\beta',-\beta+\kappa,-\gamma+\kappa}_{-\alpha',-\alpha,x,\infty}f)(x) \ \text{ for } \ \gamma > 0 \text{ and } \ \kappa = [\gamma+1]. \tag{2.5}$$

In 2019, Joshi *et al.* [5] used the the left-sided Marichev-Saigo-Maeda fractional integral operator (2.1) to introduce the generalized Marichev-Saigo-Maeda fractional integral operator with Appell function

$$(\mathscr{S}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}f)(x) = \frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{\alpha+\alpha'-\gamma}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}f)(x), \tag{2.6}$$

where $\gamma > \max\{0, \alpha+\alpha'+\beta-1, \alpha+\alpha'-1, \alpha'+\beta-1\}$ and $\beta' > \max\{-1, \alpha'-1\}$.

Remark 2.1. The relations between Appell function (2.3) and Gaussian hypergeometric function ${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty}((a)_n(b)_n z^n)/((c)_n n!)$ for $a,b,c \in \mathbb{R}$ have been pointed out as follows

(A1) $F_3(\alpha,\gamma-\alpha,\beta,\gamma-\beta;\gamma;x,y) = {}_2F_1(\alpha,\beta;\gamma;x+y-xy)$.

(A2) $F_3(\alpha,0,\beta,\beta';\gamma;x,y) = {}_2F_1(\alpha,\beta;\gamma;x)$ and $F_3(0,\alpha',\beta,\beta';\gamma;x,y) = {}_2F_1(\alpha',\beta';\gamma;y)$.

Remark 2.2. The Marichev-Saigo-Maeda fractional integral operators given in (2.1) and (2.2) concretely produce some known fractional integral operators according to the different settings of the function $\mathscr{F}_3$.

(B1) Let $\alpha' = 0$, then, from Remark 2.1, the operator presented in (2.1) reduces to the following left-sided Saigo fractional integral operators [13]

$$(\mathscr{I}^{\alpha,\beta,\gamma}_{0,x}f)(x) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,0,0,x}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\gamma;\alpha;1-\tfrac{x}{t})f(t)dt \ \text{ for } \alpha > 0. \tag{2.7}$$

(B2) Let $\alpha' = \beta = 0$, then, from Remark 2.1, the operator defined in (2.1) develops into the following left-sided Erdélyi-Kober fractional integral operators [10]

$$(\mathscr{I}^{\alpha,\gamma}_{0,x}f)(x) = (\mathscr{I}^{\alpha,0,\gamma}_{0,x}f)(x) = \frac{x^{-\alpha-\gamma}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\gamma}f(t)dt \ \text{ for } \alpha > 0. \tag{2.8}$$

(B3) Let $\alpha' = 0$ and $\beta = -\alpha$, then the operators given in (2.1) and (2.2) are converted to the well-known classical left and right-sided Riemann-Liouville fractional integrals in the literature [6].

For convenience, we suppose always that the following assumptions hold throughout this paper.

(C1) $-1 < (1 - t/x) < 0$ and $0 < (1 - x/t) < 1/2$, $(fg)(x) = f(x)g(x)$ and $f^2(x) = (f(x))^2$.

(C2) $\gamma > \max\{\alpha, \alpha', \beta, \beta'\} > 0$ and $\delta > \max\{\mu, \mu', \nu, \nu'\} > 0$ for $\alpha, \alpha', \beta, \beta', \mu, \mu', \nu, \nu', \gamma, \delta \in \mathbb{R}$.

Remark 2.3. From [5, Theorem 1] and the given assumptions above, we can know easily that $(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} f)(x) > 0$ and $(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} f)(x) > 0$ for $f(x) > 0$. Therefore, the aforementioned assumptions play an important role in the proofs of main results.

3. Weighted Marichev-Saigo-Maeda fractional Young and Plóya-Szegö-type integral inequalities

In this section, we establish firstly some new weighted Young type integral inequalities involving the left-sided Marichev-Saigo-Maeda fractional integral operators.

Theorem 3.1. *Assume that f and g are two positive integrable functions on $[0,\infty)$ and u and v two nonnegative continuous functions on $[0,\infty)$. When $1/\mathbb{p} + 1/\mathbb{q} = 1$ with $\mathbb{p}, \mathbb{q} > 1$, then we have the following inequality*

$$\begin{aligned}&(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v)(x)/\mathbb{p} + (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^{\mathbb{q}})(x)/\mathbb{q} \geq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg)(x)\\ &+ \mathbb{r}_0\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v)(x) - 2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}/2})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^{\mathbb{q}/2})(x)\\ &\quad + (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^{\mathbb{q}})(x)\Big) \geq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg)(x) \text{ for } \mathbb{r}_0 = \min\{1/\mathbb{p}, 1/\mathbb{q}\}. \end{aligned} \tag{3.1}$$

Proof. According to the inequalities (1.1) and (1.3), we have the following inequalities

$$\mathbb{x}^{\mathbb{p}}/\mathbb{p} + \mathbb{y}^{\mathbb{q}}/\mathbb{q} \geq \mathbb{x}\mathbb{y} + \mathbb{r}_0(\sqrt{\mathbb{x}^{\mathbb{p}}} - \sqrt{\mathbb{y}^{\mathbb{p}}})^2 \geq \mathbb{x}\mathbb{y} \quad \text{for } \mathbb{x}, \mathbb{y} \geq 0,\ 1/\mathbb{p} + 1/\mathbb{q} = 1 \text{ with } \mathbb{p}, \mathbb{q} > 1, \tag{3.2}$$

where $\mathbb{r}_0 = \min\{1/\mathbb{p}, 1/\mathbb{q}\}$. Setting $\mathbb{x} = f(\tau)$ and $\mathbb{y} = g(\rho)$ in (3.2), we can obtain

$$f^{\mathbb{p}}(\tau)/\mathbb{p} + g^{\mathbb{q}}(\rho)/\mathbb{q} \geq f(\tau)g(\rho) + \mathbb{r}_0\Big(f^{\mathbb{p}}(\tau) + g^{\mathbb{q}}(\rho) - 2f^{\mathbb{p}/2}(\tau)g^{\mathbb{q}/2}(\rho)\Big) \geq f(\tau)g(\rho). \tag{3.3}$$

Multiplied by $v(\rho)(x^{-\mu}/\Gamma(\delta))(x-\rho)^{\delta-1}\rho^{-\mu'}F_3(\mu,\mu',\nu,\nu';\delta;1-\rho/x,1-x/\rho)$ and $u(\tau)(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha, \alpha', \beta, \beta'; \gamma; 1-\tau/x, 1-x/\tau)$ on both sides of (3.3) and integrated the presented result with respect to ρ and τ from 0 to x and 0 to x, respectively, we write

$$\begin{aligned}&(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v)(x)/\mathbb{p} + (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^{\mathbb{q}})(x)/\mathbb{q} \geq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg)(x)\\ &+ \mathbb{r}_0\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v)(x) - 2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}/2})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^{\mathbb{q}/2})(x)\\ &\quad + (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^{\mathbb{q}})(x)\Big) \geq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg)(x), \end{aligned} \tag{3.4}$$

which implies the desired inequality (3.1). The proof of Theorem 3.1 is completed. □

Remark 3.1. Along the proof of Theorem 3.1, by setting (D1) $\mathbb{x} = f(\tau)g(\rho)$ and $\mathbb{y} = f(\rho)g(\tau)$; (D2) $\mathbb{x} = f(\tau)g^{2/\mathbb{p}}(\rho)$ and $\mathbb{y} = f^{2/\mathbb{q}}(\rho)g(\tau)$; (D3) $\mathbb{x} = f^{2/\mathbb{p}}(\tau)g(\rho)$ and $\mathbb{y} = f^{2/\mathbb{q}}(\rho)g(\tau)$; (D4) $\mathbb{x} = f(\tau)/f(\rho)$ and $\mathbb{y} = g(\tau)/g(\rho)$, $f(\rho) \neq 0$, $g(\rho) \neq 0$; (D5) $\mathbb{x} = f(\tau)/g(\tau)$ and $\mathbb{y} = f(\rho)/g(\rho)$, $g(\tau) \neq 0$, $g(\rho) \neq 0$; (D6) $\mathbb{x} = f^{2/\mathbb{p}}(\tau)/f(\rho)$ and $\mathbb{y} = g^{2/\mathbb{q}}(\tau)/g(\rho)$, $f(\rho) \neq 0$, $g(\rho) \neq 0$; (D7) $\mathbb{x} = f^{2/\mathbb{p}}(\tau)/g(\tau)$ and $\mathbb{y} = f^{2/\mathbb{q}}(\rho)/g(\rho)$, $g(\tau) \neq 0$, $g(\rho) \neq 0$ for $\tau, \rho \in [0,\infty)$ in (3.2), we can obtain some fractional integral inequalities similar to inequality (3.1).

Nextly, we give some new weighted arithmetic-geometric mean type integral inequalities involving the left-sided Marichev-Saigo-Maeda fractional integral operators.

Theorem 3.2. *Assume that f and g are two positive integrable functions defined on $[0,\infty)$ and u and v two nonnegative continuous functions on $[0,\infty)$. When $\mathbb{p}+\mathbb{q}=1$ with $\mathbb{p},\mathbb{q}>0$, then we have the following inequality*

$$\begin{aligned}\mathbb{p}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)+\mathbb{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x)\geq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^{\mathbb{q}})(x)\\+\mathbb{r}_0\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)-2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^{1/2})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^{1/2})(x)\\+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x)\Big)\geq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^{\mathbb{q}})(x)\ \text{for}\ \mathbb{r}_0=\min\{\mathbb{p},\mathbb{q}\}.\end{aligned}\tag{3.5}$$

Proof. According to the enhanced weighted arithmetic-geometric mean inequality (1.3), we get

$$\mathbb{p}\mathbb{x}+\mathbb{q}\mathbb{y}\geq\mathbb{x}^{\mathbb{p}}\mathbb{y}^{\mathbb{q}}+\mathbb{r}_0(\sqrt{\mathbb{x}}-\sqrt{\mathbb{y}})^2\geq\mathbb{x}^{\mathbb{p}}\mathbb{y}^{\mathbb{q}}\ \text{ for }\ \mathbb{x},\mathbb{y}\geq0,\ \ \mathbb{p}+\mathbb{q}=1\ \text{ with }\ \mathbb{p},\mathbb{q}>0,\tag{3.6}$$

where $\mathbb{r}_0=\min\{\mathbb{p},\mathbb{q}\}$. setting $\mathbb{x}=f(\tau)$ and $\mathbb{y}=g(\rho)$ in (3.6), we can obtain

$$\mathbb{p}f(\tau)+\mathbb{q}g(\rho)\geq f^{\mathbb{p}}(\tau)g^{\mathbb{q}}(\rho)+\mathbb{r}_0(f(\tau)+g(\rho)-2f^{1/2}(\tau)g^{1/2}(\rho))\geq f^{\mathbb{p}}(\tau)g^{\mathbb{q}}(\rho).\tag{3.7}$$

Multiplied by $v(\rho)(x^{-\mu}/\Gamma(\delta))(x-\rho)^{\delta-1}\rho^{-\mu'}F_3(\mu,\mu',\nu,\nu';\delta;1-\rho/x,1-x/\rho)$ and $u(\tau)(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ on both sides of (3.7) and integrated the presented result with respect to ρ and τ from 0 to x and 0 to x, respectively, we acquire

$$\begin{aligned}\mathbb{p}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)\mathbb{q}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x)\geq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^{\mathbb{q}})(x)\\+\mathbb{r}_0\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)\Big(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)-2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^{1/2})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^{1/2})(x)\\+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x)\Big)\geq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^{\mathbb{p}})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^{\mathbb{q}})(x),\end{aligned}\tag{3.8}$$

which exhibits the desired inequality (3.5). The proof of Theorem 3.2 is completed. □

Remark 3.2. Along the proof of Theorem 3.2, by setting (E1) $\mathbb{x}=f(\tau)g(\rho)$ and $\mathbb{y}=f(\rho)g(\tau)$; (E2) $\mathbb{x}=f(\tau)g^{2/\mathbb{p}}(\rho)$ and $\mathbb{y}=f^{2/\mathbb{q}}(\rho)g(\tau)$; (E3) $\mathbb{x}=f^{2/\mathbb{p}}(\tau)g(\rho)$ and $\mathbb{y}=f^{2/\mathbb{q}}(\rho)g(\tau)$; (E4) $\mathbb{x}=f(\tau)/f(\rho)$ and $\mathbb{y}=g(\tau)/g(\rho)$, $f(\rho)\neq0$, $g(\rho)\neq0$; (E5) $\mathbb{x}=f(\tau)/g(\tau)$ and $\mathbb{y}=f(\rho)/g(\rho)$, $g(\tau)\neq0$, $g(\rho)\neq0$; (E6) $\mathbb{x}=f^{2/\mathbb{p}}(\tau)/f(\rho)$ and $\mathbb{y}=g^{2/\mathbb{q}}(\tau)/g(\rho)$, $f(\rho)\neq0$, $g(\rho)\neq0$; (E7) $\mathbb{x}=f^{2/\mathbb{p}}(\tau)/g(\tau)$ and $\mathbb{y}=f^{2/\mathbb{q}}(\rho)/g(\rho)$, $g(\tau)\neq0$, $g(\rho)\neq0$ for $\tau,\rho\in[0,\infty)$ in (3.6), we can obtain some fractional integral inequalities similar to inequality (3.5).

Now, we present some new weighted Pólya-Szegö type integral inequalities involving the left-sided Marichev-Saigo-Maeda fractional integral operators.

Theorem 3.3. *Assume that f and g are two positive integrable functions such that*

$$\mathbb{m}=\min_{\tau\in[a,x]}\{f(\tau)/g(\tau)\}\ \text{ and }\ \mathbb{M}=\max_{\tau\in[a,x]}\{f(\tau)/g(\tau)\}\ \text{ for }\ x\in[0,\infty).\tag{3.9}$$

And let u be a nonnegative continuous function on $[0,\infty)$. Then we have the following inequalities

$$0\leq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\leq[(\mathbb{m}+\mathbb{M})^2/(4\mathbb{m}\mathbb{M})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x),\tag{3.10}$$

$$0\leq\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)}\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}-(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)\leq\frac{(\sqrt{\mathbb{M}}-\sqrt{\mathbb{m}})^2}{2\sqrt{\mathbb{m}\mathbb{M}}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x),\tag{3.11}$$

$$0\leq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)-(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x)\leq[(\mathbb{M}-\mathbb{m})^2/(4\mathbb{m}\mathbb{M})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x),\tag{3.12}$$

$$0\leq[(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)]-[(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)]\leq(\sqrt{\mathbb{M}}-\sqrt{\mathbb{m}})^2.\tag{3.13}$$

Proof. According to the conditions (3.9), we can see

$$(f(\tau)/g(\tau)-\mathbb{m})(\mathbb{M}-f(\tau)/g(\tau))g^2(\tau)\geq0\ \text{ for }\ 0\leq\tau\leq x,\ \ x\in[0,\infty).\tag{3.14}$$

Multiplied by $u(\tau)(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ on both sides of (3.14) and integrated the resulting inequality corresponding to τ from 0 to x, we can establish the following weighted Diaz-Metcalf inequality of first type

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)+\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\leq(\mathbb{m}+\mathbb{M})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x). \tag{3.15}$$

On the other hand, it follows from $\mathbb{m}\mathbb{M}>0$ and

$$\left(\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)}-\sqrt{\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}\right)^2\geq 0 \ \text{ for } \ x\in[0,\infty) \tag{3.16}$$

that we observe

$$2\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)}\sqrt{\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}\leq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)+\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x). \tag{3.17}$$

On the basis of the inequalities (3.15) and (3.17), we can demonstrate

$$2\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)}\sqrt{\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}\leq(\mathbb{m}+\mathbb{M})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x). \tag{3.18}$$

Squaring on both sides of inequality (3.18), we acquire

$$4\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\leq(\mathbb{m}+\mathbb{M})^2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x), \tag{3.19}$$

which implies (3.10). The inequality (3.18) can be represented as

$$\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)}\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}\leq[(\mathbb{m}+\mathbb{M})/(2\sqrt{\mathbb{m}\mathbb{M}})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x). \tag{3.20}$$

Subtracting $(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)$ both sides of (3.20), we obtain the inequality (3.11). To give (3.12), subtracting $(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x)$ both sides of (3.10) and proving by the same proof method used in (3.11). Next, to obtain (3.13), we suppose that there exists two positive integral functions $\mathbb{x}(\tau)$ and $\mathbb{u}(\tau)$ satisfying $\mathbb{m}\leq\mathbb{x}(\tau)\leq\mathbb{M}$ for any $\tau\in[0,x]$ and $(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u})(x)=1$. Then for any $\tau\in[0,x]$, we have the following inequalities

$$(\mathbb{x}(\tau)-\mathbb{m})(\mathbb{x}(\tau)-\mathbb{M})\mathbb{x}^{-1}(\tau)\leq 0\xrightarrow{\text{multiply by }\mathbb{u}(\tau)}\mathbb{u}(\tau)\mathbb{x}(\tau)\leq(\mathbb{m}+\mathbb{M})\mathbb{u}(\tau)-\mathbb{m}\mathbb{M}\mathbb{u}(\tau)\mathbb{x}^{-1}(\tau). \tag{3.21}$$

Multiplied by $u(\tau)(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ on both side of (3.21), integrated the given equation with respect to τ from 0 to x, and subtracted from both sides of the obtained inequality, we can obtain the following inequality

$$\begin{aligned}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x})(x)-(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x}^{-1})^{-1}(x)&\leq\mathbb{m}+\mathbb{M}-\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x}^{-1})(x)-(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x}^{-1})^{-1}(x)\\&=\mathbb{m}+\mathbb{M}-2\sqrt{\mathbb{m}\mathbb{M}}-\left(\sqrt{\mathbb{m}\mathbb{M}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x}^{-1})^{1/2}(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x}^{-1})^{-1/2}(x)\right)^2\leq(\sqrt{\mathbb{M}}-\sqrt{\mathbb{m}})^2.\end{aligned} \tag{3.22}$$

Setting $\mathbb{u}(\tau)=u(\tau)f(\tau)g(\tau)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)$ and $\mathbb{x}(\tau)=f(\tau)/g(\tau)$ in (3.22), we can deduce (3.13). The proofs of Theorem 3.3 are completed. □

Corollary 3.1. *Under the assumptions of Theorem 3.3, if $\mathbb{p}+\mathbb{q}=1$ with $\mathbb{p},\mathbb{q}>0$, then it follows from the arithmetic-geometric mean inequality (3.6) that*

$$\begin{aligned}[(\mathbb{m}\mathbb{M})^{\mathbb{q}}/(\mathbb{p}^{\mathbb{p}}\mathbb{q}^{\mathbb{q}})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)^{\mathbb{p}}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)^{\mathbb{q}}(x)&\leq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)+\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\\&\leq(\mathbb{m}+\mathbb{M})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x),\end{aligned} \tag{3.23}$$

which implies further that the following inequality holds

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)^{\mathbb{p}}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)^{\mathbb{q}}(x)\leq[\mathbb{p}^{\mathbb{p}}\mathbb{q}^{\mathbb{q}}(\mathbb{m}+\mathbb{M})/(\mathbb{m}\mathbb{M})^{\mathbb{q}}](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x). \tag{3.24}$$

Remark 3.3. Under the assumptions of Theorem 3.3, let F be a positive integrable function on $[0,\infty)$, if $f = F^{-1/2}$ and $g = F^{1/2}$, then it follows from Theorem 3.3 that we have the follow results

$$0 \leq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}(u/F))(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uF)(x) \leq [(\mathbb{m}+\mathbb{M})^2/(4\mathbb{m}\mathbb{M})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)^2(x), \tag{3.25}$$

$$0 \leq \sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}(u/F))(x)}\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uF)(x)} - (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x) \leq [(\sqrt{\mathbb{M}}-\sqrt{\mathbb{m}})^2/(2\sqrt{\mathbb{m}\mathbb{M}})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x), \tag{3.26}$$

$$0 \leq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}(u/F))(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uF)(x) - (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)^2(x) \leq [(\mathbb{M}-\mathbb{m})^2/(4\mathbb{m}\mathbb{M})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)^2(x), \tag{3.27}$$

$$0 \leq [(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}(u/F))(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)] - [(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uF)(x)] \leq (\sqrt{\mathbb{M}}-\sqrt{\mathbb{m}})^2. \tag{3.28}$$

Conversely, if we take $F = g/f$ and $u = ufg$, then the inequalities (3.25)-(3.28) are reduced to the inequalities (3.10)-(3.13), respectively. Therefore, the inequalities (3.10)-(3.13) and (3.25)-(3.28) are equivalent, respectively, if F is a positive integrable function on $[0,\infty)$.

Theorem 3.4. *Assume that f and g are two positive integrable functions satisfying the condition (1.5) on $[0,x]$ and u a nonnegative continuous function on $[0,\infty)$. Then the following inequalities hold:*

$$0 \leq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) \leq [(\Phi_1\Psi_1+\Phi_2\Psi_2)^2/(4\Phi_1\Psi_1\Phi_2\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x), \tag{3.29}$$

$$0 \leq \sqrt{({}_{\xi}\mathscr{F}^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}uf^2)(x)}\sqrt{({}_{\xi}\mathscr{F}^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}ug^2)(x)} - ({}_{\xi}\mathscr{F}^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}ufg)(x)$$
$$\leq [(\sqrt{\Phi_2\Psi_2}-\sqrt{\Phi_1\Psi_1})^2/(2\sqrt{\Phi_1\Psi_1\Phi_2\Psi_2})]({}_{\xi}\mathscr{F}^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}ufg)(x), \tag{3.30}$$

$$0 \leq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) - (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x) \leq \frac{(\Phi_2\Psi_2-\Phi_1\Psi_1)^2}{4\Phi_1\Psi_1\Phi_2\Psi_2}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x), \tag{3.31}$$

$$0 \leq \frac{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)}{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)} - \frac{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)}{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)} \leq \left(\sqrt{\frac{\Phi_2}{\Psi_1}}-\sqrt{\frac{\Phi_1}{\Psi_2}}\right)^2 \text{ (Shisha−Mond type inequality).} \tag{3.32}$$

Proof. Since f and g satisfy the condition (1.1), we have

$$\Phi_1/\Psi_2 \leq f(\tau)/g(\tau) \leq \Phi_2/\Psi_1. \tag{3.33}$$

According to Theorem 3.3, we obtain the inequality (3.29) and, applying it, we have (3.30)-(3.32). □

Corollary 3.2. *Suppose f is a positive integrable function satisfying the condition (1.5) on $[0,x]$. Then we have*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}1)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}f^2)(x) \leq (\Phi_1+\Phi_2)^2/(4\Phi_1\Phi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}f)^2(x). \tag{3.34}$$

Lemma 3.1. *Suppose $\mathbb{x}$ is a continuous function on $[0,x] \to [\mathbb{m},\mathbb{M}]$ and $\Upsilon : \mathbb{I} \to \mathbb{R}$ a convex (concave) function with $[\mathbb{m},\mathbb{M}] \subseteq \mathbb{I}$ and $\mathbb{u}$ a nonnegative continuous function on $[0,\infty)$. Then the following inequality holds true*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\Upsilon(\mathbb{x}))(x) \leq (\geq)\frac{\mathbb{M}\Upsilon(\mathbb{m})-\mathbb{m}\Upsilon(\mathbb{M})}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u})(x) + \frac{\Upsilon(\mathbb{M})-\Upsilon(\mathbb{m})}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x})(x). \tag{3.35}$$

Proof. On the basis of the definition of convexity (concavity), we can get

$$\Upsilon\left(\frac{\mathfrak{M}\mathbb{a}+\mathfrak{N}\mathbb{b}}{\mathfrak{M}+\mathfrak{N}}\right) \leq (\geq)\frac{\mathfrak{M}\Upsilon(\mathbb{a})+\mathfrak{N}\Upsilon(\mathbb{b})}{\mathfrak{M}+\mathfrak{N}} \text{ for } \mathfrak{M}+\mathfrak{N}>0 \text{ with } \mathfrak{M},\mathfrak{N}\geq 0. \tag{3.36}$$

Since $\mathbb{x}$ is a continuous function on $[0,x] \to [\mathbb{m},\mathbb{M}]$, we observe for $\tau \in [a,b]$ that

$$\mathbb{x}(\tau) = ((\mathbb{M}-\mathbb{x}(\tau))\mathbb{m} + (\mathbb{x}(\tau)-\mathbb{m})\mathbb{M})/(\mathbb{M}-\mathbb{m}). \tag{3.37}$$

Putting $\mathfrak{M} = \mathbb{M}-\mathbb{x}(\tau)$, $\mathfrak{N} = \mathbb{x}(\tau)-\mathbb{m}$, $\mathbb{a} = \mathbb{m}$ and $\mathbb{b} = \mathbb{M}$, it follows from (3.36) that

$$\Upsilon(\mathbb{x}(\tau)) = \Upsilon\left(\frac{(\mathbb{M}-\mathbb{x}(t))\mathbb{m}+(\mathbb{x}(\tau)-\mathbb{m})\mathbb{M}}{\mathbb{M}-\mathbb{m}}\right) \leq (\geq)\frac{(\mathbb{M}-\mathbb{x}(\tau))\Upsilon(\mathbb{m})+(\mathbb{x}(\tau)-\mathbb{m})\Upsilon(\mathbb{M})}{\mathbb{M}-\mathbb{m}}. \tag{3.38}$$

Multiplied by $u(\tau)(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ on both side of (3.38), integrating the resulting inequality with respect to τ from 0 to x, we get

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\Upsilon(\mathbb{x}))(x) \leq (\geq)\frac{\mathbb{M}\Upsilon(\mathbb{m})-\mathbb{m}\Upsilon(\mathbb{M})}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u})(x) + \frac{\Upsilon(\mathbb{M})-\Upsilon(\mathbb{m})}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\mathbb{u}\mathbb{x})(x), \tag{3.39}$$

which implies (3.35). This completes the proof of Lemma 3.1. □

Lemma 3.2. *Assume f and g are two continuous such that $f/g : [0,x] \to [\mathbb{m},\mathbb{M}]$ with $g(t) \neq 0$ for $t \in [0,x]$ and $\Upsilon : \mathbb{I} \to \mathbb{R}$ a convex (concave) with $[\mathbb{m},\mathbb{M}] \subseteq \mathbb{I}$ and u a nonnegative continuous on $[0,\infty)$. Then we have*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2\Upsilon((f/g)))(x) \leq (\geq) \tfrac{\mathbb{M}\Upsilon(\mathbb{m})-\mathbb{m}\Upsilon(\mathbb{M})}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) + \tfrac{\Upsilon(\mathbb{M})-\Upsilon(\mathbb{m})}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.40}$$

Proof. It follows from Lemma 3.1 for the choices $\mathbb{u} = ug^2$ and $\mathbb{x} = f/g$ that we have the above inequality. □

Theorem 3.5. *Assume f and g are two positive integrable functions satisfying the condition (3.9) and u a nonnegative continuous on $[0,\infty)$. If $\mathbb{p} \in (-\infty,0) \cup [1,+\infty)$ ($\mathbb{p} \in (0,1)$), then the following inequality holds*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}}g^{2-\mathbb{p}})(x) + \tfrac{\mathbb{m}\mathbb{M}^{\mathbb{p}}-\mathbb{M}\mathbb{m}^{\mathbb{p}}}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) \leq (\geq) \tfrac{\mathbb{M}^{\mathbb{p}}-\mathbb{m}^{\mathbb{p}}}{\mathbb{M}-\mathbb{m}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.41}$$

Especially, for $\mathbb{p} = 2$, we have weighted the Diaz-Metcalf inequality of first type (3.14), that is,

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x) + \mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) \leq (\mathbb{m}+\mathbb{M})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.42}$$

Proof. Let $\Upsilon(x) = x^{\mathbb{p}}$ in (3.40), $\mathbb{p} \in (-\infty,0) \cup [1,+\infty)$ ($\mathbb{p} \in (0,1)$), we can obtain (3.41). □

Corollary 3.3. *Assume f and g are two positive integrable functions satisfying the condition (1.5) on $[0,x]$ and u a nonnegative continuous function on $[0,\infty)$. If $\mathbb{p} \in (-\infty,0) \cup [1,+\infty)$ ($\mathbb{p} \in (0,1)$), then we have*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbb{p}}g^{2-\mathbb{p}})(x) + \tfrac{\Phi_1\Phi_2(\Phi_2^{\mathbb{p}-1}\Psi_2^{\mathbb{p}-1}-\Phi_1^{\mathbb{p}-1}\Psi_1^{\mathbb{p}-1})}{\Psi_1^{\mathbb{p}-1}\Psi_2^{\mathbb{p}-1}(\Phi_2\Psi_2-\Phi_1\Psi_1)}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) \leq (\geq) \tfrac{\Phi_2^{\mathbb{p}}\Psi_2^{\mathbb{p}}-\Phi_1^{\mathbb{p}}\Psi_1^{\mathbb{p}}}{\Psi_1^{\mathbb{p}-1}\Psi_2^{\mathbb{p}-1}(\Phi_2\Psi_2-\Phi_1\Psi_1)}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x).$$

Especially, for $\mathbb{p} = 2$, the following weighted Diaz-Metcalf inequality of second type holds true

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x) + [(\Phi_1\Phi_2)/(\Psi_1\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) \leq [(\Phi_2/\Psi_1) + (\Phi_1/\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.43}$$

Proof. Applying (3.33) and Theorem 3.5, we can easily obtain the results of the above corollary. □

Remark 3.4. We can easily derive that the following inequalities $\left(\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x)} - \sqrt{\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x)}\right)^2 \geq 0$ and $\left(\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x)} - \sqrt{[(\Phi_1\Phi_2)/(\Psi_1\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x)}\right)^2 \geq 0$ hold obviously. These inequalities combining with (3.42) and (3.43) produce immediately the Plóya-Szegö type inequalities (3.10) and (3.29), respectively.

Lemma 3.3. *Assume $\mathbb{x}$ is a continuous function on $[0,x] \to [\mathbb{m},\mathbb{M}]$ and $\Upsilon : \mathbb{I} \to \mathbb{R}$ a convex (concave) function with $[\mathbb{m},\mathbb{M}] \subseteq \mathbb{I}$ and $\mathbb{u}$ a nonnegative continuous function on $[0,\infty)$. Then the following inequality holds true*

$$\Upsilon\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\mathbb{x})(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x)\Big) \leq (\geq) (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\Upsilon(\mathbb{x}))(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x). \tag{3.44}$$

Proof. Because Υ is convex (concave), from [1], there exists $\mathbb{a}_{\mathbb{t}} \in \mathbb{R}$ for $\mathbb{t} \in [\mathbb{m},\mathbb{M}]$ such that $\mathbb{a}_{\mathbb{t}}(\mathbb{x}-\mathbb{t}) \leq (\geq) \Upsilon(\mathbb{x}) - \Upsilon(\mathbb{t})$ for $\forall \mathbb{x} \in [\mathbb{m},\mathbb{M}]$. Here letting $\mathbb{t} = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\mathbb{x})(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x) \xRightarrow{\text{since } \mathbb{x}:[0,x]\to[\mathbb{m},\mathbb{M}]} \mathbb{t} \in [\mathbb{m},\mathbb{M}]$. According to the previous two equations, we can derive

$$\begin{aligned}
&(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\Upsilon(\mathbb{x}))(x) - (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x)\Upsilon\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\mathbb{x})(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x)\Big)\\
&\quad= (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\Upsilon(\mathbb{x}))(x) - (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x)\Upsilon(\mathbb{t}) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}(\Upsilon(\mathbb{x}) - \Upsilon(\mathbb{t})))(x)\\
&\quad\geq (\leq) \mathbb{a}_{\mathbb{t}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}(\mathbb{x}-\mathbb{t}))(x) = \mathbb{a}_{\mathbb{t}}\left((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u}\mathbb{x})(x) - \mathbb{t}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} \mathbb{u})(x)\right) = 0,
\end{aligned} \tag{3.45}$$

which yields immediately to the desired inequality (3.44). □

Lemma 3.4. *Assume f and g are two continuous functions such that $f/g : [0,x] \to [\mathbb{m},\mathbb{M}]$ with $g(t) \neq 0$ for $t \in [0,x]$ and $\Upsilon : \mathbb{I} \to \mathbb{R}$ a convex (concave) function with $[\mathbb{m},\mathbb{M}] \subseteq \mathbb{I}$ and u a nonnegative continuous function on $[0,\infty)$. Then the following inequality holds true*

$$\Upsilon\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x)\Big) \leq (\geq) (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2\Upsilon(f/g))(x)/(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x). \tag{3.46}$$

Proof. It follows from Lemma 3.3 for the choices $\mathbbm{u} = ug^2$ and $\mathbbm{x} = f/g$ that we get the above inequality. □

Theorem 3.6. *Assume f and g are two positive integrable functions on $[0, x]$ and u a nonnegative continuous function on $[a, b]$. If $\mathbbm{p} \in (-\infty, 0) \cup [1, +\infty)$ ($\mathbbm{p} \in (0, 1)$), then we have the following inequality*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)^{\mathbbm{p}}(x) \le (\ge) (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)^{\mathbbm{p}-1}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^{\mathbbm{p}} g^{2-\mathbbm{p}})(x). \tag{3.47}$$

Especially, for $\mathbbm{p} = 2$, we have the following weighted Cauchy-Schwarz type inequality

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)^2(x) \le (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x). \tag{3.48}$$

Proof. Let $\Upsilon(x) = x^{\mathbbm{p}}$, $\mathbbm{p} \in (-\infty, 0) \cup [1, +\infty)$ ($\mathbbm{p} \in (0, 1)$) in (3.46), we can obtain (3.47). □

Corollary 3.4. *Assume that f is a positive integrable function on $[0, x]$. If $\mathbbm{p} \in (-\infty, 0) \cup [1, +\infty)$ ($\mathbbm{p} \in (0, 1)$), then we have the following inequality*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} f)^{\mathbbm{p}}(x) \le (\ge) (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} 1)^{\mathbbm{p}-1}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} f^{\mathbbm{p}})(x). \tag{3.49}$$

Theorem 3.7. *Assume f and g are two positive integrable functions satisfying the condition (1.5) on $[0, x]$ and u be a nonnegative continuous function on $[0, x]$. If $0 < \mathbbm{p} \le \mathbbm{q} < 1$, $\mathbbm{p} + \mathbbm{q} = 1$, then the following inequalities hold*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)^{\mathbbm{q}}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} (u/f))^{\mathbbm{p}}(x) \le [(\mathbbm{p}\Phi_1 + \mathbbm{q}\Phi_2)/(\Phi_1\Phi_2)^{\mathbbm{p}}](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x), \tag{3.50}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)^{\mathbbm{q}}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)^{\mathbbm{p}}(x) \le [(\mathbbm{p}\Phi_1\Psi_1 + \mathbbm{q}\Phi_2\Psi_2)/((\Phi_1\Phi_2)^{\mathbbm{p}}(\Psi_1\Psi_2)^{\mathbbm{q}})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.51}$$

Proof. It follows from $(\mathbbm{q}f(\tau) - \mathbbm{p}\Phi_1)(f(\tau) - \Phi_2) \le 0$ for $[0, x]$ that we have by simple computation

$$\mathbbm{q}f^2(\tau) - (\mathbbm{p}\Phi_1 + \mathbbm{q}\Phi_2)f(\tau) + \mathbbm{p}\Phi_1\Phi_2 \le 0. \tag{3.52}$$

Multiplied by $u(\tau)/f(\tau)$ on both sides of (3.52), we get

$$\mathbbm{q}u(\tau)f(\tau) - (\mathbbm{p}\Phi_1 + \mathbbm{q}\Phi_2)u(\tau) + \mathbbm{p}\Phi_1\Phi_2\tfrac{u(\tau)}{f(\tau)} \le 0 \Longrightarrow \mathbbm{q}u(\tau)f(\tau) + \mathbbm{p}\Phi_1\Phi_2\tfrac{u(\tau)}{f(\tau)} \le (\mathbbm{p}\Phi_1 + \mathbbm{q}\Phi_2)u(\tau). \tag{3.53}$$

Using the arithmetic-geometric mean inequality (3.6) and (3.53), we acquire

$$\begin{aligned}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)^{\mathbbm{q}}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} (u/f))^{\mathbbm{p}}(x) &= \tfrac{1}{(\Phi_1\Phi_2)^{\mathbbm{p}}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)^{\mathbbm{q}}(x)\Big(\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} (u/f))(x)\Big)^{\mathbbm{p}} \\ &\le \tfrac{1}{(\Phi_1\Phi_2)^{\mathbbm{p}}}\Big(\mathbbm{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf)(x) + \mathbbm{p}\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} (u/f))(x)\Big) \le \tfrac{\mathbbm{p}\Phi_1+\mathbbm{q}\Phi_2}{(\Phi_1\Phi_2)^{\mathbbm{p}}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x),\end{aligned} \tag{3.54}$$

which implies the inequality (3.50).

Substituting ufg and f/g into u and f in (3.50), respectively, and $\Phi_1/\Psi_2 \le f(\tau)/g(\tau) \le \Phi_2/\Psi_1$, we obtain

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)^{\mathbbm{q}}(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)^{\mathbbm{p}}(x) \le [(\mathbbm{p}\Phi_1\Psi_1 + \mathbbm{q}\Phi_2\Psi_2)/((\Phi_1\Phi_2)^{\mathbbm{p}}(\Psi_1\Psi_2)^{\mathbbm{q}})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x), \tag{3.55}$$

which implies (3.51). The proofs of Theorem 3.7 are completed. □

Corollary 3.5. *Assume f and g are two positive integrable functions satisfying the condition (3.9) and u a nonnegative continuous function on $[0, x]$. If $0 < \mathbbm{p} \le \mathbbm{q} < 1$, $\mathbbm{p} + \mathbbm{q} = 1$, then the following inequality holds*

$$\mathbbm{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x) + \mathbbm{p}\mathbbm{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) \le (\mathbbm{p}\mathbbm{m} + \mathbbm{q}\mathbb{M})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.56}$$

Proof. Replaced Φ_1, Φ_2 and $f(\tau)$ by $\mathbbm{m}, \mathbb{M}$ and $f(\tau)/g(\tau)$ in (3.53), then multiplied by $(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'} \times F_3(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \tau/x, 1 - x/\tau)$ on both sides of the obtained result, and integrated the resulting inequality corresponding to τ from 0 to x, then we get (3.56). □

Corollary 3.6. *Assume f and g are two positive integrable functions satisfying the condition (1.5) on $[0, x]$ and u a nonnegative continuous function on $[0, x]$. If $0 < \mathbbm{p} \le \mathbbm{q} < 1$, $\mathbbm{p} + \mathbbm{q} = 1$, then the following inequality holds*

$$\mathbbm{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x) + [(\mathbbm{p}\Phi_1\Phi_2)/(\Psi_1\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ug^2)(x) \le [(\mathbbm{p}\Phi_1/\Psi_2) + (\mathbbm{q}\Phi_2/\Psi_1)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} ufg)(x). \tag{3.57}$$

Proof. It follows from (3.33) and Corollary 3.5 that we can easily obtain the above result. □

Remark 3.5. Under the assumptions of Corollary 3.5, assume that F is a positive integrable function on $[0,x]$. When $f = F^{1/2}$ and $g = F^{-1/2}$, then it follows from Corollary 3.4 that

$$\mathbb{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uF)(x) + \mathbb{p}\mathbb{m}\mathbb{M}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}(u/F))(x) \leq (\mathbb{p}\mathbb{m} + \mathbb{q}\mathbb{M})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x). \tag{3.58}$$

Conversely, when we take $F = f/g$ and $u = ufg$, then the inequality (3.56) is transformed into the inequality (3.58), respectively. Therefore, the inequality (3.56) and (3.58) are equivalent. If $\mathbb{p} = \mathbb{q} = 1/2$ in (3.56), then the inequality (3.56) is developed into the inequality (3.42). That is, the inequality (3.42) can be a special case of the inequality (3.56).

Remark 3.6. If $\mathbb{p} = \mathbb{q} = 1/2$ in (3.50), squaring both sides of the obtained inequality, then the inequality (3.50) converted into the inequality (3.25). In other words, the inequality (3.50) can be a generalization of the inequality (3.25). Applying (3.6) to the left sides of (3.56) with $\mathbb{m} = \Phi_1, \mathbb{M} = \Phi_2$ and (3.57), we can acquire the inequalities (3.50) and (3.51), respectively.

Theorem 3.8. *Assume f and g are two integrable functions on $[0,\infty)$ and u a nonnegative continuous on $[0,\infty)$.*

(L1) *If $(\Phi_2 g(\tau) - \Psi_1 f(\tau))(\Psi_2 f(\tau) - \Phi_1 g(\tau)) \geq 0$ for all $\tau \in [0,x]$ and $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R}$, then we have*

$$\begin{aligned}\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) &\leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)\\ &\leq |\Phi_1\Psi_1 + \Phi_2\Psi_2| \sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}.\end{aligned} \tag{3.59}$$

Moreover, when $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ have the same sign, then we have the following inequalities

$$\sqrt{\tfrac{\Phi_1\Phi_2}{\Psi_1\Psi_2}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) + \sqrt{\tfrac{\Psi_1\Psi_2}{\Phi_1\Phi_2}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) \leq \left(\sqrt{\tfrac{\Phi_2\Psi_2}{\Phi_1\Psi_1}} + \sqrt{\tfrac{\Phi_1\Psi_1}{\Phi_2\Psi_2}}\right)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x), \tag{3.60}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) \leq [(\Phi_1\Psi_1 + \Phi_2\Psi_2)^2/(4\Phi_1\Psi_1\Phi_2\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x). \tag{3.61}$$

(L2) *If $(\Phi_2 g(\tau) - \Psi_1 f(\rho))(\Psi_2 f(\rho) - \Phi_1 g(\tau)) \geq 0$ for all $\tau, \rho \in [0,x]$ and $\Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R}$, then we have*

$$\begin{aligned}\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)&\\ \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x).&\end{aligned} \tag{3.62}$$

(L3) *If $(\Phi_2 g(\tau) - \Psi_1 f(\tau))(\Psi_2 f(\tau) - \Phi_1 g(\tau)) \geq 0$ for all $\tau \in [0,x]$ with $\Phi_1\Phi_2 > 0$ and $\Psi_1\Psi_2 > 0$, then we have*

$$\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x). \tag{3.63}$$

(L4) *If $(\Phi_2 g(\tau) - \Psi_1 f(\rho))(\Psi_2 f(\rho) - \Phi_1 g(\tau)) \geq 0$ for all $\tau, \rho \in [0,x]$ with $\Phi_1\Phi_2 > 0$ and $\Psi_1\Psi_2 > 0$, then we have*

$$\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x). \tag{3.64}$$

Proof. To deduce (L1), it follows from the assumptions that

$$u(\tau)(\Phi_2 g(\tau) - \Psi_1 f(\tau))(\Psi_2 f(\tau) - \Phi_1 g(\tau)) \geq 0 \text{ for } \forall \tau \in [0,x], \tag{3.65}$$

which implies that

$$\Phi_1\Phi_2 u(\tau)g^2(\tau) + \Psi_1\Psi_2 u(\tau)f^2(\tau) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)u(\tau)f(\tau)g(\tau). \tag{3.66}$$

Multiplied by $(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ on both sides of (3.66) and integrated the resulting inequality with respect to τ from 0 to x, we get the left inequality of (3.59). Moreover, by employing the weighted Cauchy-Schwarz type inequality (3.48), we obtain the right inequality of (3.59).

Because $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ have the same sign, then $\Phi_1\Phi_2, \Psi_1\Psi_2, \Phi_1\Psi_1, \Phi_2\Psi_2 > 0$. Multiplying both sides of the following inequality by $1/\sqrt{\Phi_1\Phi_2\Psi_1\Psi_2}$

$$\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x) \tag{3.67}$$

which implies the inequality (3.60). On the other hand, it follows from $\Phi_1\Phi_2, \Psi_1\Psi_2 > 0$ that

$$\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) \geq 2\sqrt{\Phi_1\Psi_1\Phi_2\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)}. \tag{3.68}$$

According to the first inequality of (3.67) and (3.68), we have

$$4\Phi_1\Phi_2\Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)^2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x), \tag{3.69}$$

which implies (3.61).

To obtain (L2), it follows from the assumption that

$$u(\tau)u(\rho)(\Phi_2 g(\tau) - \Psi_1 f(\rho))(\Psi_2 f(\rho) - \Phi_1 g(\tau)) \geq 0 \;\text{ for }\; \forall\tau, \rho \in [0, x], \tag{3.70}$$

which implies that

$$\Phi_1\Phi_2 u(\rho)u(\tau)g^2(\tau) + \Psi_1\Psi_2 p(\tau)p(\rho)f^2(\rho) \leq \Phi_1\Psi_1 u(\rho)f(\rho)u(\tau)g(\tau) + \Phi_2\Psi_2 u(\rho)f(\rho)u(\tau)g(\tau). \tag{3.71}$$

Multiplied by $(x^{-\alpha}/\Gamma(\gamma))^2(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)(x-\rho)^{\gamma-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\rho/x,1-x/\rho)$ on both sides of (3.71) and integrated the obtained result with respect to τ and ρ from 0 to x, respectively, we acquire (3.62).

To obtain (L3) and (L4), from weighted Cauchy-Schwarz type inequality (3.48), we deduce

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x) \leq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x), \;\; (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x) \leq (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x). \tag{3.72}$$

It follows from (3.59) and (3.72) that

$$\begin{aligned}\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x) &\leq \Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\\ &+ \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x),\end{aligned} \tag{3.73}$$

which implies (3.63). Moreover, it follows from (3.62) that we obtain

$$\begin{aligned}\Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x) + \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x) &\leq \Phi_1\Phi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\\ &+ \Psi_1\Psi_2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) \leq (\Phi_1\Psi_1 + \Phi_2\Psi_2)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x),\end{aligned} \tag{3.74}$$

which implies (3.64). □

Let $\mathscr{T}_\pm(f,g,u) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)$, where f and g are two integrable functions on $[0,\infty)$ and u is a nonnegative continuous on $[0,\infty)$. Based on the left-sided Marichev-Saigo-Maeda fractional integral operators, we have $\mathscr{T}(f,g,u) = (1/2)(x^{-\alpha}/\Gamma(\gamma))^2\int_0^x\int_0^x(x-\tau)^{\gamma-1}\tau^{-\alpha'}(x-\rho)^{\gamma-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\rho/x,1-x/\rho)u(\rho)u(\tau)(f(\rho)\pm f(\tau))(g(\rho)\pm g(\tau))d\rho d\tau$. Then the generalized Cauchy-Schwarz type inequality $\mathscr{T}_\pm^2(f,g,u) \leq \mathscr{T}_\pm(f,f,u)\mathscr{T}_\pm(g,g,u)$ holds, i.e.,

$$\begin{aligned}\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)\Big)^2 &\leq \Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)\\ \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x)\Big)\Big((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x) &\pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x)\Big).\end{aligned} \tag{3.75}$$

Theorem 3.9. *Assume f and g are two integrable functions on $[0,\infty)$ and u and v two nonnegative continuous on $[0,\infty)$. Then we have the following inequality*

$$\mathbb{p}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}v)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)+\mathbb{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vg^2)(x)\le 2\mathbb{r}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vg)(x), \quad (3.76)$$

where $\mathbb{r}^2\le\mathbb{pq}$ with $\mathbb{p},\mathbb{q}>0$, $\mathbb{r}\in\mathbb{R}$. Furthermore, under the assumptions of (L1) and (L1), then we have

$$(\Phi_1\Psi_1+\Phi_2\Psi_2)^2\mathscr{T}_+^2(u,f,g)\ge 4\Phi_1\Psi_1\Phi_2\Psi_2\mathscr{T}_+(u,f,f)\mathscr{T}_+(u,g,g), \quad (3.77)$$

$$1\le\mathscr{T}_+(u,f,f)\mathscr{T}_+(u,g,g)/\mathscr{T}_+^2(u,f,g)\le[(\Phi_1\Psi_1+\Phi_2\Psi_2)^2/(4\Phi_1\Psi_1\Phi_2\Psi_2)]. \quad (3.78)$$

Proof. Because $\mathbb{r}^2\le\mathbb{pq}$ for $\mathbb{p},\mathbb{q}>0$ and $\mathbb{r}\in\mathbb{R}$, it follows from arithmetic-geometric mean inequality that

$$\mathbb{p}\mathbb{x}^2+\mathbb{q}\mathbb{y}^2\ge 2\mathbb{r}\mathbb{x}\mathbb{y}\ \text{ for any }\ \mathbb{x},\mathbb{y}\in\mathbb{R}. \quad (3.79)$$

Setting $\mathbb{x}=f(\tau)$ and $\mathbb{y}=g(\rho)$ for $\rho,\rho\in[0,x]$ and multiplied by $u(\tau)v(\rho)$ on both side of the obtained result, we derive

$$\mathbb{p}u(\tau)v(\rho)f^2(\tau)+\mathbb{q}u(\tau)v(\rho)g^2(\rho)\ge 2\mathbb{r}u(\tau)v(\rho)f(\tau)g(\rho)\ \text{ for }\ \tau,\rho\in[0,x]. \quad (3.80)$$

Multiplied by $(x^{-\alpha}/\Gamma(\gamma))^2(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)(x-\rho)^{\gamma-1}\rho^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\rho/x,1-x/\rho)$ on both sides of (3.66) and integrated the given result with respect to τ and ρ from 0 to x respectively, we acquire the inequality (3.76).

To obtain (3.77), from (3.75) and Theorem 3.8, we can give

$$\Phi_1\Phi_2\mathscr{T}_+(u,g,g)+\Psi_1\Psi_2\mathscr{T}_+(u,f,f)\le(\Phi_1\Psi_1+\Phi_2\Psi_2)\mathscr{T}_+(u,f,g). \quad (3.81)$$

It follows from (3.81) that $(\Phi_1\Psi_1+\Phi_2\Psi_2)^2\mathscr{T}_+^2(u,f,g)\ge(\Phi_1\Phi_2\mathscr{T}_+(u,g,g)+\Psi_1\Psi_2\mathscr{T}_+(u,f,f))^2\ge 4\Phi_1\Psi_1\Phi_2\Psi_2$ $\times\mathscr{T}_+(u,f,f)\mathscr{T}_+(u,g,g)$, which gives (3.77). It follows from (3.75) and (3.77) that we can obtain (3.78). □

From (3.76) of Theorem 3.9, we can get directly the following corollary.

Corollary 3.7. *Let f and g be two positive integrable functions on $[0,\infty)$ and let u and v be two nonnegative continuous on $[0,\infty)$. If $\mathbb{r}^2\le\mathbb{pq}$ for $\mathbb{p},\mathbb{q}>0$ and $\mathbb{r}\in\mathbb{R}$, then the following inequalities hold*

$$\mathbb{p}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^3)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vg)(x)+\mathbb{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vg^3)(x)\le 2\mathbb{r}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vg^2)(x),$$

$$\mathbb{p}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2g)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vf)(x)+\mathbb{q}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vfg^2)(x)\le 2\mathbb{r}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}vfg)(x),$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\ge(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x).$$

Theorem 3.10. *Assume f and g are two positive integrable functions satisfying the condition (1.5) on $[0,x]$ and u a nonnegative continuous on $[0,\infty)$. Then we have the following inequalities*

$$|\mathscr{T}_-(u,f,g)|\le(\Phi_2-\Phi_1)/(2\sqrt{\Phi_1\Phi_2})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)\sqrt{\mathscr{T}_-(u,g,g)}, \quad (3.82)$$

$$|\mathscr{T}_-(u,f,g)|\le(\sqrt{\Phi_2}-\sqrt{\Phi_1})\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)}\sqrt{\mathscr{T}_-(u,g,g)}, \quad (3.83)$$

$$|\mathscr{T}_-(u,f,g)|\le[(\Phi_2-\Phi_1)(\Psi_2-\Psi_1)/(4\sqrt{\Phi_1\Phi_2\Psi_1\Psi_2})](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x), \quad (3.84)$$

$$|\mathscr{T}_-(u,f,g)|\le(\sqrt{\Phi_2}-\sqrt{\Phi_1})(\sqrt{\Psi_2}-\sqrt{\Psi_1})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)}. \quad (3.85)$$

Proof. According to the inequality (3.12) and the assumptions of Theorem 3.10, we observe

$$\mathscr{T}_-(u,f,f)\le\tfrac{(\Phi_2-\Phi_1)^2}{4\Phi_1\Phi_2}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x),\quad \mathscr{T}_-(u,g,g)\le\tfrac{(\Psi_2-\Psi_1)^2}{4\Psi_1\Psi_2}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)^2(x). \quad (3.86)$$

Combining (3.75) and (3.86) yield immediately the inequalities (3.82) and (3.84), respectively.

From Shisha-Mond type inequality (3.13) and the assumptions of Theorem 3.10, we deduce

$$\begin{aligned}\mathscr{T}_{-}(u,f,f) &\le (\sqrt{\Phi_2}-\sqrt{\Phi_1})^2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x),\\ \mathscr{T}_{-}(u,g,g) &\le (\sqrt{\Psi_2}-\sqrt{\Psi_1})^2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x).\end{aligned} \tag{3.87}$$

Combining (3.75) and (3.87) yield immediately the inequalities (3.83) and (3.85), respectively. □

Remark 3.7. Let $\mathscr{C}_{\mp}(f,g,u,v) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x) + (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vfg)(x) \mp (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x) \times(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x) \mp (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)$, where f and g are two integrable functions on $[0,x)$ and u and v are two nonnegative continuous on $[0,x)$. Then, we can get the inequalities $\mathscr{C}^2_{\mp}(f,g,u,v) \le \mathscr{C}_{\mp}(f,f,u,v)\mathscr{C}_{\mp}(g,g,u,v)$, which can be seen as the generalizations of inequalities (3.75), respectively.

Lemma 3.5. *Assume f and g are two positive integrable functions on $[0,x)$ and u and v two nonnegative continuous functions on $[0,x)$. Furthermore, assume that there exist four positive integrable functions $\Phi_1,\Phi_2,\Psi_1,\Psi_2$ such that*

$$0<\Phi_1(\tau)\le f(\tau)\le\Phi_2(\tau)\ \ \textit{and}\ \ 0<\Psi_1(\tau)\le g(\tau)\le\Psi_2(\tau)\ \ \textit{for}\ \ \forall\tau\in[0,x). \tag{3.88}$$

Then the following inequalities hold:

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Psi_1\Psi_2f^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2g^2)(x) \le \tfrac{1}{4}((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\Phi_1\Psi_1+\Phi_2\Psi_2)fg)(x))^2, \tag{3.89}$$

$$\frac{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1\Psi_2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)}{((\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1f)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1g)(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_2f)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_2g)(x))^2} \le \frac{1}{4}, \tag{3.90}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x) \le (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_2fg/\Psi_1)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_2fg/\Phi_1)(x). \tag{3.91}$$

Proof. It follows from (3.88) that

$$(\Phi_2(\tau)/\Psi_1(\tau)-f(\tau)/g(\tau))(f(\tau)/g(\tau)-\Phi_1(\tau)/\Psi_2(\tau))\ge 0\ \text{ for }\ \forall\tau\in[0,x], \tag{3.92}$$

which implies that the following inequality holds by multiplying by $\Psi_1(\tau)\Psi_2(\tau)g^2(\tau)$

$$(\Phi_1(\tau)\Psi_1(\tau)+\Phi_2(\tau)\Psi_2(\tau))f(\tau)g(\tau)\ge\Psi_1(\tau)\Psi_2(\tau)f^2(\tau)+\Phi_1(\tau)\Phi_2(\tau)g^2(\tau)\text{ for }\forall\tau\in[0,x]. \tag{3.93}$$

Multiplied by $u(\tau)(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ on both sides of (3.93) and integrated the given result with respect to τ from 0 to x, we obtain

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\Phi_1\Psi_1+\Phi_2\Psi_2)fg)(x)\ge(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Psi_1\Psi_2f^2)(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2g^2)(x). \tag{3.94}$$

Applying the arithmetic-geometric mean inequality to the right of inequality (3.94) results in the following inequality

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\Phi_1\Psi_1+\Phi_2\Psi_2)fg)(x)\ge 2\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Psi_1\Psi_2f^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2g^2)(x)}. \tag{3.95}$$

Squaring both sides of inequality (3.95), we can acquire inequality (3.89). To acquire (3.90), from (3.88), we obtain

$$(\Phi_2(\tau)/\Psi_1(\rho)-f(\tau)/g(\rho))(f(\tau)/g(\rho)-\Phi_1(\tau)/\Psi_2(\rho))\ge 0\ \text{ for }\ \forall\tau,\rho\in[0,x], \tag{3.96}$$

which implies that the following inequality holds by multiplying by $\Psi_1(\rho)\Psi_2(\rho)g^2(\rho)$

$$\Phi_1(\tau)\Psi_1(\rho)f(\tau)g(\rho)+\Phi_2(\tau)\Psi_2(\rho)f(\tau)g(\rho)\ge\Psi_1(\rho)\Psi_2(\rho)f^2(\tau)+\Phi_1(\tau)\Phi_2(\tau)g^2(\rho)\text{ for }\forall\tau,\rho\in[0,x]. \tag{3.97}$$

Multiplied by $(x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)(x^{-\mu}/\Gamma(\delta))(x-\rho)^{\delta-1}\rho^{-\mu'}F_3(\mu,\mu',\nu,\nu';\delta;1-\rho/x,1-x/\rho)u(\tau)v(\rho)$ on both sides of (3.66) and integrated the resulting inequality with respect to τ and ρ from 0 to x respectively, we achieve

$$\begin{aligned}&(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1 f)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1 g)(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_2 f)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_2 g)(x)\\&\qquad\geq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1\Psi_2)(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x).\end{aligned}\tag{3.98}$$

Applying the arithmetic-geometric mean inequality to the right of the previous inequality (3.98) leads to the following inequality

$$\begin{aligned}&(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1 f)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1 g)(x)+(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_2 f)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_2 g)(x)\\&\qquad\geq 2\sqrt{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1\Psi_2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)}\end{aligned}\tag{3.99}$$

Square on both sides of inequality (3.99), we can receive (3.90). To obtain (3.91), from (3.88), we gain

$$f^2(\tau)\leq\Phi_2(\tau)f(\tau)g(\tau)/\Psi_1(\tau)\ \text{ and }\ g^2(\rho)\leq\Psi_2(\rho)f(\rho)g(\rho)/\Phi_1(\rho)\ \text{ for }\ \forall\tau,\rho\in[0,x].\tag{3.100}$$

Multiplied by $u(\tau)x^{-\alpha}/\Gamma(\gamma))(x-\tau)^{\gamma-1}\tau^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\tau/x,1-x/\tau)$ and $v(\rho)(x^{-\mu}/\Gamma(\delta))(x-\rho)^{\delta-1}\rho^{-\mu'}F_3(\mu,\mu',\nu,\nu';\delta;1-\rho/x,1-x/\rho)$ on both sides of two equations in (3.100) and integrated the resulting inequalities with respect to τ and ρ from 0 to x, respectively, we gain

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)\leq(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_2 fg/\Psi_1)(x)\ \text{ and }\ (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)\leq(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_2 fg/\Phi_1)(x).\tag{3.101}$$

Multiplying the two inequalities of (3.101), we derive the desired inequality (3.91). The proofs of Lemma 3.5 are completed. □

It follows from Lemma 3.5 that we can get directly the following corollary.

Corollary 3.8. *Assume f and g are two positive integrable functions satisfying the condition (1.5) on $[0,x]$ and u and v two nonnegative continuous functions on $[0,\infty)$. Then the following inequalities hold:*

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)\leq[(\Phi_1\Psi_1+\Phi_2\Psi_2)^2/(4\Phi_1\Psi_1\Phi_2\Psi_2)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)^2(x)\ \ (\textit{see}\ (3.29)),\tag{3.102}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)\frac{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)}{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)^2(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)^2(x)}\leq\frac{(\Phi_1\Psi_1+\Phi_2\Psi_2)^2}{4\Phi_1\Psi_1\Phi_2\Psi_2},\tag{3.103}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)\leq[(\Phi_2\Psi_2)/(\Phi_1\Psi_1)](\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ufg)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vfg)(x).\tag{3.104}$$

Theorem 3.11. *Assume f and g are two positive integrable functions satisfying the condition (3.88) and u and v two nonnegative continuous functions on $[0,\infty)$. Then the following inequality holds:*

$$|\mathscr{C}_\pm(f,g,u,v)|\leq\sqrt{\mathscr{M}^\pm_1(f,\Phi_1,\Phi_2)+\mathscr{M}^\pm_2(f,\Phi_1,\Phi_2)}\sqrt{\mathscr{M}^\pm_1(g,\Psi_1,\Psi_2)+\mathscr{M}^\pm_2(g,\Psi_1,\Psi_2)},\tag{3.105}$$

where $\mathscr{C}_-(f,g,u,v)$ is defined in Remark 3.7,

$$\begin{aligned}\mathscr{M}^\pm_1(\mathbb{h},\mathbb{A},\mathbb{B})&=(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\mathbb{A}+\mathbb{B})\mathbb{h})^2(x)/(4(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\mathbb{A}\mathbb{B})(x))\pm(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\mathbb{h})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\mathbb{h})(x),\\ \mathscr{M}^\pm_2(\mathbb{h},\mathbb{A},\mathbb{B})&=(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v(\mathbb{A}+\mathbb{B})\mathbb{h})^2(x)/(4(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\mathbb{A}\mathbb{B})(x))\pm(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\mathbb{h})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\mathbb{h})(x).\end{aligned}$$

Proof. It follows from Remark 3.7 that

$$(\mathscr{C}_\pm(f,g,u,v))^2\leq\mathscr{C}_\pm(f,f,u,v)\mathscr{C}_\pm(g,g,u,v),\tag{3.106}$$

where

$$\mathscr{C}_\pm(f,f,u,v) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x) + (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x) \pm 2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf)(x), \quad (3.107)$$

$$\mathscr{C}_\pm(g,g,u,v) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x) + (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x) \pm 2(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x). \quad (3.108)$$

According to the inequality (3.89) with $g(\tau) = \Psi_1(\tau) = \Psi_2(\tau) = 1$, we can derive the following inequalities

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x) \le \frac{(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\Phi_1+\Phi_2)f)^2(x)}{4(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2)(x)}, \quad (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf^2)(x) \le \frac{(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v(\Phi_1+\Phi_2)f)^2(x)}{4(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Phi_1\Phi_2)(x)}, \quad (3.109)$$

which reduces to the following inequalities

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf)(x) \le (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)/(4(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Phi_1\Phi_2)(x)) \cdot (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\Phi_1+\Phi_2)f)^2(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf)(x) = \mathscr{M}_1^\pm(f,\Phi_1,\Phi_2), \quad (3.110)$$

$$(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf)(x) \le (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)/(4(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Phi_1\Phi_2)(x)) \cdot (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v(\Phi_1+\Phi_2)f)^2(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}uf)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vf)(x) = \mathscr{M}_2^\pm(f,\Phi_1,\Phi_2). \quad (3.111)$$

Similarly, putting $f(\tau) = \Phi_1(\tau) = \Phi_2(\tau) = 1$ in the inequality (3.89), we derive the inequalities as follows

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x) \le (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v)(x)/(4(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\Psi_1\Psi_2)(x)) \cdot (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\Psi_1+\Psi_2)g)^2(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x) = \mathscr{M}_1^\pm(g,\Psi_1,\Psi_2), \quad (3.112)$$

$$(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg^2)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x) \le (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)/(4(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v\Psi_1\Psi_2)(x)) \cdot (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}v(\Psi_1+\Psi_2)g)^2(x) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}ug)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x}vg)(x) = \mathscr{M}_2^\pm(g,\Psi_1,\Psi_2). \quad (3.113)$$

Finally, by combining the inequalities (3.106)-(3.108) and (3.110)-(3.113), we can get the desired inequality (3.105). The proof of Theorem 3.11 is completed. □

Corollary 3.9. *Assume f and g are two positive integrable functions satisfying the condition (3.88) and u a nonnegative continuous function on $[0,\infty)$. Then the following inequalities hold:*

$$|\mathscr{T}_\pm(f,g,u)| \le \sqrt{\mathscr{M}_\pm(f,\Phi_1,\Phi_2)}\sqrt{\mathscr{M}_\pm(g,\Psi_1,\Psi_2)}, \quad (3.114)$$

where $\mathscr{S}(f,g,u)$ and $\mathscr{T}(f,g,u)$ are defined as before,

$$\mathscr{M}_\pm(\mathbb{h},\mathbb{A},\mathbb{B}) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u(\mathbb{A}+\mathbb{B})\mathbb{h})^2(x)/(4(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\mathbb{A}\mathbb{B})(x)) \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\mathbb{h})^2(x). \quad (3.115)$$

Proof. Applying Theorem 3.11 for $(\mu,\mu',\nu,\nu',\delta) = (\alpha,\alpha',\beta,\beta',\gamma)$, we get the desired inequalities (3.114). □

Remark 3.8. From Remark 2.2, Lemma 3.5 and Theorem 3.11, our results can reduce into Pólya-Szegö-type integral inequalities for Riemann-Liouville fractional integral operator obtained by Ntouyas *et al.* [11].

4. Applications

In this section, certain estimates of Chebyshev type weighted left-sided Marichev-Saigo-Maeda fractional integral inequalities with two unknown functions are obtained by Lemma 3.5 and Theorem 3.11.

We define the Heaviside unit step function $\ell_\eta(\theta)$ by

$$\ell_\eta(\theta) = 1 \text{ if } \theta > \eta \text{ and } \ell_\eta(\theta) = 0 \text{ if } \theta \leq \eta. \tag{4.1}$$

Applying the above function, we introduce the following piecewise continuous function Φ_1 on $[0, x]$ by

$$\begin{aligned}\Phi_1(\tau) =& \Phi_{1,1}(\ell_{\tau_0}(\tau) - \ell_{\tau_1}(\tau)) + \Phi_{1,2}(\ell_{\tau_1}(\tau) - \ell_{\tau_2}(\tau)) + \Phi_{1,3}(\ell_{\tau_2}(\tau) - \ell_{\tau_3}(\tau)) + \cdots + \Phi_{1,m+1}\ell_{\tau_m}(\tau)\\ =& \Phi_{1,1}\ell_{\tau_0}(\tau) + (\Phi_{1,2} - \Phi_{1,1})\ell_{\tau_1}(\tau) + \cdots + (\Phi_{1,m+1} - \Phi_{1,m})\ell_{\tau_m}(\tau) = \textstyle\sum_{i=0}^m (\Phi_{1,i+1} - \Phi_{1,i})\ell_{\tau_i}(\tau),\end{aligned} \tag{4.2}$$

where $\Phi_{1,0} = 0$ and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = x$. Similarly, we can define three piecewise continuous functions

$$\Phi_2(\tau) = \textstyle\sum_{i=0}^m (\Phi_{2,i+1} - \Phi_{2,i})\ell_{\tau_i}(\tau), \Psi_1(\tau) = \sum_{i=0}^m (\Psi_{1,i+1} - \Psi_{1,i})\ell_{\tau_i}(\tau), \Psi_2(\tau) = \sum_{i=0}^m (\Psi_{2,i+1} - \Psi_{2,i})\ell_{\tau_i}(\tau), \tag{4.3}$$

where $\Phi_{2,0} = \Psi_{1,0} = \Psi_{2,0} = 0$. Let f and g be two positive integrable functions satisfying the condition (3.88) with the functions $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ defined in (4.2) and (4.3), respectively. Then we can derive

$$\Phi_{1,i+1} \leq f(\tau) \leq \Phi_{2,i+1} \text{ and } \Psi_{1,i+1} \leq g(\tau) \leq \Psi_{2,i+1} \text{ for each } \tau \in (\tau_i, \tau_{i+1}],\ i = 0, 1, 2, \ldots, m. \tag{4.4}$$

From Definition 2.1, the left-sided Marichev-Saigo-Maeda fractional integral operator can be written as

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}\psi)(x) = \textstyle\sum_{i=0}^m \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_{\tau_i}^{\tau_{i+1}} (x-t)^{\gamma-1} t^{-\alpha'} \mathscr{F}_3(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t)dt. \tag{4.5}$$

Proposition 4.1. *Assume f and g are two positive integrable functions satisfying the condition (3.88) with the functions Φ_1, Φ_2, Ψ_1, Ψ_2 in (4.2) and (4.3), and u and v two nonnegative continuous on $[0, \infty)$. Then we have*

$$\frac{(\sum_{i=0}^m \Psi_{1,i+1}\Psi_{2,i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} uf^2)(x))(\sum_{i=0}^m \Phi_{1,i+1}\Phi_{2,i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} ug^2)(x))}{(\sum_{i=0}^m(\Phi_{1,i+1}\Psi_{1,i+1}+\Phi_{2,i+1}\Psi_{2,i+1})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} ufg)(x))^2} \leq \frac{1}{4}, \tag{4.6}$$

$$\frac{\Big(\sum_{i=0}^m \Phi_{1,i+1}\Phi_{2,i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} u)(x)\Big)\Big(\sum_{i=0}^m \Psi_{1,i+1}\Psi_{2,i+1}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} v)(x)\Big)(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} vg^2)(x)}{\Big(\Big(\sum_{i=0}^m \Phi_{1,i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} uf)(x)\Big)\Big(\sum_{i=0}^m \Psi_{1,i+1}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} vg)(x)\Big)\Big(\sum_{i=0}^m \Phi_{2,i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} uf)(x)\Big)\Big(\sum_{i=0}^m \Psi_{2,i+1}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} vg)(x)\Big)\Big)^2} \leq \frac{1}{4}, \tag{4.7}$$

$$(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} uf^2)(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} vg^2)(x) \leq \Big(\sum_{i=0}^m \frac{\Phi_{2,i+1}}{\Psi_{1,i+1}}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} ufg)(x)\Big)\Big(\sum_{i=0}^m \frac{\Psi_{2,i+1}}{\Phi_{1,i+1}}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} vfg)(x)\Big). \tag{4.8}$$

Proof. From (4.5), we obtain $(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u\Psi_1\Psi_2 f^2)(x) = \sum_{i=0}^m \Psi_{1,i+1}\Psi_{2,i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} uf^2)(x)$ and $(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v\Psi_1\Psi_2)(x)$ $= \sum_{i=0}^m \Psi_{1,i+1}\Psi_{2,i+1}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} v)(x)$. Similarly, we can obtain other equations. Then substituting the above equalities for the results in Lemma 3.5 yield immediately the desired results (4.6)-(4.8). □

Proposition 4.2. *Assume that f and g are two positive integrable functions satisfying the conditions (3.88) with the functions Φ_1, Φ_2, Ψ_1, Ψ_2 in (4.2) and (4.3), and u and v two nonnegative continuous on $[0, \infty)$. Then we have*

$$|\mathscr{C}_\pm(f, g, u, v)| \leq \sqrt{\mathscr{N}_1^\pm(f, \Phi_1, \Phi_2) + \mathscr{N}_2^\pm(f, \Phi_1, \Phi_2)}\sqrt{\mathscr{N}_1^\pm(g, \Psi_1, \Psi_2) + \mathscr{N}_2^\pm(g, \Psi_1, \Psi_2)}, \tag{4.9}$$

where $\mathscr{C}_-(f, g, u, v)$ is defined in Remark 3.7,

$$\mathscr{N}_1^\pm(\mathbb{h}, \mathbb{A}, \mathbb{B}) = (\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v)(x)\frac{(\sum_{i=0}^m(\mathbb{A}_{i+1}+\mathbb{B}_{i+1})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} u\mathbb{h})(x))^2}{4\sum_{i=0}^m \mathbb{A}_{i+1}\mathbb{B}_{i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}} u)(x)} \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u\mathbb{h})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v\mathbb{h})(x), \tag{4.10}$$

$$\mathscr{N}_2^\pm(\mathbb{h}, \mathbb{A}, \mathbb{B}) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u)(x)\frac{(\sum_{i=0}^m(\mathbb{A}_{i+1}+\mathbb{B}_{i+1})(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} v\mathbb{h})(x))^2}{4\sum_{i=0}^m \mathbb{A}_{i+1}\mathbb{B}_{i+1}(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',\tau_i,\tau_{i+1}} v)(x)} \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x} u\mathbb{h})(x)(\mathscr{I}^{\nu,\nu',\delta}_{\mu,\mu',0,x} v\mathbb{h})(x). \tag{4.11}$$

Proof. It follows from equation (4.5) that we acquire the related equalities. Then substituting obtained equalities for the results in Theorem 3.11 yields immediately the desired result (4.9). □

Corollary 4.1. *Assume that f and g are two positive integrable functions satisfying the condition (3.88) with the functions $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ in (4.2) and (4.3), and u a nonnegative continuous on $[0, \infty)$. Then we have*

$$|\mathscr{T}_{\pm}(f, g, u)| \leq \sqrt{\mathscr{N}_{\pm}(f, \Phi_1, \Phi_2)}\sqrt{\mathscr{N}_{\pm}(g, \Psi_1, \Psi_2)}, \tag{4.12}$$

where $\mathscr{T}_{\pm}(f, g, u)$ is defined as before,

$$\mathscr{N}_{\pm}(\mathbb{h}, \mathbb{A}, \mathbb{B}) = (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u)(x)\frac{(\sum_{i=0}^{m}(\mathbb{A}_{i+1}+\mathbb{B}_{i+1})(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}}u\mathbb{h})(x))^2}{4\sum_{i=0}^{m}\mathbb{A}_{i+1}\mathbb{B}_{i+1}(\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',\tau_i,\tau_{i+1}}u)(x)} \pm (\mathscr{I}^{\beta,\beta',\gamma}_{\alpha,\alpha',0,x}u\mathbb{h})^2(x). \tag{4.13}$$

Proof. Applying Proposition 4.2 for $(\mu, \mu', \nu, \nu', \delta) = (\alpha, \alpha', \beta, \beta', \gamma)$, we can obtain the desired results (4.12). □

Remark 4.1. Based on Remark 2.2, our main results can produce some new weighted Young and Pólya-Szegö-type inequalities for Saigo, Erdélyi-Kober and Riemann-Liouville fractional integral operators, respectively. Meanwhile, some estimates of Chebyshev type weighted left-sided Saigo and Erdélyi-Kober fractional integral inequalities with two unknown functions are also established. Furthermore, some new weighted Young and Pólya-Szegö-type inequalities involving the generalized Marichev-Saigo-Maeda fractional integral operators (2.6) and applications similar to the main results in Sections 3 and 4 can be deduced using conventional methods, respectively.

5. Conclusion

Based on the classical Young and arithmetic-geometric mean inequalities, we have investigated certain new weighted Young and Pólya-Szegö-type inequalities for Marichev-Saigo-Maeda fractional integral operators. Meanwhile, some new related weighted Cauchy-Schwarz type inequalities, weighted Shisha-Mond type inequalities and weighted Diaz-Metcalf type inequalities for Marichev-Saigo-Maeda fractional integral operators haven been also established. As applications, some estimates of Chebyshev-type weighted Marichev-Saigo-Maeda fractional integral inequalities with two unknown functions have been obtained based on the Heaviside unit step function and Pólya-Szegö-type inequalities. The main results of this paper are more general and extend some classical inequalities in the existing literature. Based on the main results of this paper, other fractional integral inequalities and fractional differential systems for Marichev-Saigo-Maeda fractional integral operators will be our future research topics.

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