Filomat 36:15 (2022), 5149–5159
https://doi.org/10.2298/FIL2215149C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Character Space of Vector-Valued Lipschitz Algebras

Sakineh Chameh[a], Ali Rejali[a]

[a]*Department of Pure Mathematics, Faculty of Mathmatics and Statistics, University of Isfahan, Isfahan 81746-37441, Iran*

Abstract. In this paper, we determine the character space of the vector-valued Lipschitz algebra $\mathrm{Lip}_d(X, E)$, where (X, d) is any metric space and E is a certain unital semisimple commutative $*$-Banach algebra.

1. Introduction and preliminaries

Let (X, d) be a metric space and E be a commutative Banach algebra. The E-valued function f on X is called Lipschitz function, if there exists a positive number K such that

$$\|f(x) - f(y)\|_E \leq Kd(x, y) \qquad (x, y \in X).$$

The infimum of such constant K is called the Lipschitz constant of f and will be denoted by $P_{d,E}(f)$. Evidently,

$$P_{d,E}(f) := \sup\{\frac{\|f(x) - f(y)\|_E}{d(x, y)} : x, y \in X \ \ and \ \ x \neq y\}.$$

For any bounded $E-$valued function f, let

$$\|f\|_{\infty,E} = \sup\{\|f(x)\|_E : x \in X\}.$$

Then $\mathrm{Lip}_d(X.E)$ is the collection of all bounded E-valued Lipschitz functions, which is a Banach algebra under pointwise multiplication and the norm

$$\|f\|_{d,E} := \|f\|_{\infty,E} + P_{d,E}(f) \qquad (f \in Lip_d(X, E)).$$

It is important to know that the algebraic properties of these algebras, depend on the algebraic properties of E together with the topological properties of metric space (X, d), see [3, 5, 6]. For example, it is easy to see that $\mathrm{Lip}_d(X, E)$ is a commutative (unital) Banach algebra if and only if E is a commutative (unital) Banach algebra. Note that, whenever $E = \mathbb{C}$, we denote the classical complex-valued Lipschitz algebra by $\mathrm{Lip}_d X$, which was first studied by Sherbert, see [17, 18] for more information. Johnson [12] introduced and studied

2020 *Mathematics Subject Classification*. Primary: 46J10, 46E40; Secondary: 46J05.
Keywords. Banach algebra, Vector-valued Lipschitz function, Vector-valued Lipschitz algebra, Character space, Inverse-closed algebra, Vector-valued Lipschitz compactification.
Received: 09 September 2021; Revised: 05 October 2021; Accepted: 21 March 2022
Communicated by Mohammad Sal Moslehian
Corresponding author: Ali Rejali
Email addresses: S.chameh@sci.ui.ac.ir, Sakineh.chameh@gmail.com (Sakineh Chameh), rejali@sci.ui.ac.ir, ali.rejali2020@gmail.com (Ali Rejali)

some basic properties of vector-valued Lipschitz algebras, see also [3, 8, 11]. Moreover, the amenability of vector-valued Lipschitz algebras was investigated by some authors [1, 2, 10].

Here, we provide some preliminaries, which we require throughout the paper. The character space on a commutative Banach algebra A is the space, consisting of all nonzero multiplicative linear mappings on A, denoted by $\Delta(A)$. We equipe $\Delta(A)$ with the Gelfand topology, which is a relative topology on $\Delta(A)$, induced by the weak*-topology inherited from A^*, where A^* is the dual space of A. If $\Delta(A)$ is not empty, then for every $a \in A$, the mapping $\widehat{a} : \Delta(A) \longrightarrow \mathbb{C}$ is defined by $\widehat{a}(\varphi) = \varphi(a)$ for all $\varphi \in \Delta(A)$. It is called the Gelfand transform of a. The Banach algebra $\mathrm{C}_b(X)$ is the family of all bounded, continuous complex-valued functions on the topological space X. A subalgebra A of $\mathrm{C}_b(X)$ is called inverse-closed if for $f \in A$ with $f(x) \neq 0$, $(x \in X)$, then there is a $g \in A$ such that $f(x)g(x) = 1$, $(x \in X)$.

In this paper, we introduce the term inverse-closed of a Banach subalgebra B with respect to Banach algebra A. Then, we investigate this notion for vector-valued Lipschitz algebras. Afterward, we derive a formula which shows that how the character space of unital commutative Banach algebra and the character space of its unital subalgebra are related to each other.

Let E_1 and E_2 be Banach spaces. Then we denote the injective tensor product by $E_1 \check{\otimes} E_2$. Some authors show it by $E_1 \check{\otimes}_\varepsilon E_2$ where the $\|\cdot\|_\varepsilon$ is the samllest cross norm on $E_1 \otimes E_2$, for more details see [13, 16]. In [13, Theorem 2.11.2] the character space of the tensor product of Banach algebras is charactized. In fact, if A and B are commutative Banach algebras and $A \otimes_\gamma B$ are a tensor product of Banach algebras, for the cross norm $\gamma \geq \varepsilon$, then $\Delta(A \otimes_\gamma B)$ is homeomorphic to the cartesian product $\Delta(A) \times \Delta(B)$.

Honary and et.al. in [9, Theorem 2.1] showed that the character space of vector-valued Lipschitz algebra $\mathrm{Lip}_d(X.E)$, is homeomorphic to the cartesian product $X \times \Delta(E)$, where (X, d) is a compact metric space and E is a unital commutative Banach algebra. It is worth to point out that Essmaeili and Mahyar obtained the same result by some different method for non-unital commutative Banach algebra E and compact metric space (X, d). For more details see [7, Corollary 2.3].

In this paper, we introduce the concept of vector-valued Lipschitz compactification of a metric space (X, d), which is denoted by $l_E(X)$. We prove that $\Delta(Lip_d(X, E)) = l_E(X)$, where (X, d) is a metric space and E is a unital semisimple commutative $*$-Banach algebra, which its character space preserves involution.

2. The vector-valued Lipschitz compactification

In this section, we introduce and study some kinds of compactifications. In particular, the notion of vector-valued Lipschitz compactification of a metric space (X, d), is investigated.

Let (X, d) be a metric space and E be a commutative Banach algebra with a nonempty character space. Let

$$X \otimes \Delta(E) = \{x \otimes \varphi : x \in X, \varphi \in \Delta(E)\}.$$

(i). It is easy to show that $X \otimes \Delta(E)$ can be embedded into $(\mathrm{Lip}_d(X, E))^*$ by

$$x \otimes \varphi(f) = \varphi(f(x)) \quad (f \in Lip_d(X, E)).$$

We call the weak*-closure of the set $X \otimes \Delta(E)$ in $(\mathrm{Lip}_d(X, E))^*$, the vector-valued Lipschitz compactification of (X, d) and denote it by $l_E(X)$. In particular, whenever $E = \mathbb{C}$, we call it the classical Lipschitz compactification, denoted by $l(X)$. It should be noted that $l(X)$ is the weak*-closure of the set $\{\delta_x : x \in X\}$ in $(Lip_d X)^*$, where $\delta_x(f) = f(x)$, for all $f \in Lip_d X$. In addition, one can show that the vector-valued Lipschitz compactification $l_E(X)$ is weak*-compact in the unit ball of $(\mathrm{Lip}_d(X, E))^*$.

(ii). The set $X \otimes \Delta(E)$ may be regarded as a subset of $(Lip_d X)^* \times E^*$ by

$$x \otimes \varphi(g, a) = g(x)\varphi(a) \quad (g \in Lip_d X, a \in E).$$

Suppose that $r_E(X)$ is the weak*-closure of the set $X \otimes \Delta(E)$ in $(Lip_d X)^* \times E^*$ with product topology.

(iii). Moreover, $X \otimes \Delta(E)$ can be embedded in $(Lip_d X \check{\otimes} E)^*$ by

$$x \otimes \varphi(g \otimes a) = g(x)\varphi(a) \quad (g \in Lip_d X, a \in E).$$

The set $k_E(X)$ is stand for the weak*-closure of the set $X \otimes \Delta(E)$ in $(Lip_d X\check{\otimes}E)^*$.

Throughout this paper, we use Z instead of $Lip_d X\check{\otimes}E$, for convenience. Consider the vector-valued Lipschitz algebra $Lip_d(X, E)$ with the norm

$$|||f|||_{d,E} = \max\{\|f\|_{\infty,E}, P_{d,E}(f)\},$$

which is equivalent to the ordinary norm $\|f\|_{d,E} = \|f\|_{\infty,E} + P_{d,E}(f)$.

In the next lemma, we show that the Banach algebra Z may be regarded as a closed subalgebra of $(Lip_d(X, E), |||f|||_{d,E})$.

Lemma 2.1. *Let (X, d) be a metric space and E be a commutative Banach algebra. Then Z can be embedded isometrically into $(Lip_d(X, E), |||.|||_{d,E})$.*

Proof. Define the map $F : Lip_d X \otimes E \longrightarrow (Lip_d(X, E), |||.|||_{d,E})$ by

$$F(f \otimes a) = f \odot a,$$

where $f \odot a(x) = f(x)a$, $(x \in X)$. It is easy to see that F is homomorphism. Thus we need only to show that F is isometric. Take $u = \sum_{i=1}^n f_i \otimes a_i$. We have

$$\begin{aligned}
|||F(u)|||_{d,E} &= \max\{\|F(u)\|_{\infty,E}, P_{d,E}(F(u))\} \\
&= \max\{\sup_{x\in X}\{\|\sum_{i=1}^n a_i f_i(x)\|_{\infty,E}\}, \sup_{x\neq y}\{\frac{\sum_{i=1}^n \|f_i(x)a_i - f_i(y)a_i\|_E}{d(x,y)}\}\} \\
&= \max\{\sup_{x\in X}\sup_{g}\{|\sum_{i=1}^n g(a_i)f_i(x)| : g \in E^*, \|g\| \leq 1\} \\
&\quad , \sup_{x\neq y}\sup_{g}\{\frac{|\sum_{i=1}^n f_i(x)g(a_i) - \sum_{i=1}^n f_i(y)g(a_i)|}{d(x,y)} : g \in E^*, \|g\| \leq 1\}\} \\
&= \max\{\sup_{g}\{\|\sum_{i=1}^n g(a_i)f_i\|_\infty : g \in E^*, \|g\| \leq 1\}, \sup_{g}\{P_d(\sum_{i=1}^n g(a_i)f_i) : g \in E^*, \|g\| \leq 1\}\} \\
&= \sup_{g}\{\max\{\|\sum_{i=1}^n g(a_i)f_i\|_\infty, P_{d,E}(\sum_{i=1}^n g(a_i)f_i)\} : g \in E^*, \|g\| \leq 1\} \\
&= \sup_{g}\{|||\sum_{i=1}^n g(a_i)f_i|||_d : g \in E^*, \|g\| \leq 1\} \\
&= \sup_{g}\sup_{G}\{|\sum_{i=1}^n g(a_i)G(f_i)| : g \in E^*, \|g\| \leq 1, G \in (Lip_d X)^*, \|G\| \leq 1\} \\
&= \|\sum_{i=1}^n f_i \otimes a_i\|_\varepsilon = \|u\|_\varepsilon
\end{aligned}$$

Thus the proof is completed. □

We state here the following result, which will be used to establish the main results in section 3.

Lemma 2.2. *Let A and B be commutative Banach algebras and $F : A \longrightarrow B$ be an isomorphism. Then*

$$\Delta(A) = \{\overline{\varphi} : \varphi \in \Delta(B)\},$$

where, $\overline{\varphi}(a) = \varphi(F(a))$, for all $a \in A$.

Proof. It is easy to see that $\overline{\varphi}$ is a character on A and

$$\{\overline{\varphi} : \varphi \in \Delta(B)\} \subseteq \Delta(A).$$

Conversely, take $\psi \in \Delta(A)$. Define

$$\widetilde{\psi}(b) = \psi(F^{-1}(b)) \quad (b \in B).$$

It is clear that $\widetilde{\psi}$ is a character on B such that $\overline{\widetilde{\psi}} = \psi$. Thus

$$\Delta(A) \subseteq \{\overline{\varphi} : \varphi \in \Delta(B)\}.$$

□

It is worth mentioning that Sherbert showed $\Delta(Lip_d X) = l(X)$, see[17, Proposition 2.1]. We generalize this fact for some vector-valued cases.

In the next result, we investigate the relation between $l_E(X), k_E(X)$ and $r_E(X)$.

Lemma 2.3. *Suppose that (X, d) is a metric space and E is a Banach algebra with nonempty character space. Then the following statements hold:*
(i) $l_E(X) \subseteq k_E(X) \subseteq r_E(X)$;
(ii) If E is a Banach algebra with identity e, then

$$l_E(X) = k_E(X) = r_E(X).$$

Proof. (*i*) We first prove that $l_E(X)$ is subset of $k_E(X)$. For each $\psi \in l_E(X)$, there exists a net $(x_\alpha \otimes \varphi_\alpha)$ in $X \otimes \Delta(E)$ such that $(x_\alpha \otimes \varphi_\alpha)$ is weak *-convergence to ψ in $(Lip_d(X, E))^*$. Thus we have $(x_\alpha \otimes \varphi_\alpha(f))$ converges to $\psi(f)$, for every $f \in Lip_d(X, E)$. By applying Lemma 2.1, we conclude that $(x_\alpha \otimes \varphi_\alpha)$ is weak*-convergence to ψ in Z^*. It follows that $\psi \in k_E(X)$.
We now apply a similar argument again, with $l_E(X)$ replaced by $k_E(X)$ to show that $k_E(X)$ is a subset of $r_E(X)$. In fact, for any $\eta \in k_E(X)$, there is a net $(y_\beta \otimes \psi_\beta)$ in $X \otimes \Delta(E)$ which is weak*-convergence to η in Z^*. Thus $(y_\beta \otimes \psi_\beta(g \otimes a))$ converges to $\psi(g \otimes a)$, for all $g \in Lip_d X$ and $a \in E$. By an easy argument, one can show that $(g(y_\beta)\psi_\beta)$ converges to $\eta(g)$, for all $g \in Lip_d X$ in E^*. Therefore $(y_\beta \otimes \psi_\beta)$ is weak*-convergence to η in $(Lip_d X)^* \times E^*$, which implies that $\eta \in r_E(X)$.
(*ii*) Let us first show that $r_E(X) = \Delta(Z)$. Take $\eta \in Z$. By using [13, Theorem 2.11.2], one can show that $\Delta(Z)$ is homeomorphic with the tensor product $l(X) \otimes \Delta(E)$. Therefore $\eta = \psi \otimes \varphi$, for some $\psi \in \Delta(Lip_d X) = l(X)$ and $\varphi \in \Delta(E)$. Since $\psi \in l(X)$, there exists a net (x_α) in X such that (δ_{x_α}) is weak*-convergence to ψ in $(Lip_d X)^*$. For every $g \in Lip_d X$ and $a \in E$, we have

$$x_\alpha \otimes \varphi(g, a) = g(x_\alpha)\varphi(a) \longrightarrow \psi(g)\varphi(a) = \psi \otimes \varphi(g, a),$$

in $(Lip_d X)^* \times E^*$. It follows that $\eta \in \omega^* - cl(X \otimes \Delta(E)) = r_E(X)$.
Conversely, let $\lambda \in r_E(X)$. There exists a net $(x_\alpha \otimes \varphi_\alpha)$ in $X \otimes \Delta(E)$ such that $(x_\alpha \otimes \varphi_\alpha)$ is weak *-convergence to λ in $(Lip_d X)^* \times E^*$. Therefore, $(x_\alpha \otimes \varphi_\alpha(g, a))$ converges to $\lambda(g, a)$, for all $g \in Lip_d X$ and $a \in E$. Put $a = e$. We conclude that $(g(x_\alpha))$ converges to $\lambda(g, e)$, for all $g \in Lip_d X$. In other words, the net (δ_{x_α}) is weak *-convergence to k, where

$$k(g) = \lambda(g, e) \quad (g \in Lip_d X).$$

Now, in the same manner, we can see that if $g = 1_{Lip_d X}$, then the net $(\varphi_\alpha(a))$ converges to $\lambda(1_{Lip_d X}, a)$. Put $\varphi(a) = \lambda(1_{Lip_d X}, a)$. By easy computations, we obtain

$$\lambda(g, a) = k(g)\varphi(a) \quad (g \in Lip_d X, a \in E).$$

It follows that $(x_\alpha \otimes \varphi_\alpha)$ is weak*-convergence to $k \otimes \varphi$ in $(Lip_d X)^* \times E^*$. Thus, $\lambda \in l(X) \otimes \Delta(E) = \Delta(Z)$.
The rest of the proof is devoted to prove that $l_E(X) = \Delta(Z)$. According to part (*i*) and previous arguments,

we have $l_E(X) \subseteq \Delta(Z)$. Take $\eta \in Z$. Hence $\eta = \psi \otimes \varphi$, for some $\psi \in \Delta(Lip_d X) = l(X)$ and $\varphi \in \Delta(E)$. Since $\psi \in l(X)$, there exists a net (x_α) in X such that (δ_{x_α}) is weak*-convergence to ψ in $(Lip_d X)^*$. For every $g \in Lip_d X$, we have

$$\delta_{x_\alpha} \otimes \varphi(g) = \varphi(g(x_\alpha)) = \delta_{x_\alpha}(\varphi \circ g) \longrightarrow \psi(\varphi \circ g) = \eta(g).$$

Put $\varphi_\alpha := \delta_{x_\alpha} \otimes \varphi$. We conclude that $\{\varphi_\alpha\}$ is a net in $X \otimes \Delta(E)$, which is weak*-convergence to η in $(Lip_d(X, E))^*$. It implies that $\eta \in l_E(X)$. □

3. The inverse-closed Banach algebras

Throughout this section, A is a commutative Banach algebra with the nonempty character space. We first introduce the concept of inverse-closed for a Banach algebra. Then we proceed with stating the formula which shows that how the character space of unital commutative Banach algebra and the character space of its unital subalgebra are related to each other.

For convenience we use some notations which are described as follows.

Notation 3.1. *For all $a_1, \cdots, a_n \in A$ we write $a := (a_1, \cdots, a_n)$ and $\widehat{a} := (\widehat{a_1}, \cdots, \widehat{a_n})$, where $\widehat{a}$ is the Gelfand transform of a and defined by*

$$\widehat{a}(\varphi) := (\widehat{a_1}(\varphi), \cdots, \widehat{a_n}(\varphi)) \quad (\varphi \in \Delta(A)).$$

We say $\widehat{a}$ is nonzero, if for every $\varphi \in \Delta(A)$, we have $\widehat{a_i}(\varphi) \neq 0$, for some $i \in \{1, 2, ..., n\}$.
If B is a subalgebra of Banach algebra A and

$$\{\varphi\big|_B : \varphi \in \Delta(A), \ \ \varphi\big|_B \neq 0\} = \Delta(B),$$

then we say A and B have the same character space.

Definition 3.2. *Let A be a commutative Banach algebra with a nonempty character space. A Banach subalgebra B of A is called inverse-closed with respect to A, if for every $b = (b_1, b_2, ..., b_n)$, where $b_i \in B$, whenever $\widehat{b}$ is nonzero, there exists $a = (a_1, a_2, ..., a_n)$, where $a_i \in A$ such that*

$$\sum_{i=1}^{n} \varphi(a_i)\varphi(b_i) = 1 \qquad (\varphi \in \Delta(A)).$$

In particular, if Banach algebra A is inverse-closed with respect to A, we say A is inverse-closed Banach algebra.

The following example illustrates the concept of inverse-closed.

Example 3.3. *Let (X, d) be a compact metric space. We show that the classical Lipschitz Banach algebra $Lip_d X$ is inverse-closed. Take $f := (f_1, \cdots, f_n)$, where $f_i \in Lip_d X$ such that $\widehat{f} \neq 0$. In fact, for all $x \in X$, we have $f_j(x) \neq 0$, for some $j \in \{1, 2, ..., n\}$. We define $g := (g_1, \cdots, g_n)$, by $g_j := \frac{1}{k}\overline{f}_j$, where $\overline{f}_j$ is conjugate of f_j and $k := \sum_{i=1}^{n} \mid f_i \mid^2$. Since (X, d) is a compact metric space and $\widehat{f} \neq 0$, there exists $x_\circ \in X$ such that*

$$|k(x)| \geqslant |k(x_\circ)| \geqslant |f_j(x_\circ)|^2 > 0.$$

It follows that $g_j \in Lip_d X$. By some easy calculations, we obtain $\sum_{i=1}^{n} f_i g_i = 1_{Lip_d X}$. Now, for every $\varphi \in \Delta(Lip_d X)$, we have $\sum_{i=1}^{n} \varphi(f_i)\varphi(g_i) = 1$.

Example 3.4. *We show that $C_0(\mathbb{R})$ is not inverse-closed. To this end, we define*

$$f_\circ(x) = \begin{cases} 1 & |x| \leq 1; \\ \frac{1}{x} & |x| > 1. \end{cases}$$

For $f := (f_\circ)$, it is clear that $\widehat{f} \neq 0$. If $C_0(\mathbb{R})$ is inverse-closed, then there exists $g := (g_\circ)$, where $g_\circ \in C_0(\mathbb{R})$ such that

$$f_\circ(x)g_\circ(x) = 1 \qquad (x \in \mathbb{R})$$

Taking $n \in \mathbb{N}$, we have $g_\circ(n) = n$ which contradicts with the boundedness of g.

Theorem 3.5. *Let A be a unital commutative Banach algebra with the nonempty character space. Then A is inverse-closed.*

Proof. Take $b := (b_1, b_2, ..., b_n)$, where $b_i \in A$, with $\widehat{b} \neq 0$. Let I be an ideal of A which is generated by $\{b_1, b_2, ..., b_n\}$. We claim that $I = A$, otherwise we could find a maximal ideal M of A such that $I \subset M$. We have M=ker$\varphi_\circ$, for some $\varphi_\circ \in \Delta(A)$. Since $\widehat{b}(\varphi_\circ) \neq 0$, we have $\varphi_\circ(b_j) \neq 0$, for some $j \in \{1, 2, ..., n\}$. But $b_j \in I \subset M$=ker$\varphi_\circ$, it follows that $\varphi_\circ(b_j) = 0$, which is a contradiction. This implies that $I = A$. If 1_A is the identity of A, then $1_A \in I$. We have actually proved that there exist $a_1, a_2, ..., a_n \in A$ such that $\sum_{i=1}^n a_i b_i = 1_A$. Consequently, for every $\varphi \in A$, we obtain $\sum_{i=1}^n \varphi(a_i)\varphi(b_i) = 1$. This completes the proof . □

Corollary 3.6. *(i) If E is a unital commutative Banach algebra and (X, d) is a metric space, then $Lip_d(X, E)$ and $Lip_d X\check{\otimes}E$ are inverse-closed. In particular, the classical Lipschitz Banach algebra $Lip_d X$ is inverse-closed.*
(ii) For every compact metric space (X, d), $C(X)$ is inverse-closed.

Remark 3.7. *Point out that if X is a Hausdorff space and A is a unital, inverse-closed subalgebra of $C_b(X)$ such that $\Delta(A) = X$, then A is inverse-closed, based on the ordinary definition of inverse-closed.*

Now, the question arises whether the inverse-closed Banach algebra is unital. The following result gives a positive answer to this question for a semisimple Banach algebra.

Theorem 3.8. *Suppose that A is a commutative semisimple Banach algebra with an element $a := (a_1, a_2, ..., a_n)$ such that $\widehat{a}$ is nonzero. Then the following statements are equivalent:*
(i) A is inverse-closed ;
(ii) A is unital.

Proof. $(i) \Rightarrow (ii)$. By the hypothesis, there exists an element $b := (b_1, b_2, ..., b_n)$, where $b_i \in A$ such that

$$\sum_{i=1}^{n} \varphi(a_i)\varphi(b_i) = \varphi(\sum_{i=1}^{n} a_i b_i) = 1,$$

for every $\varphi \in \Delta(A)$.If $c := \sum_{i=1}^n a_i b_i$, then $c \in A$. By easy calculations, one can show that $\varphi(cx - x) = 0$, for every $\varphi \in \Delta(A)$ and every $x \in A$. Now, we conclude from semisimplicity of A that c is the identity for A.
$(ii) \Rightarrow (i)$. Choose $b := (b_1, b_2, ..., b_n)$, where $b_i \in A$ with $\widehat{b} \neq 0$. Let I be an ideal of A which is generated by $\{b_1, b_2, ..., b_n\}$. Now, with a similar method in Theorem 3.5, we have $I = A$. Futhermore, there exists $c := (c_1, c_2, ..., c_n)$, where $c_i \in A$ such that

$$\sum_{i=1}^{n} \varphi(b_i)\varphi(c_i) = 1.$$

It follows that A is inverse-closed. □

By a similar argument, as is used in Theorem 3.8 one can obtained the following result. Note that the theorem discusses the condition under which A is unital.

Theorem 3.9. *Let A be a commutative semisimple Banach algebra and B be a Banach subalgebra of A which is inverse-closed with respect to A. If there is an element $b := (b_1, b_2, ..., b_n)$, where $b_i \in B$ with $\widehat{b} \neq 0$, then A is unital.*

Proof. Since B is inverse-closed with respect to A, there exists an element $a := (a_1, a_2, ..., a_n)$, where $a_i \in A$ such that

$$\sum_{i=1}^{n} \varphi(a_i)\varphi(b_i) = \varphi(\sum_{i=1}^{n} a_i b_i) = 1,$$

for every $\varphi \in \Delta(A)$. Put $c := \sum_{i=1}^{n} a_i b_i$. By applying similar arguments as in Theorem 3.8, c is the identity of A. $\square$

It should be noted that the inverse-closed property is not necessarily inherited by the Banach subalgebras. The next example introduces an Banach algebra A which is inverse-closed but it has a Banach subalgebra that is neither inverse-closed nor inverse-closed with respect to A.

Example 3.10. *Let $B := C_0(\mathbb{R})$ and $A = C_0(\mathbb{R}) \oplus \mathbb{C}$. Then B is a Banach subalgebra of A which is not inverse-closed, by Example 3.3. Using Theorem 3.5, A is inverse-closed. Suppose that B is inverse-closed with respect to A. For $f := (f_\circ)$, where that*

$$f_\circ(x) = \begin{cases} 1 & |x| \leq 1; \\ \frac{1}{x} & |x| > 1, \end{cases}$$

$\widehat{f} \neq 0$ and so there exists $g := (g_\circ)$, where $g_\circ \in C_0(\mathbb{R})$ such that

$$\psi(f_\circ)\psi(g_\circ) = 1 \qquad (\psi \in \Delta(A)) \qquad \star .$$

Take $\varphi_\circ \in \Delta(A)$ which is defined by $\varphi_\circ(x, \lambda) = \lambda$, for all $(x, \lambda) \in A$. Obliviously, we have $\varphi_\circ(f_\circ) = 0$, which is a contradiction with $\star$.

If I is a closed ideal of a commutative Banach algebra A with a nonempty character space, then

$$\Delta(I) = \{\varphi\big|_I : \varphi \in \Delta(A), \;\; and \;\; \varphi\big|_I \neq 0\},$$

see [13, Lemma 2.2.15]. In other words, I and A have the same character space. The next example introduces a specific instance of a Banach subalgebra for which the same result is established. Recall that

$$Lip_d^\alpha(X, E) = \{f : X \to E : f \;\; is \;\; bounded \;\; and \;\; \sup_{x \neq y} \frac{\|f(x) - f(y)\|_E}{d^\alpha(x, y)} < \infty\}.$$

Example 3.11. *Let $A := Lip_d\mathbb{R}$, $B := Lip_d^2\mathbb{R}$ and d be Euclidean metric. It is easy to see that B is exactly the set of constant functions on $\mathbb{R}$ [3, Corollary 2.5]. Hence, B is Banach subalgebra of A and $1_A = 1_B$, where $1_A(x) = 1$ for all $x \in \mathbb{R}$. Also, we have $\Delta(B) = \{I\}$, which $I(b) = b$ for all $b \in B$. It is a simple matter to show that*

$$\left\{\varphi\big|_B : \varphi \in \Delta(A), \;\; \varphi\big|_B \neq 0\right\} = \left\{\varphi\big|_B : \varphi \in \Delta(A)\right\} = \Delta(B).$$

Therefore by the definition, A and B have the same character space. Note that B is not an ideal of A.

Now, we state the following example which shows that the Banach algebra A and its Banach subalgebras can not have the same character space, generally.

Example 3.12. *Let X be a nonempty compact subset of $\mathbb{C}$ and $B = P(X)$ the Banach subalgebra of $A = C(X)$, generated by polynomails on X. It is known $\Delta(A) = X$, [13, Example 2.2.8]. By using [13, Theorem 2.5.7(i)], $\Delta(B) = \widehat{X_p}$, where $\widehat{X_p}$ is the polynomailly convex hull of X. In particular, Let X be Swiss cheese discussed in [13, Example 2.5.11]. X is proper subset of $\widehat{X_p}$. Therefore,*

$$\left\{\varphi\big|_B : \varphi \in \Delta(A), \;\; \varphi\big|_B \neq 0\right\} \subsetneq \Delta(B).$$

Theorem 3.13. *Let B be a unital Banach subalgebra of a unital commutative Banach algebra A, with the same identity and the same character space. Then B is inverse-closed with respect to A.*

Proof. Choose $b := (b_1, b_2, ..., b_n)$, $b_i \in A$, such that $\widehat{b} \neq 0$. Let I be an ideal of A which is generated by $\{b_1, b_2, ..., b_n\}$. Then $A = I$, otherwise we can find a maximal ideal M such that $I \subsetneq$ M. We have M = ker $\varphi_\circ$, for some $\varphi_\circ \in \Delta(A)$. Since B and A have the same character space, there is $\psi_\circ \in \Delta(B)$ such that $\psi_\circ = \varphi_\circ|_B$. Since $\hat{b}(\psi_\circ) \neq 0$, we have $\psi_\circ(b_j) \neq 0$ for some j. In addition, there exists $a_\circ \in A$ such that $\varphi_\circ(a_\circ) \neq 0$. It is clear that $a_\circ b_j \in I \subseteq$ M = ker $\varphi_\circ$. Hence $\varphi_\circ(a_\circ b_j) = 0$, which is impossible. Thus, $I = A$ and so $1_A \in I$. Therefore, there exists $a = (a_1, a_2, ..., a_n)$, where $a_i \in A$, such that $\sum_{i=1}^n a_i b_i = 1_A$, this completes the proof. □

Now as a consequence of Theorem 3.13 and Lemma 2.1, we can state the following result on vector-valued Lipschitz algebras.

Corollary 3.14. *Suppose that (X, d) is a compact metric space and E is a unital commutative Banach algebra with nonempty character space. Then Z is inverse-closed with respect to $Lip_d(X, E)$.*

Proof. According Lemma 2.1, Z is a Banach subalgebra of $\mathrm{Lip}_d(X, E)$. By applying [9, Theorem 2.9], we have

$$\Delta(Z) = \Delta(Lip_d(X, E)) = X \otimes \Delta(E).$$

Now, by using Theorem 3.13, Z is inverse-closed with respect to $\mathrm{Lip}_d(X, E)$. □

In the following result, we provide some conditions on the Banach subalgebra B of the Banach algebra A, under which A and B have the same character spaces.

Theorem 3.15. *Let $(A, \|.\|_A)$ be a commutative Banach algebra and $(B, \|.\|_B)$ be a Banach subalgebra of A such that B be dense in $(A, \|.\|_A)$. Then the following statements hold:*
(i) If φ is a character on A, then $\varphi\big|_B$ is a character on B.
(ii) If there exists positive number M such that

$$\|b\|_B \leq M\|b\|_A \quad (b \in B),$$

then A and B have the same character space.

Proof. (i)It is sufficient to show that $\varphi\big|_B$ is nonzero. Choose $a \in A$ such that $\varphi(a) \neq 0$. Since B is dense in $(A, \|.\|_A)$, there exists a sequence $\{b_n\}$ such that $\{b_n\}$ converges to a in $(A, \|.\|_A)$. By continuity of φ, we have $\varphi(b_{n_\circ}) \neq 0$ for some $n_\circ$ and it follows $\varphi\big|_B \neq 0$ on B.
(ii) Let $\psi \in \Delta(B)$ and $a \in A$. There exists a sequence $\{b_n\}$ which converges to a in $(A, \|.\|_A)$. By the assumption

$$|\psi(b_n) - \psi(b_m)| \leq \|b_n - b_m\|_B \leq M\|b_n - b_m\|_A,$$

for each m and n, which guarantees that $\{\psi(b_n)\}$ is a cauchy sequence in $\mathbb{C}$. Hence we may define $\varphi(a) = \lim_{n\longrightarrow\infty} \psi(b_n)$. We show that $\varphi(a)$ does not depend on the choice of $\{b_n\}$. Let $\{b_n\}$ and $\{c_n\}$ converge to a in $(A, \|.\|_A)$. Then we have

$$|\psi(b_n) - \psi(c_n)| \leq \|b_n - c_n\|_B \leq M\|b_n - c_n\|_A.$$

It follows

$$\lim_{n\longrightarrow\infty} \psi(b_n) = \lim_{n\longrightarrow\infty} \psi(c_n).$$

Clearly, φ is a character on A such that $\varphi\big|_B = \psi$. □

Remark 3.16. *Under the conditions of (ii), B is an abstract Segal algebra with respect to A. Moreover, in [4, Lemma 2.2], the authors proved that if B is an abstract Segal algebra with respect to A, then $\Delta(A) = \Delta(B)$.*

We now express the relationship between the character space of a unital commutative Banach algebra and its unital Banach subalgebra.

Lemma 3.17. *Let B be a unital Banach subalgebra of a unital commutative Banach algebra A with the same identity. Then*

$$\Delta(A) = \Delta(B) \oplus \Delta(A)\Big|_{B^c},$$

where B^c is the complement of B in A.

Proof. Consider the map $\Theta : \Delta(A) \longrightarrow \Delta(B) \oplus \Delta(A)\Big|_{B^c}$ defined by

$$\Theta(\varphi) = (\varphi\Big|_{B}, \varphi\chi_{B^c}).$$

It is easy to see that Θ is a bijective map. □

Now, we are ready to identify the character space of vector-valued Lipschitz algebra $Lip_d(X, E)$.

Theorem 3.18. *Suppose that (X, d) is a metric space and E is a unital commutative Banach algebra with nonempty character space. Then*

$$\Delta(Lip_d(X, E)) = (l(X) \otimes \Delta(E)) \oplus W;$$

where

$$W = \{\varphi\chi_{Z^c} : \varphi \in \Delta(Lip_d(X, E))\}.$$

Proof. By using Lemma 2.1 and Lemma 3.17, the result is immediate. □

4. The character space of vectore-valued Lipschitz algebras

In this section we determine the structure of the character space of vector-valued Lipschitz algebra $\mathrm{Lip}_d(X, E)$, where (X, d) is a metric space and E is a unital semisimple commutative $*$-Banach algebra.

Definition 4.1. *Assume that (X, d) is a metric space and E is a commutative Banach algebra. The Banach algebra $Lip_d(X, E)$ is called weakly inverse-closed, if for $f \in Lip_d(X, E)$ with $|\varphi(f(x))| \geq \varepsilon$, for all $x \in X$, $\varphi \in \Delta(E)$ and some $\varepsilon > 0$, then there exists $g \in Lip_d(X, E)$ such that*

$$\varphi(f(x))\varphi(g(x)) = 1$$

for all $x \in X$ and $\varphi \in \Delta(E)$.

Clearly if $\mathrm{Lip}_d(X, E)$ is inverse-closed, then it is weakly inverse-closed.

Definition 4.2. *We say the character space of $*$-Banach algebra E preserves involution, if $\varphi(a^*) = \overline{\varphi(a)}$, for all $\varphi \in \Delta(E)$ and $a \in E$.*

Example 4.3. *(i) Suppose that (X, d) is a metric space and E=$C_0(X)$. Then the character space of E preserves the involution.*
(ii) Let (X, d) be a metric space. We know Lip_dX is a $*$- Banach algebra with involution $f^*(x) = \overline{f(x)}$, for all $x \in X$. Then $\Delta(Lip_dX) = l(X)$ preserves the involution on Lip_dX.

Remark 4.4. *It should be noted that if (X, d) is a compact metric space and E is a $*$- Banach algebra, then the character space of $Lip_d(X, E)$ preserves involution on $Lip_d(X, E)$ if and only if the character space of E preserves involution on E. Note that $Lip_d(X, E)$ is a $*$- Banach algebra whenevere E is a $*$- Banach algebra.*

In order to prove our main result, we need to state some lemmas. We state here the following assumptions:
Let (X, d) be a metric space, E be a unital commutative Banach algebra . We consider the map

$$\begin{aligned} \Theta : Lip_d(X, E) &\longrightarrow C(l_E(X)) \\ f &\longmapsto \theta_f \end{aligned}$$

where $\theta_f(w) = w(f)$, for all $w \in l_E(X)$.

Lemma 4.5. *If E is a semisimple Banach algebra, then $Lip_d(X,E) \cong \Theta\ (Lip_d(X,E))$ as two Banach algebras with equivalent norms.*

Proof. It is easy to see that Θ is a homomorphism and continuous map. We show that Θ is injective. Suppose that $\theta_f = \theta_g$, for some $f, g \in \mathrm{Lip}_d(X,E)$. Then we have $w(f) = w(g)$, for all $w \in l_E(X)$. We may consider $w = x \otimes \varphi$, for $x \in X$ and $\varphi \in \Delta(E)$. Hence $\varphi(f(x)) = \varphi(g(x))$, for all $x \in X$ and $\varphi \in \Delta(E)$. Since E is semisimple, we have $f = g$. Moreover, according to the open mapping theorem, there exists $L > 0$ such that

$$L\|f\|_{d,E} \leq \|\Theta(f)\|_\infty \leq \|f\|_{d,E} \quad (f \in Lip_d(X,E)).$$

This completes the proof. □

Lemma 4.6. *Let E be a $*$-Banach algebra which its character space preserves the involution. Then $\Theta(Lip_d(X,E))$ is self-adjoint.*

Proof. Let $f^*(x) = (f(x))^*$, for all $x \in X$ and $f \in \mathrm{Lip}_d(X,E)$. It is clear that $(\mathrm{Lip}_d(X,E), *)$ is a $*$-Banach algebra. The proof is completed by showing that $\theta_{f^*} = \overline{\theta_f}$, for all $f \in \mathrm{Lip}_d(X,E)$. Taking $w \in l_E(X)$, there exists a net $(x_\alpha \otimes \varphi_\alpha)$ in $X \otimes \Delta(E)$ such that weak*-convergence to w. For $f \in \mathrm{Lip}_d(X,E)$

$$\begin{aligned}\theta_{f^*}(w) = w(f^*) &= \lim_\alpha x_\alpha \otimes \varphi_\alpha(f^*)\\ &= \lim_\alpha \varphi_\alpha((f(x_\alpha))^*)\\ &= \lim_\alpha \overline{\varphi_\alpha(f(x_\alpha))}\\ &= \overline{\theta_f(w)} = \overline{\theta_f}(w).\end{aligned}$$

Thus the result is obtained. □

Lemma 4.7. *The algebra $\Theta(Lip_d(X,E))$ is seprating and inverse-closed Banach algebra which contains constant functions.*

Proof. Let $w_1, w_2 \in l_E(X)$ with $w_1 \neq w_2$. Then there exists $f \in \mathrm{Lip}_d(X,E)$ such that $w_1(f) \neq w_2(f)$. It follows that $\theta_f(w_1) \neq \theta_f(w_2)$. Thus $\Theta(\mathrm{Lip}_d(X,E))$ separates the points of $l_E(X)$. Now, suppose that e is the identity of E. Suppose that $f_\lambda : X \to E$ is the constant function, defined by $f_\lambda(x) = \lambda e$, for all $x \in X$ and $\lambda \in \mathbb{C}$. We can write that $f_\lambda = g_\lambda \otimes e$, for some constant function $g_\lambda \in Lip_d X$. By easy computation, one can show that $\theta_{f_\lambda}(w) = \lambda$, for all $w \in l_E(X)$. Hence $\Theta(\mathrm{Lip}_d(X,E))$ contains constant functions.
In the following, we show that $\Theta(\mathrm{Lip}_d(X,E))$ is weakly inverse-closed Bnach algebra. Let $F \in \Theta(\mathrm{Lip}_d(X,E))$ such that $|F(w)| > \varepsilon$, for all $w \in l_E(X)$ and some $\varepsilon > 0$. Since $F \in \Theta(\mathrm{Lip}_d(X,E))$, there exists $f \in \mathrm{Lip}_d(X,E)$ such that $\theta_f = F$. Thus $|x \otimes \varphi(f)| > \varepsilon$, for all $x \in X$ and $\varphi \in \Delta(E)$. According to Corollary 3.6, $\mathrm{Lip}_d(X,E)$ is inverse-closed. Thus $\mathrm{Lip}_d(X,E)$ is weakly inverse-closed. Hence, there exists $g \in \mathrm{Lip}_d(X,E)$ such that

$$\varphi(f(x))\varphi(g(x)) = 1 \qquad (x \in X, \varphi \in \Delta(E)).$$

Set $G = \theta_g$. By easy computations, one can show that G is the inverse of F in $\Theta(\mathrm{Lip}_d(X,E))$. This completes the proof. □

Now, we are ready to state our main result of this paper.

Theorem 4.8. *Suppose that (X,d) is a metric space and E is a unital semisimple commutative $*$-Banach algebra, which its character space preserves the involution. Then $\Delta(Lip_d(X,E)) = l_E(X)$.*

Proof. By using Lemma 4.5 and Lemma 2.2, $\Delta(\mathrm{Lip}_d(X,E)) = \Delta(\Theta(\mathrm{Lip}_d(X,E)))$. According to Lemma 4.6 and Lemma 4.7, $\Theta(\mathrm{Lip}_d(X,E))$ is a separation, self-adjoint, inverse-closed subalgebra of $C(l_E(X))$ and $l_E(X)$ is weak*-compact. Therefore by applying Lemma [14, p.55], we have $\Delta(\Theta(\mathrm{Lip}_d(X,E)) = l_E(X)$. It follows that $\Delta(\mathrm{Lip}_d(X,E)) = l_E(X)$. □

Corollary 4.9. *Let (X, d) be a metric space and E be a unital commutative C^*-algebra. Then $\Delta(Lip_d(X, E)) = l_E(X)$. In particular, if $E = \mathbb{C}$, then $\Delta(Lip_d X) = l(X)$.*

Acknowledgement. The authors would like to express their sincere thanks to Dr. F. Abtahi for her useful comments and discussions.
The authors express their sincere gratitude to the reviewer for her/ his constructive comments on the manuscript.

References

[1] F. Abtahi, M. Azizi and A. Rejali, Character amenability of the intersection Lipschitz algebras, Canad. Math. Bull. Vol. 60 (4) (2017), 673-689.
[2] F. Abtahi, E. Biyabani and A. Rejali, Some algebraic and homological properties of Lipschitz algebras and their second duals, Archivum Mathematicum (Brno), Tomus 55 (2019), 211-224.
[3] M. Azizi, E. Biyabani and A. Rejali, Some properties of vector-valued Lipschitz algebras, IJMSI Vol. 15 No. 2 (2020), 191-205.
[4] M. Alaghmandan, R. Nasr-Isfahani and M. Nemati, Character amenability and contractibility of abstract Segal algebras, Bull. Aust. Math. Soc. 82 (2010), 274-281.
[5] E. Biyabani and A. Rejali, Approximate and character amenability of vector-valued Lipschitz algebras, Bull. Korean Math. Soc. 55 No. 4 (2018), 1109-1124.
[6] E. Biyabani and A. Rejali, The second dual vector-valued Lipschitz algebras, U. P. B. Sci. Bull., Series A, Vol. 80, Iss 3 (2018), 103-110.
[7] K. Esmaeili and H. Mahyar, The character space and silov bounderies of vector-valued Lipschitz function algebras, Indian J. Pure Appl. Math. (2014), 977-988.
[8] O. Hatori and S. Oi, Hermitian operators on Banach algebras of vector-valued Lipschitz maps, J. Math. Anal. Appl. 452 (2017), 378-387.
[9] T. G. Honary, A. Nikou and A. H. Sanatpour, On the character space of vector-valued Lipschitz algebras, Bull. Iranian Math. Soc. Vol. 40 No. 6 (2014), 1453-1468.
[10] Z. Hu, M. S. Monfared and T. Traynor, On character amenable Banach algebras, Studia. Math. 193 No. 1 (2009), 53-78.
[11] A. Jimenez, L. Li, A. M. Peralta, L. Wang and Y. Shu Wang, 2-Local standard isometries on vector-valued Lipschitz function space, J. Math. Anal. Appl. Vol. 461 (2) (2018), 1287-1298.
[12] J. A. Johnson, Banach space of Lipschitz functions and vector-valued Lipschitz functions, Trans. Amer. Math. Soc. Vol. 148 No. 1 (1970), 147-169.
[13] E. Kanuith, A Course in Commutative Banach Algebra, Graduate Text in Mathematics, Springer, New York (2009).
[14] L. Loomis, An Introduction to Abstract Harmonic Analysis, D. Van Nostrand Company, New York (1953).
[15] A. Ranjbari and A. Rejali, Feréchet α- Lipschitz vector-valued operators algebras, U. P. B. Sci. Bull. Series A Vol. 80 (2018), 141-151.
[16] V. Runde, Lecture on Amenability, Lecture Note in Mathematics, Springer (2002).
[17] D. R. Sherbert, The structure of ideals and point derivations in Banach algebras of Lipschitz functions, Trans. Amer. Math. Soc. Vol. 111 (1964), 240-272.
[18] D. R. Sherbert, Banach algebras of Lipschitz functions, Pacific J. Math. Vol. 13 (1963), 1387-1399.