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On the Distribution of Zeros of All Solutions of a First Order Nonlinear Neutral Differential Equation

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Abstract. We investigate the distribution of zeros of all solutions of a non-autonomous nonlinear neutral differential equation that generalizes a lossless transmission network model. The neutral term is taken to be positive. We give several new estimations of the gap between adjacent zeros. The obtained results are supported by illustrative examples.

1. Introduction

Neutral differential equations have been the subject of intensive investigations due to their suitability to model many real life phenomena, see [12, 19, 21, 22]. Generally, these equations can not be integrated in an exact explicit form. Therefore, performing qualitative analysis, such as oscillation, is very important to understand the dynamics of the modeled phenomena.

Oscillation theory focuses on the existence of an infinite number of large zeros of all solutions, see [1, 2, 4, 5, 19, 20]. Estimating the locations of zeros of the solutions is a basic problem in this theory. In fact, for first order delay differential equations, this problem did not receive the deserved attention in comparison with the existence of zeros. As far as these authors know, less than fifty papers have been published on this problem, see for example [2, 4, 6, 7, 13, 14, 16, 17, 25, 26, 28, 30, 31, 33–35]. But for the existence of infinite large zeros of these equations, a huge number of papers and several monographs have been published. For example the reader is referred to the references [2, 5, 11, 15, 16, 19, 20, 23, 27] and the extended list of papers cited therein. We relate this clear difference to the lack of techniques for studying the distribution of zeros as well as the complexity of the problem itself in the sense that some solutions may stick with zero on some intervals as established by [8]. This property blocks the way of obtaining positive lower bound of the distance between consecutive zeros; except for certain classes of initial functions, see [13].

Aiming to provide an easy tool to study the distribution of zeros of functional differential equations, Baker and El-Morshedy [6, 7] established several sufficient conditions for the non-existence of positive solutions, on certain intervals, of the delay differential inequality

$$y'(t) + q(t)y(t-r) \leq 0, \tag{1.1}$$

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as well as the advanced type inequality $y'(t) - q(t)y(t+r) \geq 0$, where $q \in C([t_0, \infty), [0, \infty))$. In this work, we use some of such results, obtained for (1.1), to investigate the distance between adjacent zeros of all solutions of the following nonlinear neutral equation

$$\frac{d}{dt}\left(y(t) + k(t)y(t-\tau)\right) = -a(t)y(t) - b(t)y(t-\alpha) + c(t)\tanh y(t) + d(t)\tanh y(t-\nu), \quad t \geq t_0 \tag{1.2}$$

where $k \in C([t_0, \infty), [0, \infty))$, $a, b, c, d \in C([t_0, \infty), \mathbb{R})$ and $\alpha, \nu, \tau \geq 0$. This equation is a generalization of the autonomous equation

$$y'(t) + \bar{k}y'(t-\tau) = \bar{a}y(t) + \bar{b}y(t-\tau) + \bar{c}\tanh y(t) + \bar{d}\tanh y(t-\nu), \quad t \geq t_0,$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{k}, \tau$ and ν are non-negative constants, which has been proposed by Brayton [9] as a model of a lossless transmission lines and studied further by [10, 18, 29].

An important prototype of (1.2) is the equation

$$\frac{d}{dt}\Big(y(t) + \bar{k}y(t-\tau)\Big) = -\bar{a}y(t) + \bar{d}\tanh y(t-\nu) = 0, \quad t \geq t_0$$

which was proposed by [15] as a possible generalization of a single neuron model of Hopfield type and investigated by many authors; see for example [3, 24].

Let $t_{-1} = t_0 - \max\{\alpha, \nu, \tau\}$. By a solution of Eq.(1.2), we mean a continuous function $y : [t_0 - t_1, \infty) \to R$ such that $y(t) + k(t)y(t-\tau)$ is continuously differentiable and $y(t)$ satisfies Eq.(1.2) on $[t_0, \infty)$. The existence of a solution of Eq.(1.2) is guaranteed by [20, Theorem 1.1.2]. A solution of Eq.(1.2) is called oscillatory if it has arbitrary large zeros.

In the sequel, $d_s(y)$ denotes the least upper bound of the distances between all pairs of adjacent zeros for any solution $y(s)$ of equation (1.2) on the interval $[s, \infty)$.

This paper is organized as follows. In addition to this introduction, we have two other sections. In Section 2, we collect some interesting auxiliary results. Section 3 contains the main results.

2. Auxiliary Results

Our technique depends on some properties of the inequality (1.1). Therefore, to make this work self-contained, we list the following crucial results. We need the sequences $\{u_n(\rho)\}$ and $\{v_m(\rho)\}$ from [32, 33] which are defined, for $0 < \rho < 1$, as follows

$$u_0(\rho) = 1, \quad u_1(\rho) = \frac{1}{1-\rho}, \quad u_{n+2}(\rho) = \frac{u_n(\rho)}{u_n(\rho) + 1 - e^{\rho u_n(\rho)}}, n = 0, 1, \ldots,$$

and

$$v_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \quad v_{m+1}(\rho) = \frac{2(1 - \rho - \frac{1}{v_m(\rho)})}{\rho^2}, \; m = 1, 2, \ldots.$$

Lemma 2.1. *[6, Lemma 2.3] Assume that*

$$\int_{t-r}^{t} q(s)ds \geq \rho > \frac{1}{e}, \quad t \geq t_0 + r, \tag{2.1}$$

and $y(t)$ is a function satisfying inequality (1.1) *on $[T_1, T]$ with $y'(t) \leq 0$ for $t \in [T_1 - \delta, T]$ where $T \geq T_1 + (k_\rho + 1)r - \delta$, $|\delta| \leq r$, $T_1 \geq t_0 + r$ and k_ρ is defined by*

$$k_\rho = \begin{cases} 1, & \rho \geq 1, \\ \min\{\alpha, \beta\}, & \frac{1}{e} < \rho < 1 \end{cases} \tag{2.2}$$

where

$$\alpha = \min_{n\geq 1, m\geq 1}\{n+m|u_n(\rho)\geq v_m(\rho)\},$$

$$\beta = 1+\min_{n\geq 1}\{n|u_{n+1}(\rho)<0 \quad or \quad u_{n+1}(\rho)=\infty\}.$$

Then y(t) cannot be positive on $[T_1, T]$.

Lemma 2.2. *[6, Lemma 2.4] Let n be a positive integer such that*

$$\int_{t-r}^{t} F_n(s)ds \geq 1, \quad \text{for all } t \geq t_0 + (2n+1)r, \tag{2.3}$$

where $F_0(t) = q(t)$ *for* $t \geq t_0$ *and*

$$F_n(t) = F_{n-1}(t)\int_{t-r}^{t} F_{n-1}(s)e^{\int_{s-r}^{t} F_{n-1}(u)du}ds, \quad t \geq t_0 + 2nr \text{ and } n = 1, 2, \ldots$$

If $y(t)$ *is a nonincreasing function on* $[T_1-\delta, T]$ *which satisfies* (1.1) *on* $[T_1, T]$, *then* $y(t)$ *can not be positive on* $[T_1, T]$, *where* $T > T_1 + (3n+1)r - \delta$, $T_1 \geq t_0 + (2n+1)r$ *and* $|\delta| \leq r$.

Lemma 2.3. *[6, Lemma 2.5] Let* n^* *and* n^{**} *be two positive integers, such that* $n^{**} = \min\{i : u_{i+1}(\rho) < 0 \text{ or } u_{i+1}(\rho) = \infty\}$, *and*

$$\sum_{j=1}^{n^*}\left(\prod_{i=2}^{j} u_{n^*+2-i}(\rho)\right)\int_{t-r}^{t} q_j(s)ds \geq 1, \quad \text{for all } t \geq t_0 + n^* r, \tag{2.4}$$

for $\rho \in (0,1)$ *where*

$$q_1(s) = q(s), \quad q_{n+1}(s) = q(s-nr)\int_{t-r}^{s} q_n(u)du, \quad t \geq t_0 + nr,$$

for all $s \in (t-r, t)$. *Further, assume that* $y(t)$ *is nonincreasing on* $[T_1-\delta, T]$, *where* $T_1 \geq t_0 + n^* r$ *and* $|\delta| \leq r$. *If* $y(t)$ *satisfies* (1.1) *on* $[T_1, T]$, *then* $y(t)$ *can not be positive on* $[T_1, T]$ *where* $T > T_1 + (n+2)r - \delta$ *and* $n = \min\{n^*, n^{**}\}$.

3. Main Results

Before studying the gab between adjacent zeros of (1.2), we derive some results on the existence of positive solutions for it. Thus, we role out some cases in which one can not find a global estimate of the distance between consecutive zeros of all solutions.

Theorem 3.1. *Assume that the following conditions hold*

(C_1) $\alpha = \tau$ *and* $c(t), d(t) > 0, \quad t \geq t_0$.

(C_2) $a(t)k(t) \geq b(t), \quad t \geq t_0$.

(C_3) $k(t) \leq e^{-\int_{t-\tau}^{t} a(s)ds}, \quad t \geq t_0 + \tau$.

(C_4) $k(t) \leq e^{-\int_{t_0}^{t} a(s)ds}, \quad t \in [t_0, t_0+\tau]$.

Then Eq.(1.2) *has a positive solution on* $[t_0, \infty)$.

Proof. Consider that $y(t)$ is a solution of (1.2) with initial function $\phi(s)$ satisfying that

$$(1+k(t_0))\,\lambda > \phi(s) > \lambda > 0, \text{ for all } s \in [t_0-\gamma, t_0], \quad \gamma = \max\{\nu, \tau\}, \tag{3.1}$$

where $t_0 \geq 0$. Let t_1 be the first zero of $y(t)$. Then $y(t) > 0$ for all $t < t_1$ and $y(t_1) = 0$. Assume that $w(t) = e^{\int_{t_0}^{t} a(s)ds}\,(y(t) + k(t)y(t-\tau))$. Then (1.2) can be rewritten as

$$\frac{d}{dt}w(t) = e^{\int_{t_0}^{t} a(s)ds}\,((a(t)k(t) - b(t))y(t-\tau) + c(t)\tanh y(t) + d(t)\tanh y(t-\nu))\,.$$

Thus (C_1) and (C_2) lead to

$$\frac{d}{dt}(e^{\int_{t_0}^{t} a(s)ds}(y(t) + k(t)y(t-\tau))) > 0, \ t \in [t_0, t_1]. \tag{3.2}$$

If $t_1 \geq t_0 + \tau$, by integrating (3.2) from $t_1 - \tau$ to t_1, we obtain

$$e^{\int_{t_0}^{t_1} a(s)ds}(y(t_1) + k(t_1)y(t_1-\tau)) > e^{\int_{t_0}^{t_1-\tau} a(s)ds}(y(t_1-\tau) + k(t_1-\tau)y(t_1-2\tau)),$$

and hence,

$$y(t_1) > (e^{-\int_{t_1-\tau}^{t_1} a(s)ds} - k(t_1))y(t_1-\tau) + e^{-\int_{t_1-\tau}^{t_1} a(s)ds}k(t_1-\tau)y(t_1-2\tau) > 0,$$

which contradicts the assumption that $y(t_1) = 0$. So, we consider the case when $t_1 < t_0 + \tau$. By integrating (3.2) from t_0 to t_1, we obtain

$$e^{\int_{t_0}^{t_1} a(s)ds}(y(t_1) + k(t_1)y(t_1-\tau)) > y(t_0) + k(t_0)y(t_0-\tau).$$

Because of (C_4) and (3.1), this inequality yields

$$\begin{aligned} y(t_1-\tau) &\geq e^{\int_{t_0}^{t_1} a(s)ds}k(t_1)y(t_1-\tau) \\ &> (y(t_0) + k(t_0)y(t_0-\tau)) \\ &> (1+k(t_0))\lambda. \end{aligned}$$

which is impossible since $t_1 - \tau \in [t_1 - \gamma, t_0]$. Thus, there are no zeros for $y(t)$. Therefore, due to (3.1), $y(t)$ is positive on $[t_0, \infty)$.

Theorem 3.2. *Assume that $a(t), b(t) < 0$, $c(t), d(t) > 0$ and $k(t) \leq 1$ for all $t \geq t_0$. Then Eq.(1.2) has a positive solution on $[t_0, \infty)$.*

Proof. Consider a solution $y(t)$ of (1.2) with initial function $\phi(s)$ such that (3.1) holds (with $\gamma = \max\{\alpha, \tau, \nu\}$). Let t_1 be as in the previous proof. Then (1.2) implies that

$$\frac{d}{dt}(y(t) + k(t)y(t-\tau)) > 0, \ t \in [t_0, t_1]. \tag{3.3}$$

Therefore, for the case $t_1 \geq t_0 + \tau$, we have

$$y(t_1) + k(t_1)y(t_1-\tau) > y(t_1-\tau) + k(t_1)y(t_1-2\tau),$$

and hence,

$$k(t_1)y(t_1-\tau) > y(t_1-\tau),$$

which is impossible since $k(t_1) \leq 1$. So, we consider the case when $t_1 < t_0 + \tau$. From (3.3), we deduce that

$$y(t_1) + k(t_1)y(t_1-\tau) > y(t_0) + k(t_0)y(t_0-\tau).$$

Therefore,

$$\begin{aligned} y(t_1-\tau) &\geq k(t_1)y(t_1-\tau) \\ &> y(t_0)+k(t_0)y(t_0-\tau) \\ &> (1+k(t_0))\lambda, \end{aligned}$$

which is impossible since $t_1-\tau \in [t_0-\gamma, t_0]$. Therefore, $y(t)$ is positive on $[t_0,\infty)$.

Remark 3.3. *There are apparent differences between the previous two theorems. Theorem 3.1 has weaker restrictions on the neutral coefficient $k(t)$ and does not have sign restrictions on the coefficients a, b. On the other hand, Theorem 3.2 does not require the restriction (C_2) of Theorem 3.1 or $\alpha = \tau$.*

Next, we study the distribution of zeros of all solutions of Eq.(1.2). It will be considered, without further mention, that $\alpha = \nu$ and $\nu > \tau$. The following two assumptions are very crucial for our results

(H_1) $\quad a(t) > c(t) \geq 0,\ b(t) > d(t) \geq 0, \quad t \geq t_0.$

(H_2) $\quad 0 < p'(t) + p(t)m_1(t) < m_2(t), \quad t \geq t_0.$

where p is a positive differentiable function on $[t_0,\infty)$ and

$$m_1(t) := \min\{a(t)-c(t), \tfrac{p(t)}{k(t)}(a(t-\tau)-c(t-\tau))\},\ t \geq t_0+\tau,$$

$$m_2(t) := \min\{b(t)-d(t), \tfrac{p(t)}{k(t-\nu)}(b(t-\tau)-d(t-\tau)\},\ t \geq t_0+\nu.$$

Theorem 3.4. *Assume that (H_1) and (H_2) hold. If (2.3) is satisfied for a positive integer n with $r = \nu-\tau$ and*

$$F_0(t) = \frac{e^{\int_{t+\tau-\nu}^{t} m_1(s)ds}\Big(m_2(t)-(p'(t)+p(t)m_1(t))\Big)}{1+p(t+\tau-\nu)}, \quad t \geq t_0+\nu-\tau$$

then Eq.(1.2) is oscillatory and $d_{t^*}(y) \leq 2\nu + (3n+1)(\nu-\tau)$, where $t^* \geq t_0 + (2n+2)(\nu-\tau)$.

Proof. Consider, for the sake of contradiction, that Eq.(1.2) has a solution $y(t) > 0$ on $[T_0, T]$ for some $T_0 \geq t^*$ and $T > T_0 + 2\nu + (3n+1)(\nu-\tau)$. Let

$$z(t) = y(t) + k(t)y(t-\tau). \tag{3.4}$$

Then $z(t) > 0$, for $t \in [T_0+\tau, T]$. Moreover, (1.2) implies

$$z'(t) = -a(t)y(t) - b(t)y(t-\nu) + c(t)\tanh y(t) + d(t)\tanh y(t-\nu). \tag{3.5}$$

But $\tanh y \leq y$ for $y \geq 0$, then we have

$$z'(t) \leq -(a(t)-c(t))y(t) - (b(t)-d(t))y(t-\nu) < 0, \quad \text{for } t \in [T_0+\nu, T]. \tag{3.6}$$

Therefore,

$$z'(t) + p(t)z'(t-\tau) \leq -m_1(t)z(t) - m_2(t)z(t-\nu),\ \text{for } t \in [T_0+\tau+\nu, T].$$

Let $w(t)$ be defined by

$$w(t) = e^{\int_{t_0}^{t} m_1(s)dt}(z(t) + p(t)z(t-\tau)). \tag{3.7}$$

Then

$$w'(t) \leq -e^{\int_{t_0}^{t} m_1(s)dt}\Big(m_2(t)z(t-\nu) - (p'(t)+p(t)m_1(s))z(t-\tau)\Big), \quad \text{for } t \in [T_0+\tau+\nu, T_1],$$

which can be written as follows

$$w'(t) \le -e^{\int_{t_0}^{t} m_1(s)dt}\Big(m_2(t) - (p'(t) + p(t)m_1(s))\Big)z(t-\nu), \quad \text{for } t \in [T_0 + 2\nu, T_1]. \tag{3.8}$$

It follows that $w(t) > 0$ for $t \in [T_0 + 2\tau, T]$, and

$$w'(t) < 0, \quad \text{for } t \in [T_0 + 2\nu, T]. \tag{3.9}$$

Also (3.7) and the decreasing nature of z (see (3.6)) imply that

$$w(t) \le e^{\int_{t_0}^{t} m_1(s)ds}(1 + p(t))z(t-\tau), \quad \text{for } t \in [T_0 + \tau + \nu, T] \tag{3.10}$$

which, in turn, leads to

$$z(t-\nu) \ge \frac{e^{-\int_{t_0}^{t+\tau-\nu} m_1(s)ds}}{(1 + p(t+\tau-\nu))} w(t+\tau-\nu), \quad \text{for } t \in [T_0 + 2\nu, T].$$

Combining this inequality with (3.8), we get

$$w'(t) + \frac{e^{\int_{t+\tau-\nu}^{t} m_1(s)ds}\Big(m_2(t) - (p'(t) + p(t)m_1(t))\Big)}{1 + p(t+\tau-\nu)} w(t+\tau-\nu) < 0, \quad t \ge t_0 + \nu - \tau,$$

for $t \in [T_0 + 2\nu, T]$. This differential inequality has the same form as (1.1) with $T_1 = T_0 + 2\nu$. Remember, from (3.7) and (3.9), that $w(t) > 0$ on $[T_1, T]$ and $w'(t) < 0$ on $[T_1, T]$. It follows that Lemma 2.2 holds with $\delta = 0$. Therefore, $w(t)$ can not be positive on $[T_1, T]$, where $T > T_0 + 2\nu + (3n + 1)(\nu - \tau)$. This contradiction completes the proof.

Theorem 3.5. *Assume that* (H_1) *and* (H_2) *hold. If* (2.4) *is satisfied with* $r = \nu - \tau$ *and*

$$q_1(t) = \frac{e^{\int_{t+\tau-\nu}^{t} m_1(s)ds}\Big(m_2(t) - (p'(t) + p(t)m_1(t))\Big)}{1 + p(t+\tau-\nu)}, \quad t \ge t_0 + \nu - \tau,$$

then Eq.(1.2) *is oscillatory and* $d_{t^*}(y) \le 2\nu + (n + 2)(\nu - \tau)$*, where* $t^* \ge t_0 + (n + 1)(\nu - \tau)$ *and* n *is defined as in Lemma 2.3.*

Proof. Consider, for the sake of contradiction, that Eq.(1.2) has a solution $y(t) > 0$ on $[T_0, T]$ for some $T_0 \ge t^*$ where $T > T_0 + 2\nu + (n + 2)(\nu - \tau)$. Proceeding as in the proof of Theorem 3.4, we obtain the following inequality

$$w'(t) + q_1(t)w(t+\tau-\nu) < 0, \text{ for } t \in [T_0 + 2\nu, T].$$

Taking $T_1 = T_0 + 2\nu$, then $w(t)$ satisfies the above inequality for $t \in [T_1, T]$, $w(t) > 0$, for $t \in [T_1, T]$ and $w'(t) < 0$, for $t \in [T_1, T]$. Now, Lemma 2.3 with $\delta = 0$ and $r = \nu - \tau$ implies that $w(t)$ can not be positive on $[T_1, T]$, where $T > T_1 + (n + 2)(\nu - \tau)$. This contradiction completes the proof.

The following result is proved by utilizing the reasoning of the previous proofs but with the application of Lemma 2.1. We omit the proof to avoid repetition.

Theorem 3.6. *Assume that* (H_1) *and* (H_2) *are satisfied and*

$$\int_{t+\tau-\nu}^{t} \frac{e^{\int_{s+\tau-\nu}^{s} m_1(u)du}\Big(m_2(s) - (p'(s) + p(s)m_1(s))\Big)}{1 + p(s+\tau-\nu)} ds \ge \rho, \quad t \ge t_0 + 2(\nu - \tau).$$

Then Eq.(1.2) is oscillatory and $d_{t^*}(y) \le 2\nu + (k_\rho + 1)(\nu - \tau)$*, where* $t^* \ge t_0 + 3(\nu - \tau)$ *and* k_ρ *is defined by Lemma 2.1.*

Example 3.7. *Consider the neutral differential equation*

$$\frac{d}{dt}\left(y(t)+2y(t-1)\right)=-2y(t)-8y(t-1.5)+\tanh y(t)+0.5\tanh y(t-1.5),\quad t\geq 0.$$

This equation has the form (1.2) with $a(t)\equiv 2$, $b(t)\equiv 8$, $c(t)\equiv 1$, $d(t)\equiv 0.5$, $k(t)\equiv 2$, $\tau=1$ *and* $\nu=1.5$. *Let us take* $p(t)\equiv 2$. *Then* $m_1(t)=1$ *for* $t\geq 1$, $m_2(t)=7.5$ *for all* $t\geq 2$, *and* (H_1), (H_2) *are satisfied. Furthermore,*

$$\int_{t+\tau-\nu}^{t}\frac{e^{\int_{s+\tau-\nu}^{s}m_1(u)du}\Big(m_2(s)-(p'(s)+p(s)m_1(s))\Big)}{1+p(s+\tau-\nu)}ds>1.5=\rho,\quad t\geq 1.$$

Therefore, $k_\rho=1$ *(see (2.2)) and hence Theorem 3.6 implies that* $d_{1.5}\leq 2\nu+(k_\rho+1)(\nu-\tau)=4$.

In each of the previous three estimates of the gabs between adjacent zeros we notice the leading term 2ν. In the following three results, we improve this term to $\tau+\nu$ by making use of the following condition which is a partial complement of (H_2).

(H_3) $\quad p'(t)+p(t)m_1(t)\leq 0,\quad t\geq t_0+\tau.$

Theorem 3.8. *Assume that* (H_1) *and* (H_3) *are satisfied. If* (2.3) *holds for a positive integer n with* $r=\nu-\tau$ *and*

$$F_0(t)=\frac{e^{\int_{t+\tau-\nu}^{t}m_1(s)ds}m_2(t)}{1+p(t+\tau-\nu)},\quad t\geq t_0+\nu-\tau$$

then Eq.(1.2) *is oscillatory and* $d_{t^*}(y)\leq\tau+\nu+(3n+1)(\nu-\tau)$, *where* $t^*\geq t_0+(2n+2)(\nu-\tau)$.

Proof. As usual, we assume that Eq.(1.2) has a solution $y(t)$ which is positive on $[T_0,T]$ for some $T_0\geq t^*$ where $T>T_0+\tau+\nu+(3n+1)(\nu-\tau)$. Proceeding as in the proof of Theorem 3.4, we obtain the following inequality

$$\frac{d}{dt}w(t)\leq e^{\int_{t_0}^{t}m_1(s)dt}(-m_2(t))z(t-\nu)+(p'(t)+p(t)m_1(t))z(t-\tau),\ \text{for } t\in[T_0+\tau+\nu,T].$$

Because of (H_3), we have

$$\frac{d}{dt}w(t)\leq -e^{\int_{t_0}^{t}m_1(s)dt}m_2(t)z(t-\nu),\quad \text{for } t\in[T_0+\tau+\nu,T]. \tag{3.11}$$

It follows that $w'(t)<0$, for $t\in[T_0+\tau+\nu,T]$. Remember also that $w(t)>0$ for $t\in[T_0+2\tau,T]$. Moreover, the decreasing nature of $z(t)$, by (3.6), implies that

$$w(t)=e^{\int_{t_0}^{t}m_1(s)dt}(z(t)+p(t)z(t-\tau))\leq e^{\int_{t_0}^{t}m_1(s)ds}(1+p(t))z(t-\tau)),\quad \text{for } t\in[T_0+\tau+\nu,T]$$

By combining this inequality with (3.11), we get

$$w'(t)+\frac{e^{\int_{t+\tau-\nu}^{t}m_1(s)ds}m_2(t)}{1+p(t+\tau-\nu)}w(t+\tau-\nu)<0,\quad \text{for } t\in[T_0+2\nu,T],$$

which has the same form as (1.1) with $T_1=T_0+2\nu$. Since (2.3) holds, Lemma 2.2 with $\delta=\nu-\tau$ implies that $w(t)$ can not be positive on $[T_1,T]$, where $T>T_0+\tau+\nu+(3n+1)(\nu-\tau)$. This contradiction completes the proof.

The following two results can be proved, when (H_3) holds, using the same reasoning of the previous result.

Theorem 3.9. *Assume that* (H_1) *and* (H_3) *are satisfied. If* (2.4) *holds with* $r = \nu - \tau$ *and*

$$q_1(t) = \frac{e^{\int_{t+\tau-\nu}^{t} m_1(s)ds} m_2(t)}{1 + p(t + \tau - \nu)}, \quad t \geq t_0 + \nu - \tau$$

then Eq.(1.2) *is oscillatory and* $d_{t^*}(y) \leq \tau + \nu + (n+2)(\nu - \tau)$, *where* $t^* \geq t_0 + (n+1)(\nu - \tau)$ *and* n *is defined as in Lemma 2.3.*

Theorem 3.10. *Assume that* (H_1) *and* (H_3) *hold, and*

$$\int_{t+\tau-\nu}^{t} \frac{e^{\int_{s+\tau-\nu}^{s} m_1(u)du} m_2(s)}{1 + p(s + \tau - \nu)} ds \geq \rho, \text{ for } t \geq t_0 + 2(\nu - \tau).$$

Then Eq.(1.2) is oscillatory and $d_{t^*}(y) \leq \tau + \nu + (k_\rho + 1)(\nu - \tau)$, *where* $t^* \geq t_0 + 3(\nu - \tau)$ *and* k_ρ *is defined by Lemma 2.1.*

We conclude this work with the following results concerning the case when neither (H_2) nor (H_3) is satisfied. We consider

(H_4) $\quad k(t) \geq 1, \quad t \geq t_0 + \max\{0, \nu - 2\tau\}.$

Theorem 3.11. *Assume that* (H_1) *and* (H_4) *hold. If* (2.3) *is satisfied for a positive integer* n *with* $r = \nu - \tau$ *and*

$$F_0(t) = \frac{(k(t + 2\tau - \nu) - 1)(b(t) - d(t))}{k(t + \tau - \nu)k(t + 2\tau - \nu)}, \quad t \geq t_0 + \max\{0, \nu - 2\tau\}, \tag{3.12}$$

then Eq.(1.2) *is oscillatory and* $d_{t^*}(y) \leq \tau + \nu + (3n+1)(\nu - \tau)$, *where* $t^* \geq t_0 + (2n+2)(\nu - \tau)$.

Proof. Assume that Eq.(1.2) has a solution $y(t) > 0$ on $[T_0, T]$ for some $T_0 \geq t^*$ and $T > T_0 + \tau + \nu + (3n+1)(\nu - \tau)$. Let

$$z(t) = y(t) + k(t)y(t - \tau). \tag{3.13}$$

It follows that (3.6) is satisfied, $z(t) > 0$ for $t \in [T_0 + \tau, T]$ and $z'(t) < 0$ for $t \in [T_0 + \nu, T]$. Furthermore, (3.13) implies

$$y(t - \tau) = \frac{z(t) - y(t)}{k(t)}, \quad t \in [T_0 + \tau, T],$$

and

$$y(t) = \frac{z(t + \tau) - y(t + \tau)}{k(t + \tau)}, \quad t \in [T_0 + \tau, T - \tau]. \tag{3.14}$$

Therefore,

$$\begin{aligned} y(t - \tau) &= \frac{1}{k(t)} \left(z(t) - \frac{z(t + \tau) - y(t + \tau)}{k(t + \tau)} \right) \\ &= \frac{1}{k(t)k(t + \tau)} \left(k(t + \tau)z(t) - z(t + \tau) + y(t + \tau) \right), \quad t \in [T_0 + \tau, T - \tau]. \end{aligned} \tag{3.15}$$

By using the decreasing nature of $z(t)$ on $[T_0 + \nu, T]$, we obtain

$$y(t - \tau) > \frac{k(t + \tau) - 1}{k(t)k(t + \tau)} z(t), \quad t \in [T_0 + \nu, T - \tau],$$

and hence

$$y(t-\nu) > \frac{k(t+2\tau-\nu)-1}{k(t+\tau-\nu)k(t+2\tau-\nu)}z(t+\tau-\nu).$$

By combining this inequality with (3.6),

$$z'(t) + \frac{(k(t+2\tau-\nu)-1)(b(t)-d(t))}{k(t+\tau-\nu)k(t+2\tau-\nu)}z(t+\tau-\nu) < 0, \quad t \in [T_0+2\nu-\tau, T-\tau]. \tag{3.16}$$

This inequality has the same form as (1.1) with $T_1 = T_0 + 2\nu - \tau$ and $T_2 = T - \tau$. Also, $z(t) > 0$ on $[T_1, T_2]$ and $z'(t) < 0$ on $[T_1 - (\nu-\tau), T]$. Then all requirements of Lemma 2.2 are satisfied with $\delta = \nu - \tau$. It follows that $z(t)$ can not be positive on $[T_1, T_2]$, where $T > T_0 + \tau + \nu + (3n+1)(\nu - \tau)$. This contradiction completes the proof.

Proceeding as in the proof of Theorem 3.11, we obtain (3.16). So, applying Lemma 2.1 and Lemma 2.3 leads to the following two results, respectively.

Theorem 3.12. *Assume that* (H_1) *and* (H_4) *hold, and*

$$\int_{t+\tau-\nu}^{t} \frac{(k(s+2\tau-\nu)-1)(b(s)-d(s))}{k(s+\tau-\nu)k(s+2\tau-\nu)}ds \geq \rho, \quad t \geq t_0 + \max\{0, \nu-2\tau\}.$$

Then Eq.(1.2) is oscillatory and $d_{t^*}(y) \leq \tau + \nu + (k_\rho + 1)(\nu - \tau)$*, where* $t^* \geq t_0 + 3(\nu - \tau)$ *and* k_ρ *is defined by Lemma 2.1.*

Theorem 3.13. *Assume that* (H_1) *and* (H_4) *hold, and* (2.4) *is satisfied with* $r = \nu - \tau$ *and*

$$q_1(t) = \frac{(k(t+2\tau-\nu)-1)(b(t)-d(t))}{k(t+\tau-\nu)k(t+2\tau-\nu)}, \quad t \geq t_0 + \max\{0, \nu-2\tau\}.$$

Then Eq.(1.2) *is oscillatory and* $d_{t^*}(y) \leq \tau + \nu + (n+2)(\nu - \tau)$*, where* $t^* \geq t_0 + (n+1)(\nu - \tau)$ *and* n *is defined as in Lemma 2.3.*

Example 3.14. *Consider the constant coefficients equation*

$$\frac{d}{dt}\Big(y(t) + \frac{3}{2}y(t-1)\Big) = -2y(t) - 4y(t-2) + \tanh y(t) + \tanh y(t-2), \quad t \geq 0.$$

Comparing with (1.2), it follows that $a(t) \equiv 2$, $b(t) \equiv 4$, $c(t) \equiv 1$, $d(t) \equiv 1$, $k(t) \equiv \frac{3}{2}$, $\tau = 1$ *and* $\nu = 2$. *Then* $r = \nu - \tau = 1$, *and* (H_1), (H_4) *are satisfied. Furthermore, (3.12) leads to*

$$F_0(t) = \frac{2}{3}, \quad t \geq 1.$$

Remembering the definition of F_n *given in Lemma 2.2, we obtain*

$$F_1(t) = \frac{2}{3}(e^{\frac{4}{3}} - e^{\frac{2}{3}}) \geq 1.2, \quad t \geq 2.$$

Therefore,

$$\int_{t+\tau-\nu}^{t} F_1(s)ds > 1.2, \quad t \geq 3.$$

Thus (2.3) *holds for* $n = 1$. *Consequently, Theorem (3.11) implies that the given equation is oscillatory and* $d_4(y) \leq 7$.

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