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Reverse Order Law for the Core Inverse of a Product of Two Complex Matrices

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Abstract. In this paper, the necessary and sufficient conditions for the reverse order law $(AB)^{\#\!\!\!\bigcirc} = B^{\#\!\!\!\bigcirc}A^{\#\!\!\!\bigcirc}$ are established. In addition, the hybrid reverse order laws $(AB)^{\dagger} = B^{\#\!\!\!\bigcirc}A^{\#\!\!\!\bigcirc}$ and $(AB)^{\#} = B^{\#\!\!\!\bigcirc}A^{\#\!\!\!\bigcirc}$ are also considered.Text of the abstract.

1. Introduction

Let $\mathbb{C}_{m,n}$ be the set of $m \times n$ complex matrices. The set of matrices of index one is denoted by $\mathbb{C}_n^{CM}$, i.e.,

$$\mathbb{C}_n^{CM} = \{A \in \mathbb{C}_{n,n} \mid \mathrm{rank}(A) = \mathrm{rank}(A^2)\}.$$

We denote the column space (range), row space and null space of a matrix A by $\mathcal{C}(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively.

If $A \in \mathbb{C}_{m,n}$, then the Moore-Penrose inverse $A^{\dagger}$ of A is the unique solution of the system of equations

$$(1)\ AXA = A, (2)\ XAX = X, (3)\ (AX)^* = AX, (4)\ (XA)^* = XA.$$

If $m = n$, then the group inverse $A^{\#}$ of A is the unique solution, if it exists, of the system of equations

$$AXA = A,\ XAX = X,\ AX = XA.$$

The core inverse for a complex matrix was introduced by Baksalary and Trenkler [1]. Then, Rakić et al. [2] generalized this concept to an arbitrary ring with an involution, and they used five equations to characterize the core inverse. Namely, X is the core inverse of A if it satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, $AX^2 = X$ and $XA^2 = A$. Such X, if it exists, is unique and it is denoted by $A^{\#\!\!\!\bigcirc}$. Later, Xu et al.[3] proved that these five equations can be reduced to three equations, i.e., if $A \in \mathbb{C}_{n,n}$, then the core inverse $A^{\#\!\!\!\bigcirc}$ of A is the unique solution, if it exists, of the system of equations

$$(AX)^* = AX,\ \ AX^2 = X,\ \ XA^2 = A.$$

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There is a dual concept of the core inverse which is called the dual core inverse and denoted $A_{\#}$. It is well known that $A^{\#}$ exists if and only if $A \in \mathbb{C}_n^{CM}$. Also, Baksalary and Trenkler [1] pointed out that $A^{\#}$ exists if and only if $A \in \mathbb{C}_n^{CM}$.

As is known to all, if $A, B \in \mathbb{C}_{n,n}$ are two invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. The latter equation is called the reverse order law. In 1966, Greville [4] investigated the problem when $B^{\dagger}A^{\dagger}$ is the Moore-Penrose inverse of AB. Since then, the reverse order law for the generalized inverse has been widely studied, see for example, [5] –[20]. Some conditions for the hybrid reverse order law $(AB)^{\#} = B^{\dagger}A^{\dagger}$ in rings with involution were studied in [16].

Baksalary and Trenkler [17] proposed the following problem on the reverse order law for the core inverse:

$$\text{If } A^{\#}, B^{\#} \text{ and } (AB)^{\#} \text{ exist, does it follow that } (AB)^{\#} = B^{\#}A^{\#}\ ?$$

Later, Cohen et al. [18] gave several counterexamples for the problem. In [19], Wang and Liu investigated equivalent conditions of the reverse order law $(AB)^{\#} = B^{\#}A^{\#}$ by using the ranks of matrices. Zou et al. [20] considered the reverse order law for the core inverse in rings with involution.

In this paper, we consider the problem when $(AB)^{\#} = B^{\#}A^{\#}$. Thus, we investigate equivalent conditions for the reverse order law for the core inverse. In addition, motivated by the hybrid reverse order law from [16], we give necessary and sufficient conditions for the following hybrid reverse order laws $(AB)^{\dagger} = B^{\#}A^{\#}$ and $(AB)^{\#} = B^{\#}A^{\#}$ to hold, respectively.

The following lemmas will often be used later in this paper.

Lemma 1.1. *(I) Let $A \in \mathbb{C}_n^{CM}$, then the following hold:*
(i) $\mathcal{R}(A^*) = \mathcal{R}(A^{\#})$ and $C((A^{\#})^*) = C(A) = C(A^{\#})$;
(ii) $A^{\#}B = A^{\#}C$ if and only if $A^*B = A^*C$;
(iii) $BA^{\#} = CA^{\#}$ if and only if $BA = CA$.
(II) Let $A \in \mathbb{C}_{m,n}$, then the following hold:
(i) $\mathcal{R}(A^*) = \mathcal{R}(A^{\dagger})$ and $C(A^*) = C(A^{\dagger})$;
(ii) $A^{\dagger}B = A^{\dagger}C$ if and only if $A^*B = A^*C$;
(iii) $BA^{\dagger} = CA^{\dagger}$ if and only if $BA^* = CA^*$.

Proof. Only the result (I) is proved here, the proof of the result (II) is left to the reader.

(i). Since $A^* = (AA^{\#}A)^* = A^*AA^{\#}$ and $A^{\#} = A^{\#}AA^{\#} = A^{\#}(A^{\#})^*A^*$, we obtain $\mathcal{R}(A^*) = \mathcal{R}(A^{\#})$.

$(A^{\#})^* = (A^{\#}AA^{\#})^* = AA^{\#}(A^{\#})^*$ and $A = AA^{\#}A = (A^{\#})^*A^*A$, $A = A^{\#}A^2$ and $A^{\#} = AA^{\#}A^{\#}$, so $C((A^{\#})^*) = C(A) = C(A^{\#})$.

(ii). If $A^{\#}B = A^{\#}C$, then pre-multiplying matrices in this equality by A^*A, we have $A^*B = A^*C$.

Conversely, pre-multiplying matrices in $A^*B = A^*C$ by $A^{\#}(A^{\#})^*$, we obtain $A^{\#}B = A^{\#}C$.

(iii). Assume that $BA^{\#} = CA^{\#}$, then $BA = BA^{\#}A^2 = CA^{\#}A^2 = CA$.

Conversely, if $BA = CA$, then $BA^{\#} = BAA^{\#}A^{\#} = CAA^{\#}A^{\#} = CA^{\#}$. □

Lemma 1.2. *[21, Proposition 4] Let $A \in \mathbb{C}_{m,n}$ and let $F \in \mathbb{C}_{n,n}$ be idempotent. Then*

$$\mathcal{N}(AF) = (\mathcal{N}(A) \cap C(F)) \oplus \mathcal{N}(F).$$

2. Characterizations of $(AB)^{\#} = B^{\#}A^{\#}$

In this section, we investigate the reverse order law $(AB)^{\#} = B^{\#}A^{\#}$. Before that, some auxiliary results will be presented for further reference.

Lemma 2.1. *Let $B \in \mathbb{C}_{n,n}$ and $A \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then*

$$(AB)^{\#}AA^{\#} = (AB)^{\#} = A^{\#}A(AB)^{\#}. \tag{1}$$

Proof. Firstly, we show that $(AB)^{\#}AA^{\#} = (AB)^{\#}$. Since

$$\begin{aligned}((AB)^{\#}AA^{\#})^* &= AA^{\#}((AB)^{\#})^* = AA^{\#}((AB)^{\#}AB(AB)^{\#})^* \\ &= AA^{\#}AB(AB)^{\#}((AB)^{\#})^* = AB(AB)^{\#}((AB)^{\#})^* \\ &= ((AB)^{\#})^*,\end{aligned}$$

taking an involution on this equality, we obtain $(AB)^{\#}AA^{\#} = (AB)^{\#}$.

Next, since

$$A^{\#}A(AB)^{\#} = A^{\#}AAB(AB)^{\#}(AB)^{\#} = AB(AB)^{\#}(AB)^{\#} = (AB)^{\#},$$

we get $(AB)^{\#} = A^{\#}A(AB)^{\#}$. □

Lemma 2.2. *Let $A \in \mathbb{C}_n^{CM}$, then*

$$AA^{\#} = AA^{\dagger}. \tag{2}$$

Proof. $AA^{\#} = (AA^{\#})^* = (AA^{\dagger}AA^{\#})^* = (AA^{\#})^*(AA^{\dagger})^* = AA^{\#}AA^{\dagger} = AA^{\dagger}$. □

Now, we give some necessary and sufficient conditions for $(AB)^{\#} = B^{\#}A^{\#}$ to hold.

Theorem 2.3. *Let $A, B \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then the following statements are equivalent:*
(i) $(AB)^{\#} = B^{\#}A^{\#}$;
(ii) $\mathcal{C}(B^{\#}A) = \mathcal{C}(AB)$ and $(AB)^* = (AB)^*ABB^{\#}A^{\#}$;
(iii) $\mathcal{C}(B^{\#}A) \subseteq \mathcal{C}(AB)$ and $(AB)^* = (AB)^*ABB^{\#}A^{\#}$;
(iv) $\mathcal{C}(B^{\#}A) = \mathcal{C}(AB)$ and $(AB)^*A^2 = (AB)^*ABB^{\#}A$;
(v) $\mathcal{C}(B^{\#}A) \subseteq \mathcal{C}(AB)$ and $(AB)^*A^2 = (AB)^*ABB^{\#}A$;
(vi) $\mathcal{C}(B^{\#}A) = \mathcal{C}(AB)$ and $\mathcal{C}(A^*AB) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$;
(vii) $\mathcal{C}(B^{\#}A) \subseteq \mathcal{C}(AB)$ and $\mathcal{C}(A^*AB) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$;
(viii) $\mathcal{C}(B^{\#}A) = \mathcal{C}(AB)$ and $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\right)$;
(ix) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*BAB \\ (AB)^*(AB)^2 \end{bmatrix}\right)$.

Proof. (i) ⇒ (ii). Suppose that $(AB)^{\#} = B^{\#}A^{\#}$. Then

$$(AB)^* = (AB)^*AB(AB)^{\#} = (AB)^*ABB^{\#}A^{\#}.$$

In addition,

$$AB = (AB)^{\#}(AB)^2 = B^{\#}A^{\#}(AB)^2 = B^{\#}AA^{\#}A^{\#}(AB)^2 \in \mathcal{C}(B^{\#}A),$$

$$B^{\#}A = B^{\#}A^{\#}A^2 = (AB)^{\#}A^2 = AB((AB)^{\#})^2A^2 \in \mathcal{C}(AB),$$

thus $\mathcal{C}(B^{\#}A) = \mathcal{C}(AB)$.

(ii) ⇒ (iii). Clearly.

(iii) ⇒ (i). Suppose that $\mathcal{C}(B^{\#}A) \subseteq \mathcal{C}(AB)$ and $(AB)^* = (AB)^*ABB^{\#}A^{\#}$. From the former condition we get $B^{\#}A = (AB)^{\#}ABB^{\#}A$, which is equivalent to $B^{\#}A^{\#} = (AB)^{\#}ABB^{\#}A^{\#}$ by Lemma 1.1. Moreover, since $(AB)^* = (AB)^*ABB^{\#}A^{\#}$, or equivalently, $(AB)^{\#} = (AB)^{\#}ABB^{\#}A^{\#}$, showing that

$$B^{\#}A^{\#} = (AB)^{\#}ABB^{\#}A^{\#} = (AB)^{\#}.$$

(ii) ⇒ (iv). Applying by A^2 from the right of $(AB)^* = (AB)^*ABB^{\#}A^{\#}$ leads to $(AB)^*A^2 = (AB)^*ABB^{\#}A$.

(iii) ⇒ (v). Similarly as (ii) ⇒ (iv).

(iv) ⇒ (ii). By lemma 1.1, condition $(AB)^*A^2 = (AB)^*ABB^{\#}A$ is equivalent to $(AB)^*AA^{\#} = (AB)^*ABB^{\#}A^{\#}$. Since $(AB)^*AA^{\#} = (AA^{\#}AB)^* = (AB)^*$, implying $(AB)^* = (AB)^*ABB^{\#}A^{\#}$.

(v) ⇒ (iii). Similarly as (iv) ⇒ (ii).

(iv) ⇔ (vi) and (v) ⇔ (vii). Since $(AB)^*A^2 = (AB)^*ABB^{\#}A$ can be written as $(AB)^*A(I - BB^{\#})A = O$, which is equivalent to $\mathcal{C}(A^*AB) \subseteq \mathcal{N}(A^*(I - BB^{\#}))$, implying $\mathcal{C}(A^*AB) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$ according to Lemma 1.2.

(iv) ⇒ (viii). We write the equality $(AB)^*A^2 = (AB)^*ABB^{\#}A$ as

$$[-(AB)^*ABB^{\#}(B^{\#})^*, (B^{\#}B)^*]\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix} = O. \tag{3}$$

Let $T \in \mathbb{C}_{n,2n}$ denote the matrix

$$T = [-(AB)^*ABB^{\#}(B^{\#})^*, (B^{\#}B)^*],$$

then $T^- = \begin{bmatrix} O \\ (B^{\#}B)^* \end{bmatrix}$ is an inner inverse of T and

$$I - T^-T = \begin{bmatrix} I & O \\ (AB)^*ABB^{\#}(B^{\#})^* & I - (B^{\#}B)^* \end{bmatrix}.$$

Since $\mathcal{N}(T) = \mathcal{C}(I - T^-T)$ and $(AB)^* = (B^{\#}B)^*(AB)^*$, if the condition (3) is fulfilled then

$$\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} I \\ (AB)^*ABB^{\#}(B^{\#})^* \end{bmatrix}\right).$$

Applying $(B^{\#}B)^*$ on the left leads to

$$\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix} B^{\#}(B^{\#})^*\right),$$

which shows the conclusion

$$\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\right).$$

(viii) ⇒ (ix). The hypothesis $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\right)$ follows that for any $\mathbf{x} \in \mathbb{C}^n$, there exists $\mathbf{u} \in \mathbb{C}^n$ such that $\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\mathbf{u}$. Thus $B^*A\mathbf{x} = B^*B\mathbf{u}$, or equivalently,

$$B^{\#}A\mathbf{x} = B^{\#}B\mathbf{u}.$$

Assumption $\mathcal{C}(B^{\#}A) = \mathcal{C}(AB)$ shows that

$$\mathcal{C}(AB) \subseteq \mathcal{C}(B^{\#}) = \mathcal{C}(B)$$

and

$$B^{\#}A\mathbf{x} = AB\mathbf{z}$$

for some $\mathbf{z} \in \mathbb{C}^n$. Therefore,

$$\begin{aligned}
\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} &= \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\mathbf{u} = \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix} B^{\#}B\mathbf{u} \\
&= \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix} B^{\#}A\mathbf{x} = \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix} AB\mathbf{z} \\
&= \begin{bmatrix} B^*BAB \\ (AB)^*(AB)^2 \end{bmatrix}\mathbf{z},
\end{aligned}$$

showing the conclusion $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*BAB \\ (AB)^*(AB)^2 \end{bmatrix}\right)$.

(ix) ⇒ (v). Hypotheses $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*BAB \\ (AB)^*(AB)^2 \end{bmatrix}\right)$ show that $AB = B^{\#}BAB$ and for any $\mathbf{x} \in \mathbb{C}^n$ there exists $\mathbf{u} \in \mathbb{C}^n$ such that $\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} B^*BAB \\ (AB)^*(AB)^2 \end{bmatrix}\mathbf{u}$, respectively. Thus $B^*A\mathbf{x} = B^*BAB\mathbf{u}$, or equivalently, $B^{\#}A\mathbf{x} = B^{\#}BAB\mathbf{u}$, showing that $B^{\#}A\mathbf{x} = AB\mathbf{u}$, so $\mathcal{C}(B^{\#}A) \subseteq \mathcal{C}(AB)$. Furthermore,

$$\begin{aligned} ((AB)^*A^2 - (AB)^*ABB^{\#}A)\mathbf{x} &= [-(AB)^*ABB^{\#}(B^{\#})^*, (B^{\#}B)^*]\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} \\ &= [-(AB)^*ABB^{\#}(B^{\#})^*, (B^{\#}B)^*]\begin{bmatrix} B^*BAB \\ (AB)^*(AB)^2 \end{bmatrix}\mathbf{u} \\ &= O, \end{aligned}$$

thus $(AB)^*A^2 = (AB)^*ABB^{\#}A$. □

Theorem 2.3 is based on equality $(AB)^* = (AB)^*AB(AB)^{\#}$, the following theorem is based on another fact, that is $AB = (AB)^{\#}(AB)^2$.

Theorem 2.4. *Let $A, B \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then the following statements are equivalent:*
(i) $(AB)^{\#} = B^{\#}A^{\#}$;
(ii) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$ and $AB = B^{\#}A^{\#}(AB)^2$;
(iii) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$ and $AB = B^{\#}A^{\#}(AB)^2$;
(iv) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $B^*BAB = B^*A^{\#}(AB)^2$;
(v) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $B^*BAB = B^*A^{\#}(AB)^2$;
(vi) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(BAB) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$;
(vii) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(BAB) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$.
If $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$, then the above statements are also equivalent to the following statements:
(viii) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^2, (AB)^2])$;
(ix) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^2, (AB)^2])$;
(x) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([(AB)^*A^2, (AB)^*(AB)^2])$.

Proof. Firstly, we prove that statements (i)-(vii) are equivalent.
(i) ⇒ (ii). If $(AB)^{\#} = B^{\#}A^{\#}$, then

$$AB = (AB)^{\#}(AB)^2 = B^{\#}A^{\#}(AB)^2.$$

It remains to show that $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$. On the one hand,

$$\begin{aligned} (A^{\#})^*B &= (A^{\#})^*BB^{\#}B = (A^{\#})^*(B^{\#})^*B^*B = (B^{\#}A^{\#})^*B^*B \\ &= ((AB)^{\#})^*B^*B = ((AB)^{\#}AB(AB)^{\#})^*B^*B \\ &= AB(AB)^{\#}((AB)^{\#})^*B^*B \end{aligned}$$

gives $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$. On the other hand,

$$\begin{aligned} AB &= AB(AB)^{\#}AB = ((AB)^{\#})^*(AB)^*AB = (B^{\#}A^{\#})^*(AB)^*AB \\ &= (A^{\#})^*(B^{\#})^*(AB)^*AB = (A^{\#})^*(B^{\#}BB^{\#})^*(AB)^*AB \\ &= (A^{\#})^*BB^{\#}(B^{\#})^*(AB)^*AB \end{aligned}$$

yields $\mathcal{C}(AB) \subseteq \mathcal{C}((A^{\#})^*B)$. Therefore, $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$.
(ii) ⇒ (iii). Obviously.

(iii) ⇒ (i). Assume that $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$ and $AB = B^{\#}A^{\#}(AB)^2$. The former equality yields $(A^{\#})^*B = AB(AB)^{\#}(A^{\#})^*B$. Taking an involution on this equality, we get $B^*A^{\#} = B^*A^{\#}AB(AB)^{\#}$, which is equivalent to $B^{\#}A^{\#} = B^{\#}A^{\#}AB(AB)^{\#}$ by Lemma 1.1. Therefore,

$$B^{\#}A^{\#} = B^{\#}A^{\#}AB(AB)^{\#} = B^{\#}A^{\#}(AB)^2[(AB)^{\#}]^2 = AB[(AB)^{\#}]^2 = (AB)^{\#}.$$

(ii) ⇒ (iv). It remains to show that $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $B^*BAB = B^*A^{\#}(AB)^2$. According to $AB = B^{\#}A^{\#}(AB)^2$ and Lemma 1.1,

$$\mathcal{C}(AB) = \mathcal{C}(B^{\#}A^{\#}(AB)^2) \subseteq \mathcal{C}(B^{\#}) = \mathcal{C}(B).$$

Pre-multiplying matrices in $AB = B^{\#}A^{\#}(AB)^2$ by B^*B, we obtain $B^*BAB = B^*A^{\#}(AB)^2$.

(iii) ⇒ (v). Similarly as (ii) ⇒ (iv).

(iv) ⇒ (ii). From the condition $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, we obtain $AB = B^{\#}BAB$. Using Lemma 1.1, $B^*BAB = B^*A^{\#}(AB)^2$ is equivalent to $B^{\#}BAB = B^{\#}A^{\#}(AB)^2$. Hence,

$$AB = B^{\#}A^{\#}(AB)^2.$$

(v) ⇒ (iii). Similarly as (iv) ⇒ (ii).

(iv) ⇔ (vi) and (v) ⇔ (vii). Since $B^*BAB = B^*A^{\#}(AB)^2$ can be written as $B^*(I - A^{\#}A)BAB = O$, which is equivalent to $\mathcal{C}(BAB) \subseteq \mathcal{N}(B^*(I - A^{\#}A))$, implying $\mathcal{C}(BAB) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$ according to Lemma 1.2.

Next, we show that statements (viii)-(x) are equivalent to statements (i)-(vii) when $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$. Notice that $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$ is equivalent to

$$AB = ABA^{\#}A. \tag{4}$$

(iv) ⇒ (viii). It remains to show that $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^2, (AB)^2])$. The equality $B^*BAB = B^*A^{\#}(AB)^2$ can be written as

$$[B^*A, B^*BAB]\begin{bmatrix} -A^{\#}A^{\#}(AB)^2 \\ B^{\#}B \end{bmatrix} = O. \tag{5}$$

Let $T \in \mathbb{C}_{2n,n}$ denote the matrix

$$T = \begin{bmatrix} -A^{\#}A^{\#}(AB)^2 \\ B^{\#}B \end{bmatrix}.$$

It is easy to prove that $T^- = [O, B^{\#}B]$ is an inner inverse of T and

$$I - TT^- = \begin{bmatrix} I & A^{\#}A^{\#}(AB)^2 \\ O & I - B^{\#}B \end{bmatrix}.$$

From $\mathcal{N}(T^*) = \mathcal{C}(I - (T^-)^*T^*)$ and $AB = ABB^{\#}B$, from equality (5), we obtain

$$\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([I, A^{\#}A^{\#}(AB)^2]). \tag{6}$$

According to equality (4), applying $A^{\#}A$ on the right of (6) leads to $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^{\#}A, A^{\#}A^{\#}(AB)^2]) = \mathcal{R}(A^{\#}A^{\#}[A^2, (AB)^2])$, which shows that

$$\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^2, (AB)^2]).$$

(viii) ⇒ (ix). Obviously.

(ix) ⇒ (x). Since $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^2, (AB)^2])$, then for any $\mathbf{x} \in \mathbb{C}_{1,n}$ there exists $\mathbf{u} \in \mathbb{C}_{1,n}$ such that $\mathbf{x}[B^*A, B^*BAB] = \mathbf{u}[A^2, (AB)^2]$. So $\mathbf{x}B^*A = \mathbf{u}A^2$, equivalently,

$$\mathbf{x}B^*A^{\#} = \mathbf{u}AA^{\#}. \tag{7}$$

From the condition $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, there exists $\mathbf{z} \in \mathbb{C}_{1,n}$ such that

$$\mathbf{x}B^*A^{\#} = \mathbf{z}(AB)^*. \tag{8}$$

Furthermore,

$$\begin{aligned}\mathbf{x}[B^*A, B^*BAB] &= \mathbf{u}[A^2,(AB)^2] = \mathbf{u}[AA^{\#}A^2, AA^{\#}(AB)^2]\\ &= \mathbf{u}AA^{\#}[A^2,(AB)^2] \overset{(7)}{=} \mathbf{x}B^*A^{\#}[A^2,(AB)^2]\\ &\overset{(8)}{=} \mathbf{z}(AB)^*[A^2,(AB)^2] = \mathbf{z}[(AB)^*A^2,(AB)^*(AB)^2].\end{aligned}$$

Hence, $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([(AB)^*A^2,(AB)^*(AB)^2])$.

(x) $\Rightarrow$ (v). Assume that $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([(AB)^*A^2,(AB)^*(AB)^2])$. The latter condition shows that for any $\mathbf{x} \in \mathbb{C}_{1,n}$, there exists $\mathbf{z} \in \mathbb{C}_{1,n}$ such that

$$\mathbf{x}[B^*A, B^*BAB] = \mathbf{z}[(AB)^*A^2,(AB)^*(AB)^2].$$

Thus, $\mathbf{x}B^*A = \mathbf{z}(AB)^*A^2$, equivalently,

$$\mathbf{x}B^*A^{\#} = \mathbf{z}(AB)^*AA^{\#} = \mathbf{z}(AA^{\#}AB)^* = \mathbf{z}(AB)^*,$$

taking an involution on this equality,

$$(A^{\#})^*B\mathbf{x}^* = AB\mathbf{z}^*,$$

hence, $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$. Since

$$\begin{aligned}\mathbf{x}(B^*BAB - B^*A^{\#}(AB)^2) &= \mathbf{x}[B^*A, B^*BAB]\begin{bmatrix} -A^{\#}A^{\#}(AB)^2 \\ B^{\#}B \end{bmatrix}\\ &= \mathbf{z}[(AB)^*A^2,(AB)^*(AB)^2]\begin{bmatrix} -A^{\#}A^{\#}(AB)^2 \\ B^{\#}B \end{bmatrix}\\ &= O,\end{aligned}$$

it shows that $B^*BAB = B^*A^{\#}(AB)^2$. □

Remark 2.5. *From the proof of Theorem 2.4, it is easily to see that statements (viii)-(x) are equivalent, and they lead to statements (i)-(vii) in the absence of condition* $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$.

If we suppose that matrices A and B commute, we obtain the following equivalent conditions for the reverse order law to be satisfied for the core inverse.

Corollary 2.6. *Let* $A, B \in \mathbb{C}_n^{CM}$. *If* $AB \in \mathbb{C}_n^{CM}$ *and* $AB = BA$*, then the following statements are equivalent:*
(i) $(AB)^{\#} = B^{\#}A^{\#}$;
(ii) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$;
(iii) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$;
(iv) $\mathcal{C}(A^*B) = \mathcal{C}(A^*A^*BA)$;
(v) $\mathcal{C}(A^*B) \subseteq \mathcal{C}(A^*A^*BA)$;
(vi) $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([(AB)^*A^2,(AB)^*(AB)^2])$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vi). Since $AB = BA$, $\mathcal{R}(AB) = \mathcal{R}(BA) \subseteq \mathcal{R}(A)$. Thus statements (i)-(x) in Theorem 2.4 are equivalent. By the equivalence of statements (i), (ii), (iii) and (x) in Theorem 2.4, we obtain the conclusion.

(ii) $\Rightarrow$ (iv). Since $AB = BA$, $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB) = \mathcal{C}(BA)$, multiplying this equality from the left by A^*A yields $\mathcal{C}(A^*B) = \mathcal{C}(A^*A^*BA)$.

(iv) $\Rightarrow$ (v). Obviously.

(v) $\Rightarrow$ (iii). Since $AB = BA$ and $\mathcal{C}(A^*B) \subseteq \mathcal{C}(A^*A^*BA)$, $\mathcal{C}((A^{\#})^*B) = \mathcal{C}((A^{\#}A^{\#})^*A^*B) \subseteq \mathcal{C}((A^{\#}A^{\#})^*A^*A^*BA) = \mathcal{C}(AA^{\#}BA) = \mathcal{C}(AB)$. □

We continue studying the reverse order law for the core inverse.

Theorem 2.7. *Let $A, B \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then the following statements are equivalent:*
(i) $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$;
(ii) $(AB)^{\textcircled{\#}}A = B^{\textcircled{\#}}A^{\textcircled{\#}}A$;
(iii) $(AB)^{\textcircled{\#}}A^2 = B^{\textcircled{\#}}A$.

Proof. (i) ⇒ (ii) ⇒ (iii). Obviously.
(iii) ⇒ (i). From $(AB)^{\textcircled{\#}}A^2 = B^{\textcircled{\#}}A$ and Lemma 1.1, we get $(AB)^{\textcircled{\#}}AA^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$. By Lemma 2.1, we deduce that $(AB)^{\textcircled{\#}} = A^{\textcircled{\#}}B^{\textcircled{\#}}$. □

Theorem 2.8. *Let $A, B \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$ if and only if $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and one of the following equivalent statements holds:*
(i) $BB^{\textcircled{\#}}A^{\textcircled{\#}} = B(AB)^{\textcircled{\#}}$;
(ii) $B^*A^{\textcircled{\#}} = B^*B(AB)^{\textcircled{\#}}$;
(iii) $B^*A = B^*B(AB)^{\textcircled{\#}}A^2$;
(iv) $BB^{\textcircled{\#}}A = B(AB)^{\textcircled{\#}}A^2$;
(v) $BB^{\dagger}A = B(AB)^{\textcircled{\#}}A^2$.

Proof. In the first place, we prove that statements (i)-(v) are equivalent.
(i) ⇒ (ii). Pre-multiplying matrices in $BB^{\textcircled{\#}}A^{\textcircled{\#}} = B(AB)^{\textcircled{\#}}$ by B^*, we get $B^*A^{\textcircled{\#}} = B^*B(AB)^{\textcircled{\#}}$.
(ii) ⇒ (iii). Multiplying $B^*A^{\textcircled{\#}} = B^*B(AB)^{\textcircled{\#}}$ from the right by A^2, we get $B^*A = B^*B(AB)^{\textcircled{\#}}A^2$.
(iii) ⇒ (iv). Since $B^*A = B^*B(AB)^{\textcircled{\#}}A^2$, multiplying $(B^{\textcircled{\#}})^*$ on the left leads to $BB^{\textcircled{\#}}A = B(AB)^{\textcircled{\#}}A^2$.
(iv) ⇒ (i). Using Lemma 1.1, $BB^{\textcircled{\#}}A = B(AB)^{\textcircled{\#}}A^2$ is equivalent to $BB^{\textcircled{\#}}A^{\textcircled{\#}} = B(AB)^{\textcircled{\#}}AA^{\textcircled{\#}}$. And from Lemma 2.1, $BB^{\textcircled{\#}}A^{\textcircled{\#}} = B(AB)^{\textcircled{\#}}$.
(iv) ⇔ (v). According to Lemma 2.2.
In the second place, we show that $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$ if and only if $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and the statement (ii) holds.
Let $(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}$. Then

$$B^*A^{\textcircled{\#}} = B^*BB^{\textcircled{\#}}A^{\textcircled{\#}} = B^*B(AB)^{\textcircled{\#}}.$$

According to Lemma 1.1,

$$\mathcal{C}(AB) = \mathcal{C}((AB)^{\textcircled{\#}}) = \mathcal{C}(B^{\textcircled{\#}}A^{\textcircled{\#}}) \subseteq \mathcal{C}(B^{\textcircled{\#}}) = \mathcal{C}(B).$$

Conversely assume that $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ and $B^*A^{\textcircled{\#}} = B^*B(AB)^{\textcircled{\#}}$. By Lemma 1.1, the latter equality is equivalent to

$$B^{\textcircled{\#}}A^{\textcircled{\#}} = B^{\textcircled{\#}}B(AB)^{\textcircled{\#}}.$$

From $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, we see that $AB = B^{\textcircled{\#}}BAB$. Hence, by Lemma 1.1,

$$(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}B(AB)^{\textcircled{\#}}.$$

Therefore,

$$(AB)^{\textcircled{\#}} = B^{\textcircled{\#}}A^{\textcircled{\#}}.$$

□

3. Characterizations of $(AB)^{\dagger} = B^{\textcircled{\#}}A^{\textcircled{\#}}$ and $(AB)^{\#} = B^{\textcircled{\#}}A^{\textcircled{\#}}$

In the first part of this section, we study equivalent conditions for the hybrid reverse order law $(AB)^{\dagger} = B^{\textcircled{\#}}A^{\textcircled{\#}}$ to be satisfied.

Theorem 3.1. *Let $A, B \in \mathbb{C}_n^{CM}$. Then the following statements are equivalent:*
(i) $(AB)^{\dagger} = B^{\textcircled{\#}}A^{\textcircled{\#}}$;
(ii) $\mathcal{C}(B^{\textcircled{\#}}A) = \mathcal{C}((AB)^*)$ *and* $(AB)^* = (AB)^*ABB^{\textcircled{\#}}A^{\textcircled{\#}}$;
(iii) $\mathcal{C}(B^{\textcircled{\#}}A) \subseteq \mathcal{C}((AB)^*)$ *and* $(AB)^* = (AB)^*ABB^{\textcircled{\#}}A^{\textcircled{\#}}$;
(iv) $\mathcal{C}(B^{\textcircled{\#}}A) = \mathcal{C}((AB)^*)$ *and* $(AB)^*A^2 = (AB)^*ABB^{\textcircled{\#}}A$;

(v) $\mathcal{C}(B^{\circledast}A) \subseteq \mathcal{C}((AB)^*)$ *and* $(AB)^*A^2 = (AB)^*ABB^{\circledast}A$*;*
(vi) $\mathcal{C}(B^{\circledast}A) = \mathcal{C}((AB)^*)$ *and* $\mathcal{C}(A^*AB) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$*;*
(vii) $\mathcal{C}(B^{\circledast}A) \subseteq \mathcal{C}((AB)^*)$ *and* $\mathcal{C}(A^*AB) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$*;*
(viii) $\mathcal{C}(B^{\circledast}A) = \mathcal{C}((AB)^*)$ *and* $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\right)$*;*
(ix) $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B(AB)^* \\ (AB)^*AB(AB)^* \end{bmatrix}\right)$*.*

Proof. (i) ⇒ (ii). Suppose that $(AB)^\dagger = B^{\circledast}A^{\circledast}$. Then

$$(AB)^* = (AB)^*AB(AB)^\dagger = (AB)^*ABB^{\circledast}A^{\circledast}.$$

In addition,

$$(AB)^* = (AB)^\dagger AB(AB)^* = B^{\circledast}A^{\circledast}AB(AB)^* = B^{\circledast}AA^{\circledast}A^{\circledast}AB(AB)^*$$

and

$$B^{\circledast}A = B^{\circledast}A^{\circledast}A^2 = (AB)^\dagger A^2 = (AB)^*((AB)^\dagger)^*(AB)^\dagger A^2$$

imply that $\mathcal{C}(B^{\circledast}A) = \mathcal{C}((AB)^*)$.

(ii) ⇒ (iii). Clearly.

(iii) ⇒ (i). Suppose that $\mathcal{C}(B^{\circledast}A) \subseteq \mathcal{C}((AB)^*)$ and $(AB)^* = (AB)^*ABB^{\circledast}A^{\circledast}$. From the former condition we get $B^{\circledast}A = (AB)^\dagger ABB^{\circledast}A$, which is equivalent to $B^{\circledast}A^{\circledast} = (AB)^\dagger ABB^{\circledast}A^{\circledast}$ by Lemma 1.1. Moreover, since $(AB)^* = (AB)^*ABB^{\circledast}A^{\circledast}$, or equivalently, $(AB)^\dagger = (AB)^\dagger ABB^{\circledast}A^{\circledast}$, we get

$$B^{\circledast}A^{\circledast} = (AB)^\dagger ABB^{\circledast}A^{\circledast} = (AB)^\dagger.$$

(ii) ⇒ (iv). Applying A^2 on the right of $(AB)^* = (AB)^*ABB^{\circledast}A^{\circledast}$ leads to $(AB)^*A^2 = (AB)^*ABB^{\circledast}A$.

(iii) ⇒ (v). Similarly as (ii) ⇒ (iv).

(iv) ⇒ (ii). By lemma 1.1, condition $(AB)^*A^2 = (AB)^*ABB^{\circledast}A$ is equivalent to $(AB)^*AA^{\circledast} = (AB)^*ABB^{\circledast}A^{\circledast}$. Since $(AB)^*AA^{\circledast} = (AA^{\circledast}AB)^* = (AB)^*$, showing the conclusion $(AB)^* = (AB)^*ABB^{\circledast}A^{\circledast}$.

(v) ⇒ (iii). Similarly as (iv) ⇒ (ii).

(iv) ⇔ (vi) and (v) ⇔ (vii). By the proof Theorem 2.3, $(AB)^*A^2 = (AB)^*ABB^{\circledast}A$ is equivalent to $\mathcal{C}(A^*AB) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$.

(iv) ⇒ (viii). We write the equality $(AB)^*A^2 = (AB)^*ABB^{\circledast}A$ as

$$[-(AB)^*ABB^{\circledast}(B^{\circledast})^*, (B^{\circledast}B)^*]\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix} = O. \tag{9}$$

For $T = [-(AB)^*ABB^{\circledast}(B^{\circledast})^*, (B^{\circledast}B)^*]$, $T^- = \begin{bmatrix} O \\ (B^{\circledast}B)^* \end{bmatrix}$ is an inner inverse of T and

$$I - T^-T = \begin{bmatrix} I & O \\ (AB)^*ABB^{\circledast}(B^{\circledast})^* & I - (B^{\circledast}B)^* \end{bmatrix}.$$

Since $\mathcal{N}(T) = \mathcal{C}(I - T^-T)$ and $(AB)^* = (B^{\circledast}B)^*(AB)^*$, and from the (9), we get

$$\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} I \\ (AB)^*ABB^{\circledast}(B^{\circledast})^* \end{bmatrix}\right).$$

Applying $(B^{\circledast}B)^*$ on the left leads to

$$\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix} B^{\circledast}(B^{\circledast})^*\right),$$

which shows the conclusion

$$\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\right).$$

(viii) ⇒ (ix). By the hypothesis $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\right)$, for any $\mathbf{x} \in \mathbb{C}^n$, there exists $\mathbf{u} \in \mathbb{C}^n$ such that $\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\mathbf{u}$. Thus $B^*A\mathbf{x} = B^*B\mathbf{u}$, or equivalently,

$$B^{\#}A\mathbf{x} = B^{\#}B\mathbf{u}.$$

Assumption $\mathcal{C}(B^{\#}A) = \mathcal{C}((AB)^*)$ implies

$$\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B^{\#}) = \mathcal{C}(B)$$

and

$$B^{\#}A\mathbf{x} = (AB)^*\mathbf{z}$$

for some $\mathbf{z} \in \mathbb{C}^n$. Therefore,

$$\begin{aligned} \begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} &= \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}\mathbf{u} = \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}B^{\#}B\mathbf{u} = \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}B^{\#}A\mathbf{x} \\ &= \begin{bmatrix} B^*B \\ (AB)^*AB \end{bmatrix}(AB)^*\mathbf{z} = \begin{bmatrix} B^*B(AB)^* \\ (AB)^*AB(AB)^* \end{bmatrix}\mathbf{z}, \end{aligned}$$

showing that $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B(AB)^* \\ (AB)^*AB(AB)^* \end{bmatrix}\right)$.

(ix) ⇒ (v). Hypotheses $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ and $\mathcal{C}\left(\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B(AB)^* \\ (AB)^*AB(AB)^* \end{bmatrix}\right)$ show that $(AB)^* = B^{\#}B(AB)^*$ and for any $\mathbf{x} \in \mathbb{C}^n$ there exists $\mathbf{u} \in \mathbb{C}^n$ such that $\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} B^*B(AB)^* \\ (AB)^*AB(AB)^* \end{bmatrix}\mathbf{u}$, respectively. Thus $B^*A\mathbf{x} = B^*B(AB)^*\mathbf{u}$, or equivalently, $B^{\#}A\mathbf{x} = B^{\#}B(AB)^*\mathbf{u}$, showing that $B^{\#}A\mathbf{x} = (AB)^*\mathbf{u}$, so $\mathcal{C}(B^{\#}A) \subseteq \mathcal{C}((AB)^*)$. Furthermore,

$$\begin{aligned} ((AB)^*A^2 - (AB)^*ABB^{\#}A)\mathbf{x} &= [-(AB)^*ABB^{\#}(B^{\#})^*, (B^{\#}B)^*]\begin{bmatrix} B^*A \\ (AB)^*A^2 \end{bmatrix}\mathbf{x} \\ &= [-(AB)^*ABB^{\#}(B^{\#})^*, (B^{\#}B)^*]\begin{bmatrix} B^*B(AB)^* \\ (AB)^*AB(AB)^* \end{bmatrix}\mathbf{u} \\ &= O, \end{aligned}$$

thus $(AB)^*A^2 = (AB)^*ABB^{\#}A$. □

The following result follows by the left-right symmetry of the Moore-Penrose inverse. It is worth mentioning that the core inverse has no left-right symmetry, thus the following result is different from the symmetric form of Theorem 3.1. Because the proof is similar as the previous theorem, we omit the proof.

Theorem 3.2. *Let $A, B \in \mathbb{C}_n^{CM}$. Then the following statements are equivalent:*
(i) $(AB)^\dagger = B^{\#}A^{\#}$;
(ii) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$ *and* $(AB)^* = B^{\#}A^{\#}AB(AB)^*$;
(iii) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$ *and* $(AB)^* = B^{\#}A^{\#}AB(AB)^*$;
(iv) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$, $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $B^*B(AB)^* = B^*A^{\#}AB(AB)^*$;
(v) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $B^*B(AB)^* = B^*A^{\#}AB(AB)^*$;
(vi) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$, $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $\mathcal{C}(B(AB)^*) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$;

(vii) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $\mathcal{C}(B(AB)^*) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$.
If $\mathcal{R}((AB)^*) \subseteq \mathcal{R}(A)$, *then the above statements are also equivalent to the following statements:*
(viii) $\mathcal{C}((A^{\#})^*B) = \mathcal{C}(AB)$, $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $\mathcal{R}([B^*A, B^*B(AB)^*]) \subseteq \mathcal{R}([A^2, AB(AB)^*])$;
(ix) $\mathcal{C}((A^{\#})^*B) \subseteq \mathcal{C}(AB)$, $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $\mathcal{R}([B^*A, B^*B(AB)^*]) \subseteq \mathcal{R}([A^2, AB(AB)^*])$;
(x) $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and* $\mathcal{R}([B^*A, B^*B(AB)^*]) \subseteq \mathcal{R}([(AB)^*A^2, (AB)^*AB(AB)^*])$.

Remark 3.3. *Statements (viii)-(x) in Theorem 3.2 are equivalent, and they lead to statements (i)-(vii) in the absence of condition* $\mathcal{R}((AB)^*) \subseteq \mathcal{R}(A)$.

Theorem 3.4. *Let* $A, B \in \mathbb{C}_n^{CM}$. *Then the following statements are equivalent:*
(i) $(AB)^\dagger = B^{\#}A^{\#}$;
(ii) $(AB)^\dagger A = B^{\#}A^{\#}A$;
(iii) $(AB)^\dagger A^2 = B^{\#}A$.

Proof. (i) ⇒ (ii) ⇒ (iii). Obviously.
(iii) ⇒ (i). According to Lemma 1.1, $(AB)^\dagger A^2 = B^{\#}A$ is equivalent to $(AB)^\dagger AA^{\#} = B^{\#}A^{\#}$. Since

$$\begin{aligned}((AB)^\dagger AA^{\#})^* &= AA^{\#}((AB)^\dagger)^* = AA^{\#}((AB)^\dagger ABAB)^\dagger)^* \\ &= AA^{\#}AB(AB)^\dagger((AB)^\dagger)^* = AB(AB)^\dagger((AB)^\dagger)^* \\ &= ((AB)^\dagger)^*,\end{aligned}$$

by taking an involution, we obtain $(AB)^\dagger AA^{\#} = (AB)^\dagger$. Therefore, $(AB)^\dagger = B^{\#}A^{\#}$. □

Theorem 3.5. *Let* $A, B \in \mathbb{C}_n^{CM}$. *Then* $(AB)^\dagger = B^{\#}A^{\#}$ *if and only if* $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ *and one of the following equivalent statements holds:*
(i) $B^*B(AB)^\dagger A^2 = B^*A$;
(ii) $B(AB)^\dagger A^2 = BB^{\#}A$;
(iii) $B(AB)^\dagger A^2 = BB^\dagger A$;
(iv) $B^*B(AB)^\dagger = B^*A^{\#}$.

Proof. Firstly, we show that statements (i)-(iv) are equivalent.
(i) ⇔ (ii). Applying $(B^{\#})^*$ on the left of $B^*B(AB)^\dagger A^2 = B^*A$ leads to $B(AB)^\dagger A^2 = BB^{\#}A$. Conversely, pre-multiplying matrices in $B(AB)^\dagger A^2 = BB^{\#}A$ by B^*, we get $B^*B(AB)^\dagger A^2 = B^*A$.
(ii) ⇔ (iii). According to Lemma 2.2.
(i) ⇔ (iv). By Lemma 1.1, $B^*B(AB)^\dagger A^2 = B^*A$ is equivalent to $B^*B(AB)^\dagger AA^{\#} = B^*A^{\#}$. Since $(AB)^\dagger AA^{\#} = (AB)^\dagger$, $B^*B(AB)^\dagger AA^{\#} = B^*A^{\#}$ is equivalent to $B^*B(AB)^\dagger = B^*A^{\#}$.
Next, we show that $(AB)^\dagger = B^{\#}A^{\#}$ if and only if $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$ and statement (i) holds.
Suppose that $(AB)^\dagger = B^{\#}A^{\#}$. Then

$$B^*A = B^*BB^{\#}A^{\#}A^2 = B^*B(AB)^\dagger A^2.$$

And by Lemma 1.1,

$$\mathcal{C}((AB)^*) = \mathcal{C}((AB)^\dagger) = \mathcal{C}(B^{\#}A^{\#}) \subseteq \mathcal{C}(B^{\#}) = \mathcal{C}(B).$$

Conversely, if $\mathcal{C}((AB)^*) \subseteq \mathcal{C}(B)$, then $(AB)^* = B^{\#}B(AB)^*$, which is equivalent to $(AB)^\dagger = B^{\#}B(AB)^\dagger$ by Lemma 1.1. Furthermore, $B^*B(AB)^\dagger A^2 = B^*A$ is equivalent to $B^{\#}B(AB)^\dagger AA^{\#} = B^{\#}A^{\#}$. Therefore,

$$(AB)^\dagger = B^{\#}A^{\#}.$$

□

Next, we give two results about necessary and sufficient conditions for the hybrid reverse order law $(AB)^\# = B^{\#}A^{\#}$ to hold. The proof is left to readers.

Theorem 3.6. *Let $A, B \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then the following statements are equivalent:*
(i) $(AB)^{\#} = B^{\circledast}A^{\circledast}$;
(ii) $\mathcal{C}(B^{\circledast}A) = \mathcal{C}(AB)$ *and* $AB = (AB)^2B^{\circledast}A^{\circledast}$;
(iii) $\mathcal{C}(B^{\circledast}A) \subseteq \mathcal{C}(AB)$ *and* $AB = (AB)^2B^{\circledast}A^{\circledast}$;
(iv) $\mathcal{C}(B^{\circledast}A) = \mathcal{C}(AB)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A^*)$ *and* $ABA^2 = (AB)^2B^{\circledast}A$;
(v) $\mathcal{C}(B^{\circledast}A) \subseteq \mathcal{C}(AB)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A^*)$ *and* $ABA^2 = (AB)^2B^{\circledast}A$;
(vi) $\mathcal{C}(B^{\circledast}A) = \mathcal{C}(AB)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{C}((ABA)^*) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$;
(vii) $\mathcal{C}(B^{\circledast}A) \subseteq \mathcal{C}(AB)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{C}((ABA)^*) \subseteq (\mathcal{N}(A^*) \cap \mathcal{N}(B^*)) \oplus \mathcal{C}(B)$.
If $\mathcal{C}(AB) \subseteq \mathcal{C}(B^*)$*, then the above statements are also equivalent to the following statements:*
(viii) $\mathcal{C}(B^{\circledast}A) = \mathcal{C}(AB)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{C}\left(\begin{bmatrix} B^*A \\ ABA^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*B \\ (AB)^2 \end{bmatrix}\right)$;
(ix) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$, $\mathcal{R}(AB) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{C}\left(\begin{bmatrix} B^*A \\ ABA^2 \end{bmatrix}\right) \subseteq \mathcal{C}\left(\begin{bmatrix} B^*BAB \\ (AB)^3 \end{bmatrix}\right)$.

Theorem 3.7. *Let $A, B \in \mathbb{C}_n^{CM}$. If $AB \in \mathbb{C}_n^{CM}$, then the following statements are equivalent:*
(i) $(AB)^{\#} = B^{\circledast}A^{\circledast}$;
(ii) $\mathcal{R}(B^*A^{\circledast}) = \mathcal{R}(AB)$ *and* $AB = B^{\circledast}A^{\circledast}(AB)^2$;
(iii) $\mathcal{R}(B^*A^{\circledast}) \subseteq \mathcal{R}(AB)$ *and* $AB = B^{\circledast}A^{\circledast}(AB)^2$;
(iv) $\mathcal{R}(B^*A^{\circledast}) = \mathcal{R}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ *and* $B^*BAB = B^*A^{\circledast}(AB)^2$;
(v) $\mathcal{R}(B^*A^{\circledast}) \subseteq \mathcal{R}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ *and* $B^*BAB = B^*A^{\circledast}(AB)^2$.
(vi) $\mathcal{R}(B^*A^{\circledast}) = \mathcal{R}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ *and* $\mathcal{C}(BAB) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$;
(vii) $\mathcal{R}(B^*A^{\circledast}) \subseteq \mathcal{R}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ *and* $\mathcal{C}(BAB) \subseteq (\mathcal{N}(B^*) \cap \mathcal{N}(A)) \oplus \mathcal{C}(A)$.
If $\mathcal{R}(AB) \subseteq \mathcal{R}(A)$*, then the above statements are also equivalent to the following statements:*
(viii) $\mathcal{R}(B^*A^{\circledast}) = \mathcal{R}(AB)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ *and* $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([A^2, (AB)^2])$;
(ix) $\mathcal{R}(AB) \subseteq \mathcal{R}((A)^*)$, $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$ *and* $\mathcal{R}([B^*A, B^*BAB]) \subseteq \mathcal{R}([ABA^2, (AB)^3])$.

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (2010) 681-697.
[2] D.S. Rakić, N.Č. Dinčić, D.S. Djordievic, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl 463 (2014) 115-133.
[3] S.Z. Xu, J.L. Chen, X.X. Zhang, New characterizations for core and dual core inverses in rings with involution, Front. Math. China 12 (1) (2017) 231-246.
[4] T.N.E. Greville, Note on the generalized inverse of a matrix product, SIAM Rev 8 (4) (1966) 518-521.
[5] R.E. Hartwig, The reverse order law revisited, Linear Algebra Appl 76 (1986) 241-246.
[6] J.K. Baksalary, O.M. Baksalary, An invariance property related to the reverse order law, Linear Algebra Appl 410 (2005) 64-69.
[7] C.Y. Deng, Reverse order law for the group inverses, J. Math. Anal. Appl 382 (2011) 663-671.
[8] D.S. Cvetković-Ilić, Y. Wei, Algebraic Properties of Generalized Inverses, Series: Developments in Mathematics, Vol. 52, Springer, 2017.
[9] D.S. Cvetković-Ilić, V. Pavlović, A comment on some recent results concerning the reverse order law for $\{1, 3, 4\}$-inverses, Appl. Math. Comp 217 (2010) 105-109.
[10] D.S. Cvetković-Ilić, Reverse order laws for $\{1, 3, 4\}$-generalized inverses in C^*-algebras, Appl. Math. Letters 24 (2) (2011) 210-213.
[11] X.J. Liu, S.X. Wu, D.S. Cvetković-Ilić, New results on reverse order law for $\{1, 2, 3\}$ and $\{1, 2, 4\}$-inverses of bounded operators, Math. Comp 82 (283) (2013) 1597-1607.
[12] D. Mosić, D.S. Djordjević, Reverse order law for the group inverse in rings, Appl. Math. Comput 219 (2012) 2526-2534.
[13] Y.G. Tian, Reverse order law for the generalized inverses of multiple matrix products, Linear Algebra Appl 211 (1994) 85-100.
[14] D.S. Djordjević, Further results on the reverse order law for the generalized inverses, SIAM. J. Matrix Anal. Appl 29 (4) (2007) 1241-1246.
[15] J.L. Chen, H.H. Zhu, P. Patrício, Y.L. Zhang, Characterizations and representations of core and dual core inverses, Canad. Math. Bull 60 (2) (2017) 269-282.
[16] D. Mosić, D. S. Djordjević. Some results on the reverse order law in rings with involution, Aequationes Math 83 (3) (2012) 271-282.
[17] O.M. Baksalary, G. Trenkler, Problem 48-1: reverse order law for the core inverse, Image 48 (2012) 40.
[18] N. Cohen, E.A. Herman, S. Jayaraman, Solution to problem 48C1: reverse order law for the core inverse, Image 49 (2012) 46-47.
[19] H.X. Wang, X.J. Liu, Characterizations of the core inverse and the core ordering, Linear Multilinear Algebra 63 (2015) 1829-1836.
[20] H.L. Zou, J.L. Chen, P. Patrício, Reverse order law for the core inverse in rings, Mediterr. J. Math 15 (2018) 145.
[21] A. Korporal, G. Regensburger, On the product of projections and generalized inverses, Linear Muiltilinear Algebra 62 (12) (2014) 1567-1582.