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Kantorovich Variant of Stancu Operators

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Dedicated to Professor Manuel Lopez-Pellicer on the occasion of his 78-th birthday

Abstract. Stancu type operators play a crucial role in convergence estimates. The present article concerns the convergence estimates for certain Stancu type Kantorovich operators. We first establish some direct formulas giving the local approximation theorem, Voronovskaja type asymptotic formula, bound for the second central moment with some curtailment, and the global approximation theorem by means of modulus of continuity and the Ditzian-Totik Modulus of smoothness. We also study the difference estimates between Stancu-Bernstein operators and its Kantorovich variant. Further, we show the convergence of these operators by graphics to certain functions.

1. Introduction

Stancu (see [8], [9]) proposed the Bernstein type operators based on two parameters $l, m \in \mathbf{N} \cup \{0\}$ as follows:

$$(D_n^{[l,m]}h)(x) = \sum_{\sigma=0}^{n-lm} b_{n-lm,\sigma}(x) \sum_{\tau=0}^{m} b_{m,\tau}(x) h\Big(\frac{\sigma+\tau l}{n}\Big), \quad 0 \le x \le 1, \tag{1}$$

where

$$b_{\eta,\kappa}(x) = \binom{\eta}{\kappa} x^{\kappa}(1-x)^{\eta-\kappa}.$$

As a special case when $l = m = 0$, we get the well-known Bernstein polynomials. Abel et al in [2] estimated the complete asymptotic expansion of the Durrmeyer variant of (1). Kajla in [7] considered the Kantorovich variant of the operators (1), but it was not proper Kantorovich variant. We consider below the following Stancu-Kantorovich operators

$$(K_n^{[l,m]}h)(x) = (n+1) \sum_{\sigma=0}^{n-lm} b_{n-lm,\sigma}(x) \sum_{\tau=0}^{m} b_{m,\tau}(x) \int_{\frac{\sigma+\tau l}{(n+1)}}^{\frac{\sigma+\tau l+1}{(n+1)}} h(u)du, \quad 0 \le x \le 1. \tag{2}$$

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The results concerning convergence in the theory of approximation play an important role in different areas of mathematics. Gupta and Agarwal [6] considered several such problems based on linear positive operators. Also, in the recent monograph, Gupta and Rassias [5] provided moments estimate of many operators. In the present article, we provide some direct results including simultaneous approximation with difference estimate.

2. Estimation of Moments

Lemma 2.1. *For $x \in [0,1]$ and a real parameter A, we have*

$$(D_n^{[l,m]} e^{At})(x) = (1 - x + xe^{A/n})^{n-lm}(1 - x + xe^{Al/n})^m.$$

Thus, the i^{th} order moment is given by

$$(D_n^{[l,m]} e_i(t))(x) = \left[\frac{\partial^i}{\partial A^i}(1 - x + xe^{A/n})^{n-lm}(1 - x + xe^{Al/n})^m\right]_{A=0};\ e_i(t) = t^i,\ i = 0, 1, 2, ...$$

Proof. Obviously by definition, we have

$$\begin{aligned}
(D_n^{[l,m]} e^{At})(x) &= \sum_{\sigma=0}^{n-lm} b_{n-lm,\sigma}(x) \sum_{\tau=0}^{m} b_{m,\tau}(x) e^{A\tau l/n}.e^{A\sigma/n} \\
&= \sum_{\sigma=0}^{n-lm} b_{n-lm,\sigma}(x) e^{A\sigma/n} \sum_{\tau=0}^{m} \binom{m}{\tau} (xe^{Al/n})^{\tau}(1-x)^{m-\tau} \\
&= \sum_{\sigma=0}^{n-lm} \binom{n-lm}{\sigma} (xe^{A/n})^{\sigma}(1-x)^{n-lm-\sigma} \sum_{\tau=0}^{m} \binom{m}{\tau} (xe^{Al/n})^{\tau}(1-x)^{m-\tau} \\
&= (1 - x + xe^{A/n})^{n-lm}(1 - x + xe^{Al/n})^m.
\end{aligned}$$

By the well-known property of m.g.f, the other consequences follow immediately. □

Remark 2.2. *Using Lemma 2.1, few moments are given by:*

$$\begin{aligned}
(D_n^{[l,m]} e_0)(x) &= 1 \\
(D_n^{[l,m]} e_1)(x) &= x \\
(D_n^{[l,m]} e_2)(x) &= \frac{(lm(1-l) + n(n-1))x^2}{n^2} + \frac{(lm(l-1) + n)x}{n^2} \\
(D_n^{[l,m]} e_3)(x) &= \frac{(l^2m(2l - 3n) + lm(3n-2) + n(n-1)(n-2))x^3}{n^3} \\
&\quad + \frac{3(l^2m(n-l) + n(n-1) + lm(1-n))x^2}{n^3} + \frac{(lm(l^2-1) + n)x}{n^3} \\
(D_n^{[l,m]} e_4)(x) &= \frac{1}{n^4}\Bigg[\Big(3l^4m(m-2) + 2l^3m(4n-3m) + 3l^2m(m - 2n(n-1))) + 2lm(3n^2 - 7n + 3) \\
&\quad + n(n-1)(n-2)(n-3)\Big)x^4 - 6\Big(l^4m(m-2) + 2l^3m(n-m) + l^2m(m - n(n-2)) \\
&\quad + lm(n^2 - 4n + 2) - n(n-1)(n-2)\Big)x^3 + \Big(l^4m(3m-7) - lm(10n-7) + 7n(n-1) \\
&\quad + 3l^2m(2n+m) + 2l^3m(2n-3m)\Big)x^2 + \Big(lm(l^3-1) + n\Big)x\Bigg].
\end{aligned}$$

Lemma 2.3. *The function that generates the moments of the operators (2) is given by*

$$(K_n^{[l,m]}e^{At})(x) = \frac{(n+1)}{A}(e^{A/(n+1)}-1)(1-x+xe^{A/(n+1)})^{n-lm}(1-x+xe^{Al/(n+1)})^m.$$

Thus the moments $e_i(t) = t^i, i = 0,1,2,...$ are given by

$$(K_n^{[l,m]}e_i(t))(x) = \left[\frac{\partial^i}{\partial A^i}\left(\frac{n+1}{A}(e^{A/(n+1)}-1)(1-x+xe^{A/(n+1)})^{n-lm}(1-x+xe^{Al/(n+1)})^m\right)\right]_{A=0}.$$

The proof follows by the definition of $K_n^{[l,m]}$ and Lemma 2.1, we omit the details.

Remark 2.4. *Using Lemma 2.3, few moments are given by:*

$$\begin{aligned}
(K_n^{[l,m]}e_0)(x) &= 1\\
(K_n^{[l,m]}e_1)(x) &= \frac{2nx+1}{2(n+1)}\\
(K_n^{[l,m]}e_2)(x) &= \frac{(lm-l^2m+n(n-1))x^2}{(n+1)^2} + \frac{(l^2m-lm+2n)x}{(n+1)^2} + \frac{1}{3(n+1)^2}\\
(K_n^{[l,m]}e_3)(x) &= \frac{1}{4(n+1)^3}\Big[4\Big(n^3-3n^2+n(-3l^2m+3lm+2)+2lm(l^2-1)\Big)x^3+6\Big(3n^2+n(2l^2m\\
&\quad -2lm-3)-lm(2l^2+l-3)\Big)x^2+2\Big(7n+lm(2l^2+3l-5)\Big)x+1\Big]\\
(K_n^{[l,m]}e_4)(x) &= \frac{1}{5(n+1)^4}\Big[5\Big(n^4-6n^3+n^2(-6l^2m+6lm+11)+2n(4l^3m+3l^2m-7lm-3)+3lm\\
&\quad .(l^3(m-2)-2l^2m+lm+2)\Big)x^4+10\Big(4n^3+3n^2(l^2m-lm-4)+n(-6l^3m-9l^2m\\
&\quad +15lm+8)-lm(l-1)(3l^2(m-2)-l(3m+8)-8)\Big)x^3+5\Big(15n^2+n(4l^3m+12l^2m\\
&\quad -16lm-15)+lm(l-1)(l^2(3m-7)-l(3m+13)-15)\Big)x^2+5\Big(6n+lm(l^3+2l^2\\
&\quad +2l-5)\Big)x+1\Big].
\end{aligned}$$

Lemma 2.5. *If we denote $\mu_{n,j}^{[l,m]}(x) = (K_n^{[l,m]}(t-x)^j)(x)$, $j \in \mathbf{N}$. Then*

$$\mu_{n,j}^{[l,m]}(x) = \left[\frac{\partial^i}{\partial A^i}\left(\frac{n+1}{A}e^{-Ax}(e^{A/(n+1)}-1)(1-x+xe^{A/(n+1)})^{n-lm}(1-x+xe^{Al/(n+1)})^m\right)\right]_{A=0}.$$

By basic computations, few moments are given by

$$\begin{aligned}
\mu_{n,1}^{[l,m]}(x) &= \frac{1-2x}{2(n+1)}\\
\mu_{n,2}^{[l,m]}(x) &= \frac{1}{3(n+1)^2}+\frac{(l^2m-lm+n-1)}{(n+1)^2}x(1-x)\\
\mu_{n,3}^{[l,m]}(x) &= \frac{1}{4(n+1)^3}+\frac{(2l^3m+3l^2m-5lm+5n-2)x}{2(n+1)^3}\\
&\quad+\frac{2l^3m+3l^2m-5lm+5n-1}{2(n+1)^3}x^2(2x-3)\\
\mu_{n,4}^{[l,m]}(x) &= \frac{1}{5(n+1)^4}+\frac{(l^4m+2l^3m+2l^2m-5lm+5n-1)x}{(n+1)^4}\\
&\quad+\frac{(3l^4m^2-6l^3m^2+3l^2m^2-7l^4m-10l^3m+6nl^2m-8l^2m-6nlm+25lm+3n^2-25n+2)x^2}{(n+1)^4}\\
&\quad+\frac{(3l^4m^2-6l^3m^2+3l^2m^2-6l^4m-8l^3m+6nl^2m-6l^2m-6nlm+20lm+3n^2-20n+1)}{(n+1)^4}x^3(x-2).
\end{aligned}$$

Furthermore, following limits hold true

$$\begin{aligned}
\lim_{n\to\infty} n.\mu_{n,1}^{[l,m]}(x) &= \frac{1-2x}{2}\\
\lim_{n\to\infty} n.\mu_{n,2}^{[l,m]}(x) &= x(1-x)\\
\lim_{n\to\infty} n^2.\mu_{n,4}^{[l,m]}(x) &= 3x^2(x-1)^2.
\end{aligned}$$

Lemma 2.6. *For $l,m,n\in\mathbf{N}\cup\{0\}$ and $l^2m-lm-1<0$, we have*

$$\mu_{n,2}^{[l,m]}(x)\le\frac{1}{n+1}d_n^2(x),$$

where $d_n^2(x)=\vartheta^2(x)+\frac{1}{n+1}$ and $\vartheta^2(x)=x(1-x)$.

Proof. In view of Lemma 2.5, we have

$$\begin{aligned}
\mu_{n,2}^{[l,m]}(x) &= \frac{1+3x(1-x)(l^2m-lm+n-1)}{3(n+1)^2}\\
&\le \frac{1}{n+1}\left(\vartheta^2(x)+\frac{1}{n+1}\right).
\end{aligned}$$

□

Remark 2.7. *For $l,m\in\mathbf{N}\cup\{0\}$, $l^2m-lm-1<0$ if (l,m) has either of the following forms $(l,0)$, $(0,m)$, $(1,m)$.*

3. Direct Estimates

Theorem 3.1. *For $h\in C[0,1]$, we have*

$$\lim_{n\to\infty}(K_n^{[l,m]}h)(x)=h(x),$$

uniformly in $[0,1]$.

Proof. By Remark 2.4, we have

$$(K_n^{[l,m]}e_i)(x)\to x^i;\ i=0,1,2$$

as n→ ∞, uniformly in $[0,1]$. By Bohman-Korovkin's theorem, $(K_n^{[l,m]}h)$ converges uniformly to h in [0,1]. □

Theorem 3.2. *Let $h \in C[0,1]$ and $h''(x)$ exists for $x \in [0,1]$, then we have*

$$\lim_{n\to\infty} n[(K_n^{[l,m]}h)(x) - h(x)] = (0.5-x)h'(x) + 0.5x(1-x)h''(x).$$

Proof. Using Taylor's theorem, we have

$$h(u) = \sum_{i=0}^{2} \frac{(u-x)^i}{i!} h^{(i)}(x) + \Xi(u,x)(u-x)^2;\ u \in [0,1], \tag{3}$$

where

$$2\Xi(u,x) = h''(\check{\delta}) - h''(x),$$

and $\check{\delta}$ lies between x and u. Also $\Xi(u,x)$ vanishes as $u \to x$. On applying the operators $K_n^{[l,m]}$ to (3), we have

$$(K_n^{[l,m]}(h(u) - h(x)))(x) = h'(x)\mu_{n,1}^{[l,m]}(x) + \frac{h''(x)}{2!}\mu_{n,2}^{[l,m]}(x) + \grave{R}_{n,l,m}(x), \tag{4}$$

where

$$\grave{R}_{n,l,m}(x) = (K_n^{[l,m]}\Xi(u,x)(u-x)^2)(x).$$

From (4), after applying Lemma 2.5, we obtain

$$\lim_{n\to\infty} n.[(K_n^{[l,m]}(h(u) - h(x))(x)] = (0.5-x)h'(x) + 0.5x(1-x)h''(x) + \lim_{n\to\infty} n.\grave{R}_{n,l,m}(x).$$

To estimate $\lim_{n\to\infty} n.\grave{R}_{n,l,m}(x)$, for all possible $\varepsilon > 0$, choose $\varrho > 0$ such that

$$\Xi(t,x) < \varepsilon \quad for \quad |u-x| < \varrho.$$

Therefore if $|u-x| < \varrho$, then $\left|\Xi(u,x)(u-x)^2\right| < \varepsilon(u-x)^2$ while if $|u-x| \geq \varrho$, then since $\Xi(u,x) < M$, we have

$$\left|\Xi(u,x)(u-x)^2\right| \leq \frac{M}{\varrho^2}(u-x)^4.$$

Using Lemma 2.5, we get

$$\begin{aligned}
\grave{R}_{n,l,m}(x) \leq\ & \varepsilon\left(\frac{1}{3(n+1)^2} + \frac{(l^2m - lm + n - 1)}{(n+1)^2}x(1-x)\right) \\
& + \frac{M}{\varrho^2}\Bigg(\frac{1}{5(n+1)^4} + \frac{(l^4m + 2l^3m + 2l^2m - 5lm + 5n - 1)x}{(n+1)^4} \\
& + \frac{(3l^4m^2 - 6l^3m^2 + 3l^2m^2 - 7l^4m - 10l^3m + 6nl^2m - 8l^2m - 6nlm + 25lm + 3n^2 - 25n + 2)x^2}{(n+1)^4} \\
& + \frac{(3l^4m^2 - 6l^3m^2 + 3l^2m^2 - 6l^4m - 8l^3m + 6nl^2m - 6l^2m - 6nlm + 20lm + 3n^2 - 20n + 1)}{(n+1)^4}x^3(x-2)\Bigg).
\end{aligned}$$

For arbitrarily small $\varepsilon > 0$, we get

$$\lim_{n\to\infty} n.\grave{R}_{n,l,m}(x) = 0.$$

This completes the proof of the theorem. □

Theorem 3.3. *If $h \in C[0,1]$ admits 3-rd order derivative at a fixed point $x \in [0,1]$. Then, we have*

$$\lim_{n\to\infty} n\left(\left(\frac{\partial}{\partial u}(K_n^{[l,m]}h)(u)\right)_{u=x} - h'(x)\right) = -h'(x) + (1-2x)h''(x) + 0.5x(1-x)h'''(x).$$

Proof. Using Taylor's expansion, we are allowed to write

$$h(u) = h(x) + \sum_{i=1}^{3} \frac{(u-x)^i}{i!} h^{(i)}(x) + \Xi(u,x)(u-x)^3;\ u \in [0,1], \tag{5}$$

where $\Xi(u,x)$ vanishes as $u \to x$. In view of eq. (5), we have

$$\begin{aligned}\left(\frac{\partial}{\partial u}(K_n^{[l,m]}h)(u)\right)_{u=x} &= h'(x)\left(\frac{\partial}{\partial u}\left((K_n^{[l,m]}e_1)(u)-x\right)\right)_{u=x} \\ &+\frac{h''(x)}{2}\left(\frac{\partial}{\partial u}\left((K_n^{[l,m]}e_2)(u)-2x(K_n^{[l,m]}e_1)(u)+x^2\right)\right)_{u=x} \\ &+\frac{h'''(x)}{3!}\left(\frac{\partial}{\partial u}\left((K_n^{[l,m]}e_3)(u)-3x(K_n^{[l,m]}e_2)(u)+3x^2(K_n^{[l,m]}e_1)(u)-x^3\right)\right)_{u=x} \\ &+\left(\frac{\partial}{\partial u}(K_n^{[l,m]}\Xi(u,x)(u-x)^3)(u)\right)_{u=x}.\end{aligned}$$

Taking into account Remark 2.4, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial u}(K_n^{[l,m]}h)(u)\right)_{u=x} &= h'(x)\left[\frac{n}{n+1}\right] \\ &+\frac{h''(x)}{2}\left[\frac{2(lm-l^2m+n(n-1))x}{(n+1)^2}+\frac{(l^2m-lm+2n)}{(n+1)^2}-\frac{2nx}{n+1}\right] \\ &\frac{h'''(x)}{6}\left[\frac{1}{4(n+1)^3}\Big((2lm(2l^2+3l-5)+14n)+12(3n^2+n(2l^2m\right. \\ &-2lm-3)-lm(2l^2+l-3))x++12(n^3-3n^2+n(-3l^2m+3lm \\ &+2)+2lm(l^2-1))x^2\Big)-3x\Big(\frac{2(lm-l^2m+n(n-1))x}{(n+1)^2}+\frac{(l^2m-lm+2n)}{(n+1)^2} \\ &\left.-\frac{2nx}{n+1}\Big)+\frac{3nx^2}{n+1}\right]+\left(\frac{\partial}{\partial u}(K_n^{[l,m]}\Xi(u,x)(u-x)^3)(u)\right)_{u=x}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty} n\left(\left(\frac{\partial}{\partial u}(K_n^{[l,m]}h)(u)\right)_{u=x}-h'(x)\right) &= -h'(x)+(1-2x)h''(x)+0.5x(1-x)h'''(x) \\ &+\lim_{n\to\infty} n\left(\frac{\partial}{\partial u}(K_n^{[l,m]}\Xi(u,x)(u-x)^3)(u)\right)_{u=x}.\end{aligned}$$

Stepping forward in the same aspect as in Theorem 3.2, we can show that

$$\lim_{n\to\infty} n\left(\frac{\partial}{\partial u}(K_n^{[l,m]}\Xi(u,x)(u-x)^3)(u)\right)_{u=x} = 0.$$

Thus, the validation is concluded. □

Remark 3.4. *In general, from the above asymptotic formulae, one can extend the result to higher order derivatives and the general result takes the following form:*

For $r \in \mathbf{N} \cup \{0\}$, let $h^{(r+2)}(x)$ exists at a fixed point $x \in [0,1]$. Then, we have the following:

$$\begin{aligned}\lim_{n\to\infty} n\left(\left(\frac{\partial^r}{\partial u^r}(K_n^{[l,m]}h)(u)\right)_{u=x}-h^{(r)}(x)\right) &= -\kappa_r.h^{(r)}(x)+\left(\frac{r+1}{2}\right)(1-2x)h^{(r+1)}(x) \\ &+0.5x(1-x)h^{(r+2)}(x),\end{aligned}$$

where $\kappa_r = r + \kappa_{r-1}$ *with* $\kappa_0 = 0$ *and* $h^{(r)}(x)$ *denotes the* r^{th} *order derivative of the function* h.

Theorem 3.5. *Let* $h \in C[0,1]$ *and* $x \in [0,1]$, *then there exists a constant* $M > 0$ *such that*

$$|(K_n^{[l,m]}h)(x) - h(x)| \leq M\omega_2\Big(h, \sqrt{\Psi_{n,l,m}(x)}\Big) + \omega_1\Big(h, (n+1)^{-1}\Big),$$

where

$$\Psi_{n,l,m}(x) = \frac{(l^2m - lm + n - 2)}{(n+1)^2}x(1-x) + \frac{7}{12(n+1)^2},$$

and ω_1, ω_2 *are 1st and 2nd order modulus of continuity respectively.*

Proof. Consider the auxiliary operators $\hat{K}_n^{[l,m]}$ defined as

$$(\hat{K}_n^{[l,m]}h)(x) = (K_n^{[l,m]}h)(x) + h(x) - h\Big(\frac{2nx+1}{2(n+1)}\Big). \tag{6}$$

Using Remark 2.4 and definition (6), the modified operators $\hat{K}_n^{[l,m]}$ preserve constant as well as linear functions. Define $C_2[0,1] = \{\tilde{g} : \tilde{g}'' \in C[0,1]\}$. Let $g \in C_2[0,1]$ and $u \in [0,1]$. Then by Taylor's formula

$$g(u) = g(x) + (u-x)g'(x) + \int_x^u (u-v)g''(v)dv.$$

Applying the operators $\hat{K}_n^{[l,m]}$ on both sides of the above equation, we have

$$\begin{aligned}(\hat{K}_n^{[l,m]}g)(x) &= g(x) + \Big(\hat{K}_n^{[l,m]}\Big(\int_x^u (u-v)g''(v)dv\Big)\Big)(x)\\ &= g(x) + \Big(K_n^{[l,m]}\Big(\int_x^u (u-v)g''(v)dv\Big)\Big)(x) - \int_x^{\frac{2nx+1}{2(n+1)}}\Big(\frac{2nx+1}{2(n+1)} - v\Big)g''(v)dv.\end{aligned}$$

Consider

$$\begin{aligned}|(\hat{K}_n^{[l,m]}g)(x) - g(x)| &\leq \Big(K_n^{[l,m]}\Big(\Big|\int_x^u |u-v||g''(v)|dv\Big|\Big)\Big)(x) + \Big|\int_x^{\frac{2nx+1}{2(n+1)}}\Big|\frac{2nx+1}{2(n+1)} - v\Big||g''(v)|dv\Big|\\ &\leq \Big\{\frac{1}{3(n+1)^2} + \frac{(l^2m - lm + n - 1)}{(n+1)^2}x(1-x) + \Big(\frac{2nx+1}{2(n+1)} - x\Big)^2\Big\}\|g''\|\\ &= \Big\{\frac{(l^2m - lm + n - 2)}{(n+1)^2}x(1-x) + \frac{7}{12(n+1)^2}\Big\}\|g''\|\\ &:= \Psi_{n,l,m}(x)\|g''\|.\end{aligned} \tag{7}$$

In view of definition (2) and Remark 2.4, we get

$$\|(K_n^{[l,m]}h)\| \leq \|h\|(K_n^{[l,m]}e_0)(x) = \|h\|. \tag{8}$$

Using (8), we have

$$|(\hat{K}_n^{[l,m]}h)(x)| \leq |(K_n^{[l,m]}h)(x)| + |h(x)| + \Big|h\Big(\frac{2nx+1}{2(n+1)}\Big)\Big| \leq 3\|h\|. \tag{9}$$

Now, for $g \in C_2[0,1]$ and $h \in C[0,1]$, using (7) and (9), we have

$$\begin{aligned}|(K_n^{[l,m]}h)(x) - h(x)| &= \Big|(\hat{K}_n^{[l,m]}h)(x) - h(x) + h\Big(\frac{2nx+1}{2(n+1)}\Big) - h(x)\Big|\\ &\leq |(\hat{K}_n^{[l,m]}(h-g))(x)| + |(\hat{K}_n^{[l,m]}g)(x) - g(x)| + |g(x) - h(x)| + \Big|h\Big(\frac{2nx+1}{2(n+1)}\Big) - h(x)\Big|\\ &\leq 4\|h-g\| + \Psi_{n,l,m}(x)\|g''\| + \omega_1(h, (n+1)^{-1}).\end{aligned}$$

Taking infimum over all $g \in C_2[0,1]$, we have

$$|(K_n^{[l,m]}h)(x) - h(x)| \leq 4K_2(h, \Psi_{n,l,m}(x)) + \omega_1(h, (n+1)^{-1}),$$

where K_2 is the functional on $C_2[0,1]$ defined as

$$K_2(h,\xi) = \inf_{g \in C_2[0,1]} \{\|h-g\| + \xi\|g''\|\} \ (\xi > 0).$$

By the well known property due to Devore-Lorentz [3], there exists an absolute constant M such that

$$K_2(h,\xi) \leq M\omega_2(h, \sqrt{\xi}).$$

Thus, we get

$$|(K_n^{[l,m]}h)(x) - h(x)| \leq M\omega_2\Big(h, \sqrt{\Psi_{n,l,m}(x)}\Big) + \omega_1(h, (n+1)^{-1}).$$

Hence the result follows. □

Consider

$$H^2(\vartheta) = \{f \in C[0,1] : f' \in AC_d[0,1], \vartheta^2 f'' \in C[0,1]\},$$

where $AC_d[0,1] = \{f' : f$ is differentiable and f' is absolutely continuous on every closed subinterval of $(0,1)\}$ and $\vartheta(x) = \sqrt{x(1-x)}$ for $x \in [0,1]$.

Then, for $h \in C[0,1]$ and $\xi > 0$, the second order Ditzian-Totik Modulus of smoothness $\omega_{2,\vartheta}$ and corresponding K-functional $K_2^{\vartheta(x)}$ are given by

$$\omega_{2,\vartheta}(h,\xi) = \sup_{0<p\leq\xi} \sup_{x, x\pm p\vartheta(x)\in[0,1]} |h(x+p\vartheta(x)) - 2h(x) + h(x-p\vartheta(x))|$$

and

$$K_2^{\vartheta(x)}(h,\xi) = \inf_{f\in H^2(\vartheta)} \{\|h-f\| + \xi\|\vartheta^2 f''\| + \xi^2\|f''\|\}$$

respectively. In view of [4, Theorem 1.3.1], there exists an absolute constant M such that

$$K_2^{\vartheta(x)}(h,\xi) \leq M\omega_{2,\vartheta}(h, \sqrt{\xi}).$$

Also, the first order Ditzian-Totik moduli with Θ as an admissible step weight function on $[0,1]$ is given by

$$\omega_{1,\Theta}(h,\xi) = \sup_{0<p\leq\xi} \sup_{x\pm\frac{p}{2}\Theta(x)\in[0,1]} |h(x+\frac{p}{2}\Theta(x)) - h(x-\frac{p}{2}\Theta(x))|.$$

Theorem 3.6. *Let $h \in C[0,1]$ and $l^2m - lm - 1 < 0$. Then for $\Theta(x) = |1-2x|$, $x \in [0,1]$, we have*

$$\|(K_n^{[l,m]}h) - h\| \leq M\omega_{2,\vartheta}\left(h, \frac{1}{\sqrt{n+1}}\right) + \omega_{1,\Theta}\left(h, \frac{1}{n+1}\right) + \omega_1\left(h, \frac{1}{n+1}\right),$$

where M is an absolute constant and $\|.\|$ is the uniform norm on $[0,1]$.

Proof. Consider the assisting operators defined as

$$(\hat{K}_n^{[l,m]}h)(x) = (K_n^{[l,m]}h)(x) + h(x) - h\Big(\frac{2nx+1}{2(n+1)}\Big).$$

Let $g \in H^2(\vartheta)$. On applying the operators $\hat{K}_n^{[l,m]}$ on both sides of the Taylor's expansion of g and on continuing as in the proof of Theorem 3.5, we get

$$|(\hat{K}_n^{[l,m]}g)(x) - g(x)| \leq \left(K_n^{[l,m]}\left(\left|\int_x^u |u-v||g''(v)|dv\right|\right)\right)(x) + \left|\int_x^{\frac{2nx+1}{2(n+1)}} \left|\frac{2nx+1}{2(n+1)} - v\right||g''(v)|dv\right|. \tag{10}$$

Taking into account the concavity of $d_n^2(x)$ on $[0,1]$, for $v = \xi x + (1-\xi)u$, $\xi \in [0,1]$, we have

$$d_n^2(v) \geq \xi d_n^2(x) + (1-\xi)d_n^2(u).$$

Thus, we get

$$\frac{|u-v|}{d_n^2(v)} \leq \frac{\xi|u-x|}{\xi d_n^2(x) + (1-\xi)d_n^2(u)} \leq \frac{|u-x|}{d_n^2(x)}. \tag{11}$$

Utilizing (11) in (10) implies the following:

$$\begin{aligned} |(\hat{K}_n^{[l,m]}g)(x) - g(x)| &\leq \frac{\|d_n^2 g''\|}{d_n^2(x)}\left((K_n^{[l,m]}(u-x)^2)(x) + \left(\frac{2nx+1}{2(n+1)} - x\right)^2\right) \\ &= \frac{\|d_n^2 g''\|}{d_n^2(x)}\left((K_n^{[l,m]}(u-x)^2)(x) + \left(\frac{1-2x}{2(n+1)}\right)^2\right). \end{aligned}$$

In view of Lemma 2.6, we get

$$\begin{aligned} |(\hat{K}_n^{[l,m]}g)(x) - g(x)| &\leq \left(\frac{1}{n+1} + \frac{1}{d_n^2(x)}\left(\frac{1-2x}{2(n+1)}\right)^2\right)\|d_n^2 g''\| \\ &\leq \frac{2}{n+1}\|d_n^2 g''\| \leq \frac{2}{n+1}\left[\|\vartheta^2 g''\| + \frac{1}{n+1}\|g''\|\right]. \end{aligned}$$

Using (9), we obtain

$$\begin{aligned} |(K_n^{[l,m]}h)(x) - h(x)| &\leq |(\hat{K}_n^{[l,m]}(h-g))(x)| + |(\hat{K}_n^{[l,m]}g)(x) - g(x)| + |g(x) - h(x)| + \left|h\left(\frac{2nx+1}{2(n+1)}\right) - h(x)\right| \\ &\leq 4\|h-g\| + \frac{2}{n+1}\|\vartheta^2 g''\| + \frac{2}{(n+1)^2}\|g''\| + \left|h\left(\frac{2nx+1}{2(n+1)}\right) - h(x)\right|. \end{aligned}$$

Taking the infimum on right hand side over all $g \in H^2(\vartheta)$, we get

$$|(K_n^{[l,m]}h)(x) - h(x)| \leq 4K_2^{\vartheta(x)}\left(h, \frac{1}{n+1}\right) + \left|h\left(\frac{2nx+1}{2(n+1)}\right) - h(x)\right|. \tag{12}$$

Also

$$\begin{aligned} \left|h\left(\frac{2nx+1}{2(n+1)}\right) - h(x)\right| &\leq \left|h\left(x + \frac{1-2x}{2(n+1)}\right) - h\left(x - \frac{1-2x}{2(n+1)}\right)\right| + \left|h\left(x - \frac{1-2x}{2(n+1)}\right) - h(x)\right| \\ &\leq \omega_{1,\Theta}\left(h, \frac{1}{n+1}\right) + \omega_1\left(h, \frac{1}{n+1}\right). \end{aligned} \tag{13}$$

Using (13) in (12), we get the result. □

Theorem 3.7. *Let $h'' \in C[0,1]$. Then, in support of the operators (1) and (2), we have*

$$\begin{aligned}&\left|((D_n^{[l,m]} - K_n^{[l,m]})h)(x) - \sum_{i=1}^{2} \frac{h^{(i)}(x)}{i!}((D_n^{[l,m]} - K_n^{[l,m]})(e_1 - xe_0)^i)(x)\right| \\ \leq\ & \frac{1}{2}\left(1 + \frac{1}{12(n+1)^2}\right)\omega\left(h'', \frac{1}{4\sqrt{5}(n+1)^2}\right) \\ &+\frac{1}{2}\left(1 + \frac{1}{3(n+1)^2} + \Big(lm(l-1)(2n^2+2n+1) + n(2n^2+n+1)\Big)\frac{x(1-x)}{n^2(n+1)^2}\right)\omega(h'', \sqrt{\gamma(x)}),\end{aligned}$$

where

$$\begin{aligned}\gamma(x) \ =\ & \frac{1}{12n^4(n+1)^4}\Big[n^4 + \Big(19l^2mn^4 + 24l^3mn^4 + 12l^4m(2n^4+4n^3+6n^2+4n+1) + \\ & n(67n^4+41n^3+72n^2+48n+12) - lm(67n^4+48n^3+72n^2+48n+12)\Big)x \\ & +\Big(12l^4m(3m-7)(2n^4+4n^3+6n^2+4n+1) + lm(-144n^5+91n^4-96n^3+216n^2 \\ & +264n+84) + n(72n^5-235n^4-101n^3-360n^2-300n-84) - 24l^3m(5n^4+3m(2n^4 \\ & +4n^3+6n^2+4n+1)) + l^2m(36m(2n^4+4n^3+6n^2+4n+1) + n(144n^4+197n^3 \\ & +432n^2+288n+72))\Big)x^2 - 24\Big(3l^4m(m-2)(2n^4+4n^3+6n^2+4n+1) + n(6n^5 \\ & -14n^4-5n^3-24n^2-21n-6) - 2lm(6n^5-n^4+6n^3-6n^2-9n-3) + 3l^2m(2n \\ & .(2n^4+3n^3+6n^2+4n+1) + m(2n^4+4n^3+6n^2+4n+1)) - 2l^3m(4n^4+3m(2n^4 \\ & +4n^3+6n^2+4n+1))\Big)x^3 + 12\Big(3l^4m(m-2)(2n^4+4n^3+6n^2+4n+1) + n(6n^5 \\ & -14n^4-5n^3-24n^2-21n-6) - 2lm(6n^5-n^4+6n^3-6n^2-9n-3) + 3l^2m(2n \\ & .(2n^4+3n^3+6n^2+4n+1) + m(2n^4+4n^3+6n^2+4n+1)) - 2l^3m(4n^4+3m(2n^4 \\ & +4n^3+6n^2+4n+1))\Big)x^4\Big].\end{aligned}$$

Proof. If we indicate $H_{n,\sigma,\tau}^{[l,m]}(h) = h\left(\frac{\sigma+\tau l}{n}\right)$, $J_{n,\sigma,\tau}^{[l,m]}(h) = (n+1)\int_{\frac{\sigma+\tau l}{n+1}}^{\frac{\sigma+\tau l+1}{n+1}} h(u)du$ and $p_{n,\sigma,\tau}^{[l,m]} = b_{n-lm,\sigma}(x)b_{m,\tau}(x)$, then we can show operators (1) and (2) as

$$(D_n^{[l,m]}h)(x) = \sum_{\sigma=0}^{n-lm}\sum_{\tau=0}^{m} p_{n,\sigma,\tau}^{[l,m]} H_{n,\sigma,\tau}^{[l,m]}(h)$$

and

$$(K_n^{[l,m]}h)(x) = \sum_{\sigma=0}^{n-lm}\sum_{\tau=0}^{m} p_{n,\sigma,\tau}^{[l,m]} J_{n,\sigma,\tau}^{[l,m]}(h).$$

Further, following the notations $b^H = H(e_1)$, $\mu_k^H = H(e_1 - b^H e_0)^k$ and $\mu_k^H(x) = H(e_1 - xe_0)^k$ and using the fact that $D_n^{[l,m]}$ preserves linear functions, we get

$$b^{H_{n,\sigma,\tau}^{[l,m]}} = \frac{\sigma+\tau l}{n},\ \mu_2^{H_{n,\sigma,\tau}^{[l,m]}} = 0,\ \mu_4^{H_{n,\sigma,\tau}^{[l,m]}} = 0,$$

$$b^{J_{n,\sigma,\tau}^{[l,m]}} = \frac{[1+2(\sigma+\tau l)]}{2(n+1)},\ \mu_2^{J_{n,\sigma,\tau}^{[l,m]}} = \frac{1}{12(n+1)^2},\ \mu_4^{J_{n,\sigma,\tau}^{[l,m]}} = \frac{1}{80(n+1)^4}.$$

Using Remark 2.2, we have

$$\begin{aligned}\alpha(x) &= \sum_{\sigma=0}^{n-lm}\sum_{\tau=0}^{m}(\mu_2^{H_{n,\sigma,\tau}^{[l,m]}} + \mu_2^{J_{n,\sigma,\tau}^{[l,m]}})p_{n,\sigma,\tau}^{[l,m]} = \frac{1}{12(n+1)^2}\\ \beta(x) &= \sum_{\sigma=0}^{n-lm}\sum_{\tau=0}^{m}(\mu_4^{H_{n,\sigma,\tau}^{[l,m]}} + \mu_4^{J_{n,\sigma,\tau}^{[l,m]}})p_{n,\sigma,\tau}^{[l,m]} = \frac{1}{80(n+1)^4}\\ \gamma(x) &= \sum_{\sigma=0}^{n-lm}\sum_{\tau=0}^{m}\Big((b^{H_{n,\sigma,\tau}^{[l,m]}} - x)^2\mu_2^{H_{n,\sigma,\tau}^{[l,m]}}(x) + (b^{J_{n,\sigma,\tau}^{[l,m]}} - x)^2\mu_2^{J_{n,\sigma,\tau}^{[l,m]}}(x)\Big)p_{n,\sigma,\tau}^{[l,m]}.\end{aligned}$$

Following [1, Theorem 3], we have

$$\begin{aligned}&\left|((D_n^{[l,m]} - K_n^{[l,m]})h)(x) - \sum_{i=1}^{2}\frac{h^{(i)}(x)}{i!}((D_n^{[l,m]} - K_n^{[l,m]})(e_1 - xe_0)^i)(x)\right|\\ \leq\ & \frac{\omega(h'', \sqrt{\beta(x)})}{2}[1+\alpha(x)] + \frac{\omega(h'', \sqrt{\gamma(x)})}{2}\Big[1 + (D_n^{[l,m]}(e_1 - xe_0)^2)(x) + (K_n^{[l,m]}(e_1 - xe_0)^2))(x)\Big].\end{aligned}$$

The proof follows using Lemma 2.5 and the second central moment of $D_n^{[l,m]}$ given by

$$D_n^{[l,m]}(e_1 - xe_0)^2)(x) = \frac{(lm(l-1)+n)}{n^2}x(1-x).$$

□

Remark 3.8. *As a special case with $l = m = 0$, the result reduces to the improved difference estimate between Bernstein operators B_n and Kantorovich operators K_n given by*

$$\begin{aligned}&\left|((B_n - K_n)h)(x) - \sum_{i=1}^{2}\frac{h^{(i)}(x)}{i!}((B_n - K_n)(e_1 - xe_0)^i)(x)\right|\\ \leq\ & \frac{1}{2}\Big(1 + \frac{1}{12(n+1)^2}\Big)\omega\Big(h'', \frac{1}{4\sqrt{5}(n+1)^2}\Big) + \frac{1}{2}\Big[1 + \frac{1}{3(n+1)^2} + \frac{(2n^2+n+1)x(1-x)}{n(n+1)^2}\Big]\omega(h'', \sqrt{\gamma(x)}),\end{aligned}$$

where

$$\begin{aligned}\gamma(x) &= \frac{1}{12n^3(n+1)^4}\Big[72n^5x^2(1-x)^2 + n^4x(1-x)(168x^2 - 168x + 67) + n^3(-60x^4\\ &+120x^3 - 101x^2 + 41x + 1) + 72n^2x(1-2x)^2(1-x) + 12nx(1-x)(21x^2\\ &-21x + 4) + 12x(1-x)(6x^2 - 6x + 1)\Big].\end{aligned}$$

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. M. Acu, G. B. Tunca and I. Raşa, Voronovskaja-type quantitative results for differences of positive linear operators, Symmetry 13, 1392 (2021).

[2] U. Abel , M. Ivan and R. Păltănea, The Durrmeyer variant of an operator defined by D. D. Stancu, Appl. Math. Comput. 259 (2015) 116–123.

[3] R.A. Devore and G. G. Lorentz, Constructive Approximation, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York-London, (1933).

[4] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer, New York (1987).
[5] V. Gupta and M. T. Rassias, Moments of Linear Positive Operators and Approximation, Series: SpringerBriefs in Mathematics, Springer Nature Switzerland AG (2019).
[6] V. Gupta and R. P. Agarwal, Convergence Estimates in Approximation Theory, Springer, New York, (2014).
[7] A. Kajla, The Kantorovich variant of an operator defined by D. D. Stancu, Appl. Math. Comput. 316 (2018), 400–408.
[8] D. D. Stancu, The remainder in the approximation by a generalized Bernstein operator: a representation by a convex combination of second-order divided differences, Calcolo 35 (1998) 53–62.
[9] D. D. Stancu, A note on a multiparameter Bernstein-type approximating operator. Mathematica (Cluj) 26 (49)(1984), 153–157.