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An Equivalent Condition for a Pseudo (k_0, k_1)-Covering Space

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Abstract. The paper aims at developing the most simplified axiom for a pseudo (k_0, k_1)-covering space. To make this a success, we need to strongly investigate some properties of a weakly local (*WL*-, for short) (k_0, k_1)-isomorphism. More precisely, we initially prove that a digital-topological imbedding *w.r.t.* a (k_0, k_1)-isomorphism implies a *WL*-(k_0, k_1)-isomorphism. Besides, while a *WL*-(k_0, k_1)-isomorphism is proved to be a (k_0, k_1)-continuous map, it need not be a surjection. However, the converse does not hold. Taking this approach, we prove that a *WL*-(k_0, k_1)-isomorphic surjection is equivalent to a pseudo-(k_0, k_1)-covering map, which simplifies the earlier axiom for a pseudo (k_0, k_1)-covering space by using one condition. Finally, we further explore some properties of a pseudo (k_0, k_1)-covering space regarding lifting properties. The present paper only deals with k-connected digital images.

1. Introduction

Although there are many works associated with typical covering spaces in algebraic topology [27], semicovering spaces [3], and generalized covering spaces [4, 5], it turns out that these approaches cannot facilitate the study of digital spaces (or digital images). Thus the notions of a digital (k_0, k_1)-covering space [8] and a pseudo (k_0, k_1)-covering space [11] were developed so that they can play important roles in studying several types of lifting theorems from a viewpoint of digital topology. Hence there are many works studying these topics including the papers [6–8, 10, 11]. Indeed, lifting theorems based on digital covering maps have been substantially used in calculating digital fundamental groups of digital images and classifying digital images [6, 7]. It indeed has its root in classical graph theory [1] with a certain k-adjacency (see the property (2.1) of the present paper), where $\mathbb{Z}^n$ is the set of points in the Euclidean nD space with integer coordinates, $n \in \mathbb{N}$ that is the set of natural numbers.

In digital topology, among many methods of dealing with digital images [15–19, 22, 24–26], the present paper will follow graph theoretical approach originated in [25, 26] because a digital image (X, k) can be assumed to be a set $X \subset \mathbb{Z}^n$ with one of the k-adjacency of $\mathbb{Z}^n$ (or a digital k-graph on $\mathbb{Z}^n$) [25] (see also [9]).

Motivated by a digital (k_0, k_1)-covering space in [8], a paper [11] developed a pseudo (k_0, k_1)-covering which is broader than a digital covering. Moreover, it proved that a pseudo (k_0, k_1)-covering map has

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the unique pseudo-lifting property instead of the unique lifting property. Furthermore, several kinds of local (k_0, k_1)-isomorphisms were developed in [6, 7, 11, 14] such as a pseudo-local (*PL*-, for brevity)(k_0, k_1)-isomorphism, a *WL*-(k_0, k_1)-isomorphism, and a local (k_0, k_1)-isomorphism which have been used in classifying digital images. However, in some literature we can observe some confusion and misunderstanding on certain relationships between a digital (k_0, k_1)-covering space and a pseudo (k_0, k_1)-covering space and further, among several kinds of local (k_0, k_1)-isomorphisms. Thus, a recent paper [14] established the most refined axiom for a digital (k_0, k_1)-covering space which was one of the hot issues for the last twenty years in digital topology. Motivated by this study, since a pseudo (k_0, k_1)-covering space is a weaker than a digital (k_0, k_1)-covering space, it is worthy to establish the most simplified axiom for a pseudo (k_0, k_1)-covering space. To make this work a success, the following issues might be raised, which remains open.

(Q1) What are characterizations of a *WL*-(k_0, k_1)-isomorphism ?

(Q2) What are certain relationships between a digital-topological imbedding and a *WL*-(k_0, k_1)-isomorphism ?

(Q3) What relationships exist among a *PL*-(k_0, k_1)-isomorphism, a *WL*-(k_0, k_1)-isomorphism, and a local (k_0, k_1)-isomorphism ?

(Q4) Given two simple closed k-curves, under what condition do we have a digital-topological imbedding from one to another ?

(Q5) What is an equivalent and the most simplified axiom for a pseudo (k_0, k_1)-covering space ?

To address the issues, first of all we need to make a certain distinction among several kinds of local k-isomorphisms. Naively, we need to clarify some relationships among a *PL*-k-isomorphism [6, 14], a *WL*-k-isomorphism, and a local k-isomorphism.

The paper is organized as follows. Section 2 provides some basic notions needed for the study in the paper. Section 3 investigates some properties of a *PL*-k-isomorphism and a *WL*-k-isomorphism. Furthermore, it compares among a *PL*-(k_0, k_1)-isomorphism, a *WL*-(k_0, k_1)-isomorphism, and a local (k_0, k_1)-isomorphism. Section 4 proposes an equivalent condition for a pseudo (k_0, k_1)-covering map. Besides, we further remark on the digital pseudo-lifting property associated with a pseudo (k_0, k_1)-covering space. Finally, Section 5 concludes the paper. The paper only deals with k-connected digital images and uses the notation := to introduce some terms.

2. Preliminaries

Motivated by the digital k-connectivity for low dimensional digital images $(X, k), X \subset \mathbb{Z}^3$ [25, 26], the papers [6, 8] firstly generalized it to obtain the k-adjacency relations for high dimensional lattice spaces. More precisely, when studying $X \subset \mathbb{Z}^n, n \in \mathbb{N}$, the k-adjacency (or digital k-connectivity) relations were initially considered on X [8] (see also [6, 7, 12]), as follows:

For a natural number t, $1 \leq t \leq n$, the distinct points $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{Z}^n$ are $k(t, n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide. According to this statement, the $k(t, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, are established [8] (see also [7, 10–12]) as follows:

$$k := k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n := \frac{n!}{(n-i)!\, i!}. \tag{2.1}$$

We say that the pair (X, k) is a digital image in a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$ [17, 20, 25]. Owing to the *digital k-connectivity paradox* of a digital image (X, k) [20], we remind the reader that $k \neq \bar{k}$ except the case $(\mathbb{Z}, 2, 2, X)$. However, the present paper is not concerned with the $\bar{k}$-adjacency of $\mathbb{Z}^n \setminus X$. Using these k-adjacency relations of $\mathbb{Z}^n$ stated in (2.1), $n \in \mathbb{N}$, we will call (X, k) a digital image on $\mathbb{Z}^n$, $X \subset \mathbb{Z}^n$. Besides, for $x, y \in \mathbb{Z}$ with $x \lneq y$, the set $[x, y]_{\mathbb{Z}} = \{n \in \mathbb{Z} \,|\, x \leq n \leq y\}$ with 2-adjacency is called a digital interval [20].

Hereafter, (X, k) is assumed in $\mathbb{Z}^n$ for a certain $n \in \mathbb{N}$ with one of the k-adjacency of (2.1). The following terminology and concepts [8, 9, 20, 25, 26] will be often used later. Given two non-empty digital images (A_1, k) and (A_2, k) is k-adjacent if $A_1 \cap A_2 = \emptyset$ and there are certain points $a_1 \in A_1$ and $a_2 \in A_2$ such that a_1 is k-adjacent to a_2 [20].

Consider a digital image (X, k) in $\mathbb{Z}^n, n \in \mathbb{N}$, and a point $y \in X^c$ which is the complement of X in $\mathbb{Z}^n$. The point y is said to be k-adjacent to (X, k) if X is k-adjacent to $\{y\}$, i.e., there is a point $x \in X$ which is k-adjacent to y. In a digital image (X, k), by a k-path, we mean a sequence $(c_i)_{i\in[0,l]_{\mathbb{Z}}} \subset X$ such that c_i and c_j are k-adjacent if $|i - j| = 1$ [21]. Besides, l is said to be a length of this k-path. Besides, (X, k) is said to be k-connected [21] if for any distinct points $p, q \in X$, a k-path $(c_i)_{i\in[0,l]_{\mathbb{Z}}}$ exists in X such that $c_0 = p$ and $c_l = q$ (for more details see [13]). In particular, a singleton set is assumed to be k-connected (for more details see [13]).

By a simple k-path from p to q in (X, k), we mean a finite set $(c_i)_{i\in[0,m]_{\mathbb{Z}}} \subset X$ such that c_i and c_j are k-adjacent if and only if $|i - j| = 1$, where $c_0 = p$ and $c_m = q$ [21]. Then, the length of this set $(c_i)_{i\in[0,m]_{\mathbb{Z}}}$ is said to be m and denoted by $l_k(p, q) := m$.

A simple closed k-curve (or k-cycle) with l elements in $\mathbb{Z}^n, n \geq 2$, denoted by $SC_k^{n,l}$ [8, 21], $4 \leq l \in \mathbb{N}$, is defined to be the set $(c_i)_{i\in[0,l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that c_i and c_j are k-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$. Then, the number l of $SC_k^{n,l}$ depends on both the dimension n of $\mathbb{Z}^n$ and the k-adjacency. For more details, see the property (5) in [14].

Let us recall the concept of digital (k_0, k_1)-continuity of a map $f : (X, k_0) \to (Y, k_1)$ originated by [26]. By mapping every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [26] established the notion of (digital) (k_0, k_1)-continuity. Motivated by this continuity, in order to efficiently study various properties of digital images, we have often used the following digital k-neighborhood [7, 8, 12]. For a digital image (X, k) in $\mathbb{Z}^n$, the digital k-neighborhood of $p \in X$ with radius ε is defined in X to be the following subset of X

$$N_k(p, \varepsilon) = \{x \in X \,|\, l_k(p, x) \leq \varepsilon\} \cup \{p\}, \tag{2.2}$$

where $l_k(p, x)$ is the length of a shortest simple k-path from p to x and $\varepsilon \in \mathbb{N}$.

Indeed, the digital k-neighborhood of (2.2) can be also represented by using a digital k-ball with a certain metric in (X, k) (for more details, see the notion (7) of [13]).

By using the digital k-neighborhood of (2.2), the typical continuity for digital images in [25] can be represented as the following form because every point x of a digital image (X, k) always has an $N_k(x, 1) \subset X$.

Proposition 2.1. ([8, 10, 11]) *Let (X, k_0) and (Y, k_1) be digital images. A map $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every point $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

The presentation of the digital (k_0, k_1)-continuity in Proposition 2.1 plays a crucial role in addressing the issues (Q1)-(Q5). As mentioned in the previous part, since a digital image (X, k) can be considered to be a digital k-graph [9], we have often used a (k_0, k_1)-*isomorphism* as in [9] instead of a (k_0, k_1)-*homeomorphism* as in [2], as follows:

Definition 2.2. ([2]; see also [9]) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism* and use the notation $X \approx_{k_0} Y$.

Since the concept of a digital-topological imbedding can play an important role in digital topology, a recent paper [14] proposed it, as follows:

Definition 2.3. ([14]) (Digital-topological embedding (imbedding)) Consider two digital images $(X, k := k(t, n)), X \subset \mathbb{Z}^n$ and $(Y, k' := k(t', n')), Y \subset \mathbb{Z}^{n'}$ such that there is an arbitrary (k, k')-isomorphism $h : (X, k) \to (h(X), k') \subset (Y, k')$. Then, we say that h is a (k, k')-imbedding (or embedding) of (X, k) into (Y, k') or (X, k) is a digital-topological (k, k')-imbedding into (Y, k') *w.r.t.* the (k, k')-isomorphism h.

In particular, in the case $X \subset Y \subset \mathbb{Z}^n$ with the same k-adjacency of both X and Y, a digital-topological imbedding from (X, k) to (Y, k) is simply understood to be an inclusion map from (X, k) into (Y, k).

In Definition 2.3, we observe that the dimension "n" (resp. k-adjacency) need not be equal to "n'" (resp. k'-adjacency) [14].

Definition 2.4. ([14]) In Definition 2.3, for $k := k(t, n)$ for X and $k' := k(t', n')$ for Y, if $t = t'$, then we say that the map h in Definition 2.3 is a strict (k, k')-imbedding of (X, k) into (Y, k') or (X, k) is a strictly digital-topological imbedding into (Y, k') *w.r.t.* the (k, k')-isomorphism h.

3. Comparison among several types of local (k_0, k_1)-isomorphisms and a digital-topological imbedding

This section initially makes a comparison among several types of local k-isomorphisms such as a *PL*-k-isomorphism [6], a *WL*-k-isomorphism [6], and a (strong) local k-isomorphism [7] so that we can clarify some difference among them. Indeed, this approach is essential to simplifying the axiom for a pseudo (k_0, k_1)-covering space in Section 4 and further, it can facilitate the study of the unique pseudo-lifting property which is weaker than the unique lifting property in digital covering theory in [8].

Definition 3.1. ([6, 14]) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a (k_0, k_1)-continuous map $h : X \to Y$ is called a pseudo-local (*PL*-, for brevity) (k_0, k_1)-isomorphism if for every point $x \in X$, $h(N_{k_0}(x, 1))$ is k_1-isomorphic with $N_{k_1}(h(x), 1)$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a *PL*-k_0-isomorphism.

For instance, we suggest the following example for a *PL*-(k_0, k_1)-isomorphism.

Example 3.2. Let us consider the map h in Figure 1. Then, the map h is a *PL*-$(8, 26)$-isomorphism. More precisely, assume the set $X := \{x_i \,|\, i \in [0, 10]_{\mathbb{Z}}\} \subset \mathbb{Z}^2$ in Figure 1. Then, consider the map $h : (X, 8) \to SC_{26}^{3,5} := (c_i)_{i \in [0,4]_{\mathbb{Z}}}$ defined by

$$h(x_i) = c_{i(mod\,5)}, i \in [0, 9]_{\mathbb{Z}} \text{ and } h(x_{10}) = c_0.$$

Then, the map h is a *PL*-$(8, 26)$-isomorphism.

Regarding the model of $SC_{26}^{3,5}$ in Figure 1, we can take several types of it (for more details, see [14]). Furthermore, using the notion of a digital-topological imbedding, it turns out that there are many types of $SC_{k(t,n)}^{n,5}$ and each of them is $(k(t, n), 26)$-isomorphic to $SC_{26}^{3,5}$, $3 \le t \le n$ (for more details, see Theorem 1 and Corollary 1 of [14]).

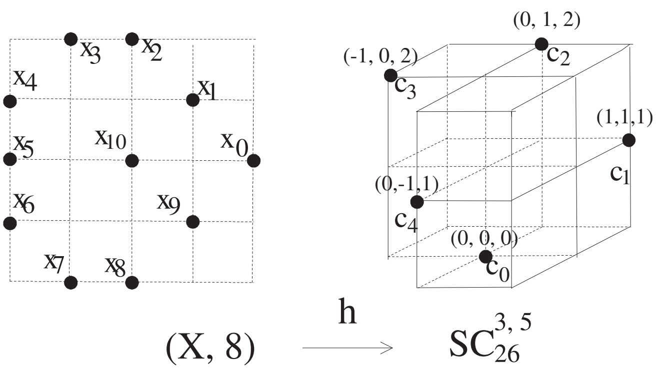

Figure 1: Configuration of a *PL*-$(8, 26)$-isomorphism h referred to in Example 3.2. However, it is not a *WL*-$(8, 26)$-isomorphism at the points x_1 and x_9 (see Definition 3.11).

Definition 3.1 is indeed admissible in studying digital images from the viewpoint of digital topology. However, we find that the condition " a (k_0, k_1)-continuous map $h : X \to Y$" is redundant for defining a "*PL*-(k_0, k_1)-isomorphism" because the condition "for every $x \in X$, $h(N_{k_0}(x, 1))$ is k_1-isomorphic with $N_{k_1}(h(x), 1)$" implies the (k_0, k_1)-continuity of the given map h. Let us now support this feature.

Lemma 3.3. *For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, assume a map $h : X \to Y$ such that for each point $x \in X$, $h(N_{k_0}(x, 1))$ is k_1-isomorphic with $N_{k_1}(h(x), 1)$. Then the map h is (k_0, k_1)-continuous. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is k_0-continuous.*

Proof. Owing to the hypothesis, for every point $x \in X$ since

$$h(N_{k_0}(x, 1)) \approx_{k_1} N_{k_1}(h(x), 1),$$

we obtain a k_1-continuous bijection between $h(N_{k_0}(x,1))$ and $N_{k_1}(h(x),1)$ so that

$$h(N_{k_0}(x,1)) \subset N_{k_1}(h(x),1),$$

because each of these sets $h(N_{k_0}(x,1))$ and $N_{k_1}(h(x),1)$ is a subset of (Y,k_1) having the element $h(x)$. □

Based on Lemma 3.3, we now have a most refined version of the PL-(k_0,k_1)-isomorphism of Definition 3.1.

Definition 3.4. (Simplification of a PL-(k_0,k_1)-isomorphism) For two digital images (X,k_0) in $\mathbb{Z}^{n_0}$ and (Y,k_1) in $\mathbb{Z}^{n_1}$, assume a map $h : X \to Y$ such that for each point $x \in X$, $h(N_{k_0}(x,1))$ is k_1-isomorphic with $N_{k_1}(h(x),1)$. Then the map h is called a PL-(k_0,k_1)-isomorphism. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is a PL-k_0-isomorphism.

Theorem 3.5. *A PL-(k_0,k_1)-isomorphism is a surjection.*

Proof. By contrary, suppose a PL-(k_0,k_1)-isomorphism $h : (X,k_0) \to (Y,k_1)$ which is not a surjection. With the hypothesis of the k_1-connectedness of (Y,k_1), take a certain point $y' \in Y \setminus h(X)$ such that y' is k_1-adjacent to $h(X)$. Hence there is a point $y \in h(X)$ which is k_1-adjacent to y' so that $y' \in N_{k_1}(y,1)$. Then, there is a point $x \in X$ such that $h(x) = y$. Owing to the hypothesis, we have

$$h(N_{k_0}(x,1)) \approx_{k_1} N_{k_1}(h(x),1) = N_{k_1}(y,1). \tag{3.1}$$

While $y' \in N_{k_1}(y,1)$, there is no point $x' \in N_{k_0}(x,1)$ such that $h(x') = y'$, which invokes a contradiction to the PL-(k_0,k_1)-isomorphism of h at the point x. □

In view of Lemma 3.3 and Theorem 3.5, we obtain the following:

Corollary 3.6. *A PL-(k_0,k_1)-isomorphism implies a (k_0,k_1)-continuous surjection. However, the converse does not hold.*

Proof. By Lemma 3.3 and Theorem 3.5, it turns out that a PL-(k_0,k_1)-isomorphism leads to a (k_0,k_1)-continuous surjection. However, using a counterexample, let us now prove that not every (k_0,k_1)-continuous surjection is always a PL-(k_0,k_1)-isomorphism. More precisely, let us consider the map

$$f : [0,5]_{\mathbb{Z}} \to SC_k^{n,4} := (c_i)_{i\in[0,3]_{\mathbb{Z}}}$$

defined by $f(t) = c_{t(mod\,4)}$, where $k := 3^n - 1$. While the map f is a $(2,k)$-continuous surjection, it is not a PL-$(2,k)$-isomorphism at the points 0 and 5 in $[0,5]_{\mathbb{Z}}$. □

Unlike Corollary 3.6, we obtain the following:

Remark 3.7. Neither of a PL-(k_0,k_1)-isomorphism and a (k_0,k_1)-continuous bijection implies the other.

Proof. Using counterexamples, we prove the assertion. First of all, consider the map

$$h : SC_k^{n,2l} := (x_i)_{i\in[0,2l-1]_{\mathbb{Z}}} \to SC_k^{n,l} := (y_i)_{i\in[0,l-1]_{\mathbb{Z}}}$$

defined by $h(x_i) = y_{i(mod\,l)}$. While the map h is a PL-(k_0,k_1)-isomorphism, it is (k_0,k_1)-continuous surjection which is not an injective map.

Next, consider the map

$$f : [0,3]_{\mathbb{Z}} \to SC_k^{n,4} := (c_i)_{i\in[0,3]_{\mathbb{Z}}},$$

defined by $f(i) = c_i$, where $k := 3^n - 1, n \geq 2$. While the map f is a $(2,k)$-continuous bijection, it is not a PL-$(2,k)$-isomorphism at the points 0 and 3 in $[0,3]_{\mathbb{Z}}$. □

To make a PL-(k_0,k_1)-isomorphism more rigid, the paper [7] defined the following:

Definition 3.8. ([7]; see also [8]) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a (k_0, k_1)-continuous map $h : X \to Y$ is called a local (k_0, k_1)-isomorphism if for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a local k_0-isomorphism.

A recent paper [14] simplified this local (k_0, k_1)-isomorphism by using the following property.

Remark 3.9. ([14]) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, consider a map $h : X \to Y$ such that for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$. Then h is a (k_0, k_1)-continuous map. In particular, in the case $n_0 = n_1$ and $k := k_0 = k_1$, the map h is a k-continuous map.

Owing to this property, we can represent the original version of a local (k_0, k_1)-isomorphism of Definition 3.8 as the most simplified version of a local (k_0, k_1)-isomorphism, as follows:

Definition 3.10. ([14]) (Simplification of a a local (k_0, k_1)-isomorphism) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, consider a map $h : (X, k_0) \to (Y, k_1)$ such that for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$. Then the map h is said to be a local (k_0, k_1)-isomorphism If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a local k_0-isomorphism.

The paper [11] defined the following notion which is weaker than a local (k_0, k_1)-isomorphism.

Definition 3.11. ([11]) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a weakly local (*WL*-, for brevity) (k_0, k_1)-isomorphism if for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $h(N_{k_0}(x, 1)) \subset (Y, k_1)$. In particular, if $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a weakly local k_0-isomorphism (or a *WL*-k_0-isomorphism).

A paper [11] proved that a *WL*-(k_0, k_1)-isomorphism is a (k_0, k_1)-continuous map. More precisely, given two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, consider any point $x \in X$. Since the given map h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $h(N_{k_0}(x, 1))$, we obtain $h(N_{k_0}(x, 1)) \subset N_{k_1}(h(x), 1)$. By Proposition 2.1, we obtain the (k_0, k_1)-continuity of h.

The recent paper [14] proved that a digital-topological embedding *w.r.t.* a (k_0, k_1)-isomorphism does not imply a local (k_0, k_1)-isomorphism. Let us now explore some relationships between a digital-topological imbedding *w.r.t.* a (k_0, k_1)-isomorphism and a *WL*-(k_0, k_1)-isomorphism.

Theorem 3.12. *Given two digital images (X, k_0) and (Y, k_1), if (X, k_0) is a digital-topological imbedding into (Y, k_1) w.r.t. a (k_0, k_1)-isomorphism, say h, then the map h is a WL-(k_0, k_1)-isomorphism from (X, k_0) to $(h(X), k_1)$. However, the converse does not hold.*

Proof. Owing to the hypothesis, we obtain a (k_0, k_1)-isomorphism $h : (X, k_0) \to (h(X), k_1) \subset (Y, k_1)$. Thus, for any element $x \in X$ the restriction of h to $N_{k_0}(x, 1)$, denoted by

$$h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to (h(N_{k_0}(x, 1)), k_1) \subset (h(X), k_1),$$

is also a (k_0, k_1)-isomorphism. Thus h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $h(N_{k_0}(x, 1)) \subset (Y, k_1)$, which implies that h is a *WL*-(k_0, k_1)-isomorphism from (X, k_0) to $(h(X), k_1)$.

However, the converse does not hold with the following counterexample. Let us consider the k-continuous surjection

$$h : SC_k^{n,2l} := (c_i)_{i\in[0,2l-1]_{\mathbb{Z}}} \to SC_k^{n,l} := (d_i)_{i\in[0,l-1]_{\mathbb{Z}}} \tag{3.2}$$

such that $h(c_i) = d_{i(mod\, l)}$. Then, it is clear that the map h is a *WL*-k-isomorphism. However, $SC_k^{n,2l}$ is not a digital-topological imbedding into $SC_k^{n,l}$ w.r.t. a k-isomorphism. □

Corollary 3.13. *A WL-(k_0, k_1)-isomorphism need neither be injective nor be surjective.*

Proof. Owing to the property of (3.2), it is clear that a *WL*-(k_0, k_1)-isomorphism need not be injective. Next, consider the two simple k-paths $C := (c_i)_{i\in[0,3]_{\mathbb{Z}}}$ and $D := (d_i)_{i\in[0,4]_{\mathbb{Z}}}$ such that $C \subset D$. Consider an inclusion map $i : C \to D$. While this inclusion is a *WL*-k-isomorphism, it is not surjective. □

Remark 3.14. Consider the map

$$h : SC_8^{2,4} := (c_i)_{i\in[0,3]_{\mathbb{Z}}} \to SC_8^{2,6} := (d_i)_{i\in[0,5]_{\mathbb{Z}}}$$

given by $h(c_i) = d_i$. While the map h is neither a WL-8-isomorphism nor a digital-topological $(8, 8)$-imbedding from $SC_8^{2,4}$ into $SC_8^{2,6}$ w.r.t. 8-isomorphism.

We can observe that a WL-(k_0, k_1)-isomorphism has its own intrinsic properties, as follows. It is clear that a (k_0, k_1)-continuous map need not be a WL-(k_0, k_1)-isomorphism because a (k_0, k_1)-continuous map which is not injective is not a WL-(k_0, k_1)-isomorphism. Besides, we can observe some difference among a PL-(k_0, k_1)-isomorphism, a WL-(k_0, k_1)-isomorphism, and a local (k_0, k_1)-isomorphism, as follows.

Theorem 3.15. *(1) A WL-(k_0, k_1)-isomorphic surjection does not imply a local (k_0, k_1)-isomorphism. However, a local (k_0, k_1)-isomorphism implies a WL-(k_0, k_1)-isomorphic surjection.*

(2) A PL-(k_0, k_1)-isomorphism is weaker than a local (k_0, k_1)-isomorphism.

(3) Neither of a PL-(k_0, k_1)-isomorphism and a WL-(k_0, k_1)-isomorphism implies the other.

Proof. (1) As a counterexample, let us consider the map

$$g : [0, l-1]_{\mathbb{Z}} \to SC_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbb{Z}}}$$

defined by $g(i) = c_i$. While the map g is a WL-$(2, k)$-isomorphic surjection, it is clear that g is not a local $(2, k)$-isomorphism at the points 0 and $l-1$.

Meanwhile, a recent paper [14] firstly proved that a local (k_0, k_1)-isomorphism is a surjection. Besides, given a local (k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$, for every point $x \in X$ we obtain $N_{k_0}(x, 1) \approx_{(k_0,k_1)} N_{k_1}(h(x), 1)$ via the given map h. Naively, we have the restriction of h to the set $N_{k_0}(x, 1)$ onto $N_{k_1}(h(x), 1)$, i.e.,

$$h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to N_{k_1}(h(x), 1),$$

which is a (k_0, k_1)-isomorphism. Hence we obtain

$$h(N_{k_0}(x, 1)) \approx_{k_1} N_{k_1}(h(x), 1).$$

Indeed, we obtain $h(N_{k_0}(x, 1)) = N_{k_1}(h(x), 1)$ so that $N_{k_0}(x, 1) \approx_{(k_0,k_1)} h(N_{k_0}(x, 1))$ via the given map h, which is a WL-(k_0, k_1)-isomorphic surjection of h.

(2) It is clear that a local (k_0, k_1)-isomorphism implies a PL-(k_0, k_1)-isomorphism. However, the converse does not hold. As a counterexample, let us consider the map h in Example 3.2. As mentioned in Example 3.2, while the map h is a PL-$(8, 26)$-isomorphism, it is not a local $(8, 26)$-isomorphism at the points x_1 and x_9.

(3) The map h in Example 3.2 is a counterexample for the assertion that a PL-(k_0, k_1)-isomorphism implies a WL-(k_0, k_1)-isomorphism (see the points x_1 and x_9). Conversely, let us consider an inclusion map which is not a surjection. Then this map is a counterexample for the assertion that a WL-(k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ implies a PL-(k_0, k_1)-isomorphism owing to the point $y(\in Y \setminus h(X))$ which is k_1-adjacent to $h(X)$. □

Example 3.16. Consider the map $g : [0, 3]_{\mathbb{Z}} \to SC_8^{2,4} := (z_i)_{i\in[0,3]_{\mathbb{Z}}}$ given by $g(i) = z_i$. Then we obtain the following:

(1) g is a WL-$(2, 8)$-isomorphism.

(2) g is not a PL-$(2, 8)$-isomorphism.

(3) g is not a local $(2, 8)$-isomorphism.

A WL-(k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ is a local version of a digital-topological imbedding *w.r.t.* $N_{k_0}(x, 1) \subset (X, k_0)$

Remark 3.17. (1) Given a digital image (X, k) and its subset $A \subset X$, the inclusion map $i : (A, k) \to (X, k)$ is a *WL*-*k*-isomorphism.

(2) Consider the map $g : [0, 3]_{\mathbb{Z}} \to SC_8^{2,4} := (Z, 8)$ given by $g(i) = z_i, i \in [0, 3]_{\mathbb{Z}}$ in Figure 2. While it is a *WL*-$(2, 8)$-isomorphism (see Example 3.16(1)), it is not a digital-topological imbedding.

Proof. (1) The proof of (1) is straightforward.

(2) While the map g is a *WL*-$(2, 8)$-isomorphism, it is clear that there is no $(2, 8)$-isomorphism supporting $(X, 2)$ to be a digital-topological imbedding into $SC_8^{2,4}$ owing to the points 0 and 3. □

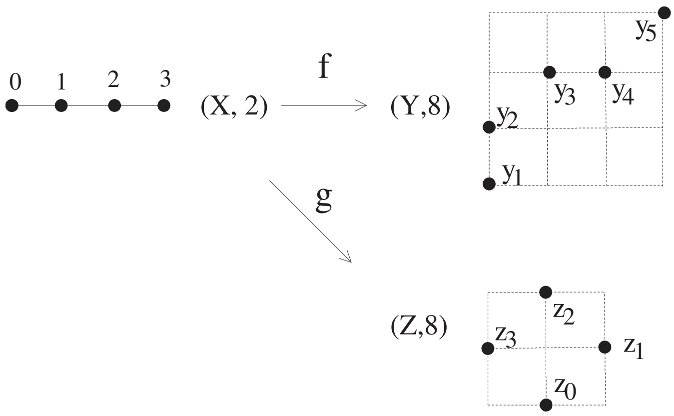

Figure 2: Explanation of a map related to the map in the proof of Remark 3.17. (1) f is a *WL*-$(2, 8)$-isomorphism (2) While the map g is a *WL*-$(2, 8)$-isomorphism, it is neither a *PL*-$(2, 8)$-isomorphism nor a digital-topological imbedding *w.r.t.* a $(2, 8)$-isomorphism.

Example 3.18. Given $l_1 \lneq l_2$, consider the map h

$$h : SC_{k_1}^{n_1,l_1} := (c_i)_{i\in[0,l_1-1]_{\mathbb{Z}}} \to SC_{k_2}^{n_2,l_2} := (d_i)_{i\in[0,l_2-1]_{\mathbb{Z}}},$$

such that $h(c_i) = d_i, i \in [0, l_1 - 1]_{\mathbb{Z}}$. Since $h(SC_{k_1}^{n_1,l_1})$ is a k_2-connected proper subset of $SC_{k_2}^{n_2,l_2}$, we obtain the following:

(1) h is not a *PL*-(k_1, k_2)-isomorphism.
(2) h is not a *WL*-(k_1, k_2)-isomorphism.
(3) h is not a local (k_1, k_2)-isomorphism.
(4) h is not a digital-topological (k_1, k_2)-imbedding.

Corollary 3.19. *Given two $SC_{k_1}^{n_1,l_1}$ and $SC_{k_2}^{n_2,l_2}$, they are PL-(k_1, k_2)-, WL-(k_1, k_2)-, and local (k_1, k_2)-isomorphic with each other if and only if $l_1 = l_2$.*

Since $SC_k^{n,l}$ plays an important role in digital topology, regarding the question (Q4), let us now explore some properties of it *w.r.t.* a digital-topological imbedding.

Theorem 3.20. *$SC_{k_1}^{n_1,l_1}$ is a digital-topological imbedding into $SC_{k_2}^{n_2,l_2}$ w.r.t. a (k_1, k_2)-isomorphism if and only if $l_1 = l_2$.*

Proof. Using the contrapositive law, we prove that a digital-topological imbedding from $SC_{k_1}^{n_1,l_1}$ into $SC_{k_2}^{n_2,l_2}$ *w.r.t.* a (k_1, k_2)-isomorphism implies the identity $l_1 = l_2$.

Naively, assume $l_1 \neq l_2$. Without loss of generality, we may take $l_1 \lneq l_2$. Then we now prove that $SC_{k_1}^{n_1,l_1}$ is not a digital-topological imbedding into $SC_{k_2}^{n_2,l_2}$ *w.r.t.* a (k_1,k_2)-isomorphism. By contrary, suppose there is a certain (k_1,k_2)-isomorphism from $SC_{k_1}^{n_1,l_1}$ into $SC_{k_2}^{n_2,l_2}$. For convenience, we may assume the map h

$$h : SC_{k_1}^{n_1,l_1} := (c_i)_{i\in[0,l_1-1]_{\mathbb{Z}}} \to SC_{k_2}^{n_2,l_2} := (d_i)_{i\in[0,l_2-1]_{\mathbb{Z}}},$$

such that $h(SC_{k_1}^{n_1,l_1})$ as a k_2-connected proper subset of $SC_{k_2}^{n_2,l_2}$. Since $h(SC_{k_1}^{n_1,l_1})^\sharp = l_1$ is less than l_2, we conclude that the map h is not (k_0,k_1)-continuous at the points c_0 and c_{l_1-1}, which implies that $SC_{k_1}^{n_1,l_1}$ is not a digital-topological imbedding into $SC_{k_2}^{n_2,l_2}$ *w.r.t.* a (k_1,k_2)-isomorphism.

Conversely, if $SC_{k_1}^{n_1,l_1}$ is a digital-topological imbedding into $SC_{k_2}^{n_2,l_2}$ *w.r.t.* a (k_1,k_2)-isomorphism, owing to a certain (k_1,k_2)-isomorphism from $SC_{k_1}^{n_1,l_1}$ to $SC_{k_2}^{n_2,l_2}$, we clearly have $l_1 = l_2$. □

4. An equivalent axiom for a pseudo (k_0,k_1)-covering space

To address the query (Q5) in Section 1, first of all we now recall the notion of a pseudo-(k_0,k_1)-covering space. While a local (k_0,k_1)-isomorphism is proved to be a surjection [14], since a *WL*-(k_0,k_1)-isomorphism need not be surjective (see Corollary 3.13), the notion of a pseudo-(k_0,k_1)-covering space is defined, as follows:

Definition 4.1. ([11]) Let (E,k_0) and (B,k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection such that for any $b \in B$,

(1) for some index set M, $p^{-1}(N_{k_1}(b,1)) = \cup_{i\in M} N_{k_0}(e_i,1)$ with $e_i \in p^{-1}(b)$;

(2) if $i,j \in M$ and $i \neq j$, then $N_{k_0}(e_i,1) \cap N_{k_0}(e_j,1)$ is an empty set; and

(3) the restriction of p to $N_{k_0}(e_i,1)$ from $N_{k_0}(e_i,1)$ to $N_{k_1}(b,1)$ is a *WL*-(k_0,k_1)-isomorphism for all $i \in M$.

Then the map p is called a pseudo-(k_0,k_1)-covering map, (E,p,B) is said to be a pseudo-(k_0,k_1)-covering and (E,k_0) is called a pseudo-(k_0,k_1)-covering space over (B,k_1).

In Definition 4.1(3), note that the set $N_{k_1}(b,1)$ need not be k_1-isomorphic to $p(N_{k_0}(e_i,1))$.

Remark 4.2. The original version of a digital (k_0,k_1)-covering space was developed in [6–8, 10]. After that, the recent paper [14] proved that a local (k_0,k_1)-isomorphism $p : (E,k_0) \to (B,k_1)$ is a surjective and further, equivalent to a digital (k_0,k_1)-covering map.

Unlike Remark 4.2, up to now we don't know if there is the most simplified axiom for a pseudo-(k_0,k_1)-covering map in Definition 4.1. Thus we need to observe the following:

Remark 4.3. (1) Neither of a *PL*-(k_0,k_1)-isomorphism and a *WL*-(k_0,k_1)-isomorphism implies a pseudo-(k_0,k_1)-covering map.

(2) Based on Remark 4.2, a digital (k_0,k_1)-covering map implies a pseudo-(k_0,k_1)-covering map. However, the converse does not hold [11].

Proof. (1) Using counterexamples, we prove these assertions. First, let us consider the *PL*-(k_0,k_1)-isomorphism p shown in Example 3.2. Then it is not a pseudo-(k_0,k_1)-covering map (see the points x_1 and x_9).

Second, as a counterexample, given $l_1 \lneq l_2 - 1$, consider the map $g : [0,l_1]_{\mathbb{Z}} \to SC_k^{n,l_2} := (c_i)_{i\in[0,l_2-1]_{\mathbb{Z}}}$ given by $g(i) = c_i, i \in [0,l_1]_{\mathbb{Z}}$. While the map g is a *WL*-$(2,k)$-isomorphism, it is not a surjection which implies that g is not a pseudo-$(2,k)$-covering map.

(2) Let E be the set (see Figure 3)

$$\{e_{2m} := (2m,0) \,|\, m \in \mathbb{N} \cup \{0\}\} \cup \{e_{2m-1} := (2m-1,1) \,|\, m \in \mathbb{N}\}.$$

Assume the map (see Figure 3)

$$\begin{Bmatrix} p : (E, 8) \to SC_8^{2,8} := (i)_{i \in [0,7]_{\mathbb{Z}}}; \\ \text{given by } p(e_i) = i \in SC_8^{2,8}, i \in \mathbb{N} \cup \{0\}. \end{Bmatrix} \tag{4.1}$$

Then it is obvious that while the map p is a pseudo-$(8, 8)$-covering map, it is not a local $(8, 8)$-isomorphism (see the point $e_0 := (0, 0)$) which implies that p is not a digital $(8, 8)$-covering map. □

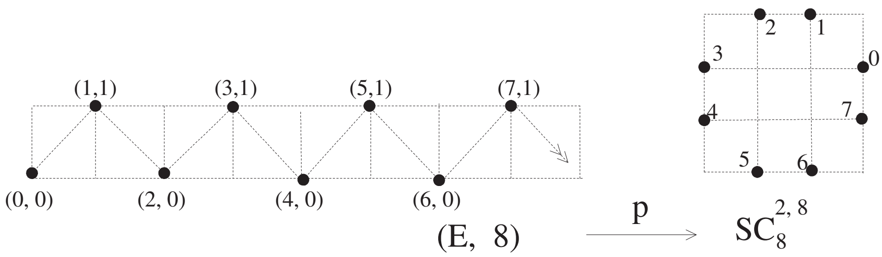

Figure 3: Comparison between a digital $(8, 8)$-covering map and a digital pseudo-$(8, 8)$-covering map. The digital pseudo-$(8, 8)$-covering map $p : (E, 8) \to SC_8^{2,8}$ in Remark 4.3(2) is not a digital $(8, 8)$-covering map.

Let us now explore some properties of a WL-(k_0, k_1)-isomorphism of Definition 3.8 which will be used in addressing the issue (Q5), as follows:

Proposition 4.4. *Let $p : (E, k_0) \to (B, k_1)$ be a WL-(k_0, k_1)-isomorphic surjection. Then, for any $b \in B$ with $e_i \in p^{-1}(b)$, for some index set M we obtain*

$$p^{-1}(N_{k_1}(b, 1)) = \cup_{i \in M} N_{k_0}(e_i, 1) \text{ with } e_i \in p^{-1}(b). \tag{4.2}$$

Then, the following hold.

(1) In (4.2), if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1)$ is an empty set;

(2) For the points e_i and b in (4.2), $N_{k_0}(e_i, 1)$ need not be (k_0, k_1)-isomorphic to $N_{k_1}(b, 1))$ so that for $i, j \in M$, $N_{k_0}(e_i, 1)$ need not be k_0-isomorphic to $N_{k_0}(e_j, 1)$.

(3) In (4.2), for distinct $i, j \in M$, $N_{k_0}(e_i, 1)$ is not k_0-adjacent to $N_{k_0}(e_j, 1)$.

The proof of this assertion is motivated by the proof of Proposition 2 of [14] regarding some properties of a local (k_0, k_1)-isomorphism which is stronger than a WL-(k_0, k_1)-isomorphism.

Proof. (1) Owing to the condition of the WL-(k_0, k_1)-isomorphic surjection of p in (4.2), it is clear that

$$\text{for } i, j \in M \text{ and } i \neq j, \; e_i \text{ is not } k_0\text{-adjacent to } e_j. \tag{4.3}$$

By contrary, suppose $e_i \in N_{k_0}(e_j, 1)$ and $e_i \neq e_j$. Then, by the hypothesis, note that $p|_{N_{k_0}(e_j,1)} : N_{k_0}(e_j, 1) \to p(N_{k_0}(e_j, 1))$ should be a (k_0, k_1)-isomorphism. From (4.2), since we have $p(e_i) = p(e_j) = b$, the map $p|_{N_{k_0}(e_j,1)}$ is not injective, which invokes a contradiction to the WL-(k_0, k_1)-isomorphism of p.

Next, in (4.2), we now prove that for any $i \neq j \in M$, the two sets $N_{k_0}(e_i, 1)$ and $N_{k_0}(e_j, 1)$ are disjoint. For the sake of a contradiction, for some $N_{k_0}(e_i, 1)$ and $N_{k_0}(e_j, 1)$, suppose

$$N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1) \neq \emptyset.$$

Then, take a certain point

$$e \in N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1). \tag{4.4}$$

As proved above, since e_i is not k_0-adjacent to e_j and $e_i \neq e_j$, we may take $e \notin \{e_i, e_j\}$. Owing to the property (4.4), it is clear that the element $e \in E$ is k_0-adjacent to both the points e_i and e_j. Naively, we obtain $e_i, e_j \in N_{k_0}(e, 1)$. Owing to the hypothesis of a *WL*-(k_0, k_1)-isomorphism of p at the point e and the property (4.2), the restriction p to $N_{k_0}(e, 1)$, i.e.,

$$p|_{N_{k_0}(e,1)} : N_{k_0}(e, 1) \to p(N_{k_0}(e, 1)) \tag{4.5}$$

should be a (k_0, k_1)-isomorphism. However, since $p(e_i) = p(e_j) = b \in p(N_{k_0}(e, 1))$, by the properties (4.2), the restriction map in (4.5) is not a (k_0, k_1)-isomorphism because it is not injective, which invokes a contradiction to the property (4.5).

(2) Note that a *WL*-(k_0, k_1)-isomorphic surjection need not support a (k_0, k_1)-isomorphism between $N_{k_0}(e_i, 1)$ and $N_{k_1}(b, 1)$ in (4.2). For instance, consider the map $p : (E, 8) \to SC_8^{2,8}$ in (4.1). Then take the point $0 \in SC_8^{2,8}$ and the set $N_8(0, 1) = \{1, 0, 7\}$. Then, take the set

$$\begin{Bmatrix} p^{-1}(N_8(0, 1)) = \cup_{i \in M} N_{k_0}(e_i, 1) \text{ with } e_i \in p^{-1}(0) \text{ as in (4.2),} \\ \text{where } M = \{8m \mid m \in \mathbb{N} \cup \{0\}\}. \end{Bmatrix}$$

Then, we obviously obtain that

$$p^{-1}(N_8(0, 1)) = \{e_0, e_1\} \cup \{e_7, e_8, e_9\} \cup \cdots \cup \{e_{8m-1}, e_{8m}, e_{8m+1}\} \cup \cdots .$$

Then it is clear that the set $\{e_0, e_1\}$ is not 8-isomorphic to $N_8(0, 1) = \{1, 0, 7\} \subset SC_8^{2,8}$.

(3) In (4.2), after recalling the fact $N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1) = \emptyset$ already proved in (1), by contrary, in (4.2), suppose that there are certain $i, j \in M$ with $i \neq j$ such that the set $N_{k_0}(e_i, 1)$ is k_0-adjacent to $N_{k_0}(e_j, 1)$. Then, owing to the facts already proved in (1) and (2), there are at least two distinct points $e, e' \in E$ such that

$$\begin{Bmatrix} e \in N_{k_0}(e_i, 1) \text{ and } e \neq e_i; \\ e' \in N_{k_0}(e_j, 1) \text{ and } e' \neq e_j; \text{ and} \\ e \text{ is } k_0\text{-adjacent to } e'. \end{Bmatrix}$$

Then, we have a simple k_0-path $E_1 := (e_i, e, e', e_j) \subset (E, k_0)$ such that $p(e_i) = p(e_j) = b \in (B, k_1)$. Let us now consider the sequence

$$(p(e_i), p(e), p(e'), p(e_j)) = (b, p(e), p(e'), b) \subset (B, k_1). \tag{4.6}$$

Regarding the sequence in (4.6), since $e' \in N_{k_0}(e, 1)$ and $e' \neq e$, owing to the hypothesis, the (k_0, k_1)-isomorphism

$$p|_{N_{k_0}(e,1)} : N_{k_0}(e, 1) \to p(N_{k_0}(e, 1))$$

is also considered. Hence we have $p(e) \neq p(e')$ and further, $p(e)$ is k_1-adjacent to $p(e')$.

Similarly, by (4.2), owing to the *WL*-(k_0, k_1)-isomorphism of p, we also obtain the following:

$$\begin{Bmatrix} p(e) \text{ is } k_1\text{-adjacent to } p(e_i); \text{and} \\ p(e') \text{ is } k_1\text{-adjacent to } p(e_j). \end{Bmatrix}.$$

Besides, it is clear that $p(E_1)$ is k_1-connected. Hence the sequence $(b, p(e), p(e'), b)$ is a k_1-cycle with three points. To be precise, since b is k_1-adjacent to both $p(e)$ and $p(e')$ and further, $p(e)$ is also k_1-adjacent to $p(e')$, the sequence $(b, p(e), p(e'), b)$ has a shape of a triangle with k_1-adjacency and it is a subset of $N_{k_1}(t, 1) \subset B$, where $t \in \{b, p(e), p(e')\} \subset (B, k_1)$. This invokes a contradiction to the hypothesis of a *WL*-(k_0, k_1)-isomorphic surjection of p at any point in (E, k_0) (see the set E_1 above). □

Owing to Definition 4.1 and Proposition 4.4, we obtain the following:

Corollary 4.5. *A WL-local* (k_0, k_1)*-isomorphic surjection is equivalent to a pseudo-*(k_0, k_1)*-covering map.*

Given a digital image (X, k), take a certain point $x_0 \in X$. Then, the pair (X, x_0) is called a pointed digital image with the given k-adjacency. We say that a k-path on (X, k), $f : [0, m]_{\mathbb{Z}} \to (X, k)$ begins at $x \in X$ if $f(0) = x$ [7]. If a (k_0, k_1)-continuous map $f : ((X, x_0), k_0) \to ((Y, y_0), k_1)$ satisfies $f(x_0) = y_0$, then we say that f is a pointed (k_0, k_1)-continuous map [8]. Since the notion of a digital lifting, the unique path lifting property [8] and the unique pseudo-path lifting property [11] play important roles in digital covering theory, let us recall them.

Definition 4.6. ([7, 8, 12]) (1) For digital images (E, k_1) in $\mathbb{Z}^{n_1}$, (B, k_2) in $\mathbb{Z}^{n_2}$, and (X, k_0) in $\mathbb{Z}^{n_0}$, let $p : (E, k_1) \to (B, k_2)$ be a (k_1, k_2)-continuous map and $f : (X, k_0) \to (B, k_2)$ be a (k_0, k_2)-continuous map. We say that a lifting of f (with respect to p) is a (k_0, k_1)-continuous map $\tilde{f} : (X, k_0) \to (E, k_1)$ such that $p \circ \tilde{f} = f$. In particular, in the case $f : [0, m]_{\mathbb{Z}} \to (B, k_2)$ be a $(2, k_2)$-continuous map, the lifting of f denoted by $\tilde{f} : [0, m]_{\mathbb{Z}} \to (X, k_1)$ is called a k_2-path lifting (with respect to p).

(2) In (1), the map p has the *unique path lifting property* if any two k_2-paths $f, g : [0, m]_{\mathbb{Z}} \to (B, k_2)$ are equal if $p \circ f = p \circ g$ and $f(0) = g(0)$.

Since a local (k_0, k_1)-isomorphism is equivalent to a digital (k_0, k_1)-covering map [14], using this fact, we can represent the unique path lifting property, as follows:

Theorem 4.7. ([8]) ([Unique path lifting property]) *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a local (k_0, k_1)-isomorphism such that $p(e_0) = b_0$. Then, any k_1-path $f : [0, m]_{\mathbf{Z}} \to B$ beginning at b_0 has a unique digital lifting to a k_0-path $\tilde{f}$ in E beginning at e_0.*

Using the most simplified version of a pseudo-(k_0, k_1)-covering map in Corollary 4.5, we can represent the pseudo-path lifting property in [11], as follows:

Theorem 4.8. (Simplified version of the pseudo-path lifting property) *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a WL-(k_0, k_1)-isomorphic surjection such that $p(e_0) = b_0$. Then, let $g : (Y, k) \to (B, k_1)$ be (k, k_1)-continuous map. If there are two (k, k_0)-continuous maps $f_0, f_1 : Y \to E$ both coinciding at one point $y_0 \in Y$ and satisfying $p \circ f_0 = p \circ f_1 = g$, then $f_0 = f_1$.*

In Theorem 4.8 and Corollary 4.9 below, all digital images are assumed to be digitally connected depending on the given digital connectivity.

Corollary 4.9. *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a WL-(k_0, k_1)-isomorphic surjection such that $p(e_0) = b_0$. Then, let $g : [0, m]_{\mathbb{Z}} \to (B, k_1)$ be $(2, k_1)$-continuous map. If there are two $(2, k_0)$-continuous maps $f_0, f_1 : [0, m]_{\mathbb{Z}} \to E$ both coinciding at one point $y_0 \in [0, m]_{\mathbb{Z}}$ and satisfying $p \circ f_0 = p \circ f_1 = g$, then $f_0 = f_1$.*

5. Summary and conclusions

After comparing among a *PL*-(k_0, k_1)-, a *WL*-(k_0, k_1)-, and a local (k_0, k_1)-isomorphism, we have proposed an equivalent and the most simplified version of a pseudo-(k_0, k_1)-covering map. As a further work, we can study some properties of a pseudo-(k_0, k_1)-covering map related to digital homotopic properties. Based on the obtained topological space in [15, 16], we can further study some covering spaces of the given spaces.

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