



Nonlinear Sequential Fractional Boundary Value Problems Involving Generalized Ψ -Caputo Fractional Derivatives

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Abstract. This paper is devoted to study the nonlinear sequential fractional boundary value problems involving generalized ψ -Caputo fractional derivatives with nonlocal boundary conditions. We investigate the Green function and some of its properties, from which we obtain a new Lyapunov-type inequality for our problem. A lower bound for the possible eigenvalues of our problem is derived. Furthermore, we apply some properties of the Green function to obtain some existence results for our problem. It is worth mentioning that our results still work with some source functions including singularities.

1. Introduction

1.1. Statement of the problem

Let $a < b$, $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let ψ be an appropriate function. We consider the sequential fractional boundary value problem involving generalized ψ -Caputo fractional derivatives

$$\left({}^C D_{a+}^{\alpha, \psi} {}^C D_{a+}^{\beta, \psi} x\right)(t) + f(t, x(t)) = 0, \quad a < t < b \quad (1)$$

subject to the generalized nonlocal conditions

$$x(a) = 0, \quad x(b) = \mathcal{G}(x), \quad (2)$$

where ${}^C D_{a+}^{\alpha, \psi}$, ${}^C D_{a+}^{\beta, \psi}$ are the left ψ -Caputo fractional derivatives. Our study focuses on discussing the following aspects: the Green function and some of its properties, the Lyapunov-type inequality, and some existence results for our problem.

1.2. Relevant works and motivations

Fractional calculus is a generalization of ordinary differentiation and integration, in which derivatives and integrals of arbitrary real or complex order are defined. This field has become a rapidly growing area and has found applications in diverse research fields such as in physics [12], fractional dynamics [18], quantum mechanics [13], bioengineering [17]. Particularly, one can gain a comprehensive view of applications of the fractional derivatives in physics from the excellent monograph [19] and the references

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therein. Definition of fractional derivatives are very rich such as Caputo, Caputo-Katugampola, Riemann-Liouville, Hadamard, etc. The concept of fractional derivatives of a function with respect another function was introduced in [15]. Based on this ideas, Almeida [2] presented the ψ -Caputo fractional derivatives which is generalized from the mentioned fractional derivatives.

Sequential fractional differential equations were firstly introduced in the literature in the book by Miller and Ross [16], in which the authors defined compositions of Riemann-Liouville fractional derivatives $\prod_{i=1}^n D^\alpha = \underbrace{D^\alpha D^\alpha \dots D^\alpha}_{n \text{ times}}$, where $n \in \mathbb{N}$ and α is a positive real number. They also investigated some linear differential equations involving these compositions.

In the past few years, there are numerous works on the sequential fractional boundary value problems such as [1, 3–9, 11, 14, 20, 21] and the references therein. In fact, Ferreira [8] and Zhang et al [21] derived some Lyapunov-type inequalities for linear sequential fractional boundary value problems involving Caputo fractional derivatives and Hilfer fractional derivatives, respectively. Fazli and Nieto [7] were obtained some existence and uniqueness results for the nonlinear initial value problems with Riemann-Liouville fractional derivatives $D^{2\alpha} = D^\alpha D^\alpha$. Kassymov and Torebek [14] obtained a Lyapunov-type and a Hartman-Wintner-type inequalities for a nonlinear fractional hybrid equation with left Riemann-Liouville and right Caputo fractional derivatives. Ferreira [9] also obtained a Lyapunov-type inequality and a existence result for a nonlinear fractional derivatives with Riemann-Liouville and the Caputo fractional derivatives. Very recently, in [4–6], we investigated the existence and continuity results for the fractional Langevin problems involving ψ -Caputo fractional derivatives with weakly singular sources.

To the best of our knowledge, Lyapunov-type inequalities for nonlinear sequential fractional differential equations involving ψ -Caputo fractional derivatives associated with generalized nonlocal conditions are still not considered. Furthermore, the existence results for the problem including some singularities in the source function are under consideration. Motivated by the above discussions, in this paper, we consider the sequential fractional boundary value problem (1) and (2). In our work, the main new features are that:

- We establish the Green function and investigate some of its properties. It is worth noting that the Green function in the present paper is different from the Green function in recent papers.
- We derive a new Lyapunov-type inequality for our problem.
- We obtain some existence results for our problem. Unlike previous papers, our obtained results still work with some source functions including singularities.

1.3. Outline

The present paper is organized as follows. In section 2, we introduce the reader to some basic concepts of fractional integral and ψ -Caputo fractional derivative. Some well-known results are also recalled before proceeding to the main results. In section 3, we establish the Green function and some of its properties. In section 4, we present a new Lyapunov-type inequality and some existence results for our problem.

2. Preliminaries

We begin by setting some notations. For convenience, we denote the class of increasing and differentiable functions on $[a, b]$ by $C_+^1[a, b]$, i.e.,

$$C_+^1[a, b] = \{\psi \in C^1[a, b] : \psi'(t) > 0 \text{ for all } t \in [a, b]\}.$$

For z belong to $C([a, b], \mathbb{R})$, we define by $\|z\| = \sup_{a \leq t \leq b} |z(t)|$. We also denote the set of non-negative real numbers by \mathbb{R}_+ .

To present the concept of ψ -Caputo fractional derivative, we start with the definition of fractional integral of a function with respect to another function.

Definition 2.1 (see [2, 15]). For $\alpha > 0$, $\psi \in C^1_+[a, b]$, and $x \in L^1[a, b]$, the left fractional integral of a function x depending another function ψ is given by

$$I^{\alpha, \psi}_{a+} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} x(\tau) \, d\tau,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2 (see [2]). For $n - 1 < \alpha \leq n$, and $x, \psi \in C^n[a, b]$ with $\psi'(t) > 0$ for all $t \in [a, b]$, the left-side ψ -Caputo fractional derivative of x of order α is defined by

$${}^C D^{\alpha, \psi}_{a+} x(t) = I^{n-\alpha, \psi}_{a+} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n x(t).$$

For complete surveys of basic properties of the fractional operators $I^{\alpha, \psi}_{a+}$ and ${}^C D^{\psi, \alpha}_{a+}$, we refer to [2, 15]. In this paper, we will use only the following properties.

Lemma 2.3 (see [2, 15]). We have

- (i). For $x, y \in C^n[a, b]$ then ${}^C D^{\alpha, \psi}_{a+} x(t) = {}^C D^{\alpha, \psi}_{a+} y(t)$ if and only if $x(t) = y(t) + \sum_{k=0}^{n-1} c_k(\psi(t) - \psi(a))^k$.
- (ii). For $x \in C^1[a, b]$, $\alpha > 0$ then ${}^C D^{\alpha, \psi}_{a+} I^{\alpha, \psi}_{a+} x(t) = x(t)$.
- (iii). For $\alpha, \beta > 0$, we have $I^{\alpha, \psi}_{a+} I^{\beta, \psi}_{a+} x(t) = I^{\alpha+\beta, \psi}_{a+} x(t)$.

At the end of this section, we present two well-known results that we will use to prove the main results of our paper.

Lemma 2.4 (Jensen’s inequality). Let μ be a positive measure and let Ω be a measurable set with $\mu(\Omega) = 1$. If x is a real function in $L^1(\mu)$, if $a < x(t) < b$ for all $t \in \Omega$, and if f is a convex on (a, b) , then

$$f\left(\int_{\Omega} x \, d\mu\right) \leq \int_{\Omega} (f \circ x) \, d\mu. \tag{3}$$

If f is concave on (a, b) , then the inequality (3) holds with \leq reversed.

Lemma 2.5 (The nonlinear Leray-Schauder alternatives fixed point theorem [10]). Let \mathbb{B} be a Banach space, and let W be a closed convex subset of \mathbb{B} . Let V be a relatively open subset of W and $0 \in V$. Suppose that $Q : \bar{V} \rightarrow W$ is a continuous compact mapping. Then we have either

- (i). Q has a fixed point in \bar{V}
- or
- (ii). There exist $\lambda \in (0, 1)$ and $u \in \partial V$ such that $u = \lambda Qu$.

3. Green function and some of its properties

In this section, we establish the Green function for our problem and some of its properties. We start with constructing the Green function for our problem.

Lemma 3.1. Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let $\psi \in C^1_+[a, b]$. If x is a solution of the problem (1) and (2) such that ${}^C D^{\beta, \psi}_{a+} x, f(\cdot, x(\cdot)) \in C^1[a, b]$, then x is a solution of the following non-local integral equation

$$x(t) = \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \mathcal{G}(x) + \int_a^b G(\tau, t) \psi'(\tau) f(\tau, x(\tau)) \, d\tau \tag{4}$$

where

$$G(\tau, t) = \begin{cases} g_1(\tau, t), & a \leq \tau \leq t \leq b, \\ g_2(\tau, t), & a \leq t \leq \tau \leq b, \end{cases} \tag{5}$$

where

$$g_1(\tau, t) = C(\psi(t) - \psi(a))^\beta(\psi(b) - \psi(\tau))^{\alpha+\beta-1} - D(\psi(t) - \psi(\tau))^{\alpha+\beta-1},$$

$$g_2(\tau, t) = C(\psi(t) - \psi(a))^\beta(\psi(b) - \psi(\tau))^{\alpha+\beta-1},$$

and $C = 1 / [\Gamma(\alpha + \beta)(\psi(b) - \psi(a))^\beta]$, $D = 1/\Gamma(\alpha + \beta)$.

Proof. From the assumption $f(\cdot, x(\cdot)) \in C^1[a, b]$, we obtain from Lemma 2.3 that

$${}^C D_{a+}^{\alpha, \psi} I_{a+}^{\alpha, \psi} f(t, x(t)) = f(t, x(t)).$$

This leads to Eq. (1) equivalent to the following equation

$$({}^C D_{a+}^{\alpha, \psi} {}^C D_{a+}^{\beta, \psi} x)(t) = -{}^C D_{a+}^{\alpha, \psi} I_{a+}^{\alpha, \psi} f(t, x(t)).$$

By virtue of Lemma 2.3, one has

$$x(t) = c_2 + (I_{a+}^{\beta, \psi} c_1)(t) - I_{a+}^{\alpha+\beta, \psi} f(t, x(t)) = c_2 + \frac{c_1}{\Gamma(\beta + 1)}(\psi(t) - \psi(a))^\beta + I_{a+}^{\alpha+\beta, \psi} f(t, x(t)). \tag{6}$$

Here we have used the fact that $(I_{a+}^{\beta, \psi} c_1)(t) = (c_1/\Gamma(\beta + 1))(\psi(t) - \psi(a))^\beta$.

We now find the coefficients c_1, c_2 . Since $u(a) = 0$, we find out $c_2 = 0$. Continuously, we have

$$x(b) = \frac{c_1}{\Gamma(\beta + 1)}(\psi(b) - \psi(a))^\beta - \frac{1}{\Gamma(\alpha + \beta)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha+\beta-1} f(\tau, x(\tau)) \, d\tau.$$

Using the nonlocal condition $x(b) = \mathcal{G}(x)$, we get

$$c_1 = \frac{\Gamma(\beta + 1)}{(\psi(b) - \psi(a))^\beta} \left(\mathcal{G}(x) + \frac{1}{\Gamma(\alpha + \beta)} \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha+\beta-1} f(\tau, x(\tau)) \, d\tau \right).$$

Substituting the obtained coefficients c_1 and c_2 into (6), we obtain

$$\begin{aligned} x(t) &= \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \mathcal{G}(x) + \frac{1}{\Gamma(\alpha + \beta)} \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{\alpha+\beta-1} f(\tau, x(\tau)) \, d\tau \\ &\quad - \frac{1}{\Gamma(\alpha + \beta)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha+\beta-1} f(\tau, x(\tau)) \, d\tau \\ &= \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \mathcal{G}(x) + \int_a^b G(\tau, t) \psi'(\tau) f(\tau, x(\tau)) \, d\tau. \end{aligned}$$

The proof of Lemma is completed. \square

Definition 3.2. The function $G(\cdot, \cdot)$ given by (5) is called the Green function of the problem (1) and (2).

Proposition 3.3. Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let $\psi \in C_+^1[a, b]$ and the Green function G be defined as in Lemma 3.1. Then

$$\max_{a \leq \tau, t \leq b} |G(\tau, t)| = \frac{1}{\Gamma(\alpha + \beta)} (\psi(b) - \psi(a))^{\alpha+\beta-1} \max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha+\beta-1}, \frac{\beta(\alpha + \beta - 1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha + 2\beta - 1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\}.$$

Proof. We consider the functions g_1 and g_2 which are defined in Lemma 3.1. We divide the proof into two steps.

Step 1. We show that

$$\max_{a \leq t \leq \tau \leq b} g_2(\tau, t) = g(\tau_0) = \frac{1}{\Gamma(\alpha + \beta)} \beta^\beta (\alpha + \beta - 1)^{\alpha + \beta - 1} (\psi(b) - \psi(a))^{\alpha + \beta - 1}.$$

Since ψ is an increasing function, hence, we have $0 \leq \psi(t) - \psi(a) \leq \psi(\tau) - \psi(a)$ for all $a \leq t \leq \tau \leq b$. This leads to

$$0 \leq g_2(\tau, t) \leq C(\psi(\tau) - \psi(a))^\beta (\psi(b) - \psi(\tau))^{\alpha + \beta - 1} := g(\tau) \tag{7}$$

for any $a \leq t \leq \tau \leq b$. By direct computations, we have

$$g'(\tau) = C(\psi(\tau) - \psi(a))^{\beta - 1} (\psi(b) - \psi(\tau))^{\alpha + \beta - 2} (\beta(\psi(b) - \psi(\tau)) - (\alpha + \beta - 1)(\psi(\tau) - \psi(a))),$$

where C defined in Lemma 3.1. Since $\beta - 1 \leq 0$ and $\alpha + \beta - 2 \leq 0$, we can find that $g'(\tau) = 0$ at $\tau_0 = \psi^{-1}\left(\frac{\beta\psi(b) + (\alpha + \beta - 1)\psi(a)}{\alpha + 2\beta - 1}\right)$. Observe that $\psi(a) < \frac{\beta\psi(b) + (\alpha + \beta - 1)\psi(a)}{\alpha + 2\beta - 1} < \psi(b)$, this implies that $\tau_0 \in (a, b)$ and $\max_{a \leq \tau \leq b} g(\tau) = g(\tau_0) = C\beta^\beta (\alpha + \beta - 1)^{\alpha + \beta - 1} (\psi(b) - \psi(a))^{\alpha + 2\beta - 1}$. This leads to the desired result of *Step 1*.

Step 2. We prove that

$$\max_{a \leq \tau \leq t \leq b} |g_1(\tau, t)| = \frac{1}{\Gamma(\alpha + \beta)} (\psi(b) - \psi(a))^{\alpha + \beta - 1} \max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha + \beta - 1}, \frac{\beta(\alpha + \beta - 1)^{\frac{\alpha + \beta - 1}{\beta}}}{(\alpha + 2\beta - 1)^{\frac{\alpha + 2\beta - 1}{\beta}}} \right\}.$$

We firstly verify that $g_1(\tau, t)$ is an increasing function with respect to τ on $[a, t]$. Indeed, using the fact that $C/D = (\psi(b) - \psi(a))^\beta$ and the following estimate

$$\left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta (\psi(b) - \psi(\tau))^{\alpha + \beta - 2} - (\psi(t) - \psi(\tau))^{\alpha + \beta - 2} \leq (\psi(b) - \psi(\tau))^{\alpha + \beta - 2} - (\psi(t) - \psi(\tau))^{\alpha + \beta - 2} \leq 0,$$

we get

$$\begin{aligned} \frac{\partial g_1}{\partial \tau}(\tau, t) &= -(\alpha + \beta - 1)\psi'(\tau) \left[C(\psi(t) - \psi(a))^\beta (\psi(b) - \psi(\tau))^{\alpha + \beta - 2} - D(\psi(t) - \psi(\tau))^{\alpha + \beta - 2} \right] \\ &= -D(\alpha + \beta - 1)\psi'(\tau) \left[\left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta (\psi(b) - \psi(\tau))^{\alpha + \beta - 2} - (\psi(t) - \psi(\tau))^{\alpha + \beta - 2} \right] \\ &\geq 0 \text{ for all } \tau \in [a, t]. \end{aligned}$$

This implies that

$$\max_{a \leq \tau \leq t} |g_1(\tau, t)| = \max\{|g_1(a, t)|, |g_1(t, t)|\}.$$

Thus,

$$\max_{a \leq \tau \leq t \leq b} |g_1(\tau, t)| = \max\{\max_{a \leq t \leq b} |g_1(a, t)|, \max_{a \leq t \leq b} |g_1(t, t)|\}. \tag{8}$$

Note that $g_1(t, t) = g(t)$ with g defined in (7), this gives $g_1(t, t) \geq 0$ and

$$\max_{a \leq t \leq b} |g_1(t, t)| = \frac{1}{\Gamma(\alpha + \beta)} \beta^\beta (\alpha + \beta - 1)^{\alpha + \beta - 1} (\psi(b) - \psi(a))^{\alpha + \beta - 1}. \tag{9}$$

We now consider the function $h(t) := g_1(a, t)$. By directly computes, one has

$$h'(t) = \frac{(\psi(t) - \psi(a))^{\alpha+\beta-2}}{\Gamma(\alpha + \beta)} \left((\alpha + 2\beta - 1) \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta - (\alpha + \beta - 1) \right).$$

We deduce $h'(t) = 0$ at t_0 such that $\psi(t_0) = \psi(a) + \left(\frac{\alpha+\beta-1}{\alpha+2\beta-1}\right)^{1/\beta} (\psi(b) - \psi(a)) := \psi_0$, or $t_0 = \psi^{-1}(\psi_0)$. Since $0 < \left(\frac{\alpha+\beta-1}{\alpha+2\beta-1}\right)^{1/\beta} < 1$, then we have $\psi(a) < \psi_0 = \psi(a) + \left(\frac{\alpha+\beta-1}{\alpha+2\beta-1}\right)^{1/\beta} (\psi(b) - \psi(a)) < \psi(b)$. This gives $a < t_0 < b$. On the other hand, we have $h(a) = 0$ and

$$h(t) = \frac{1}{\Gamma(\alpha + \beta)} (\psi(t) - \psi(a))^{\alpha+\beta-1} \left(\left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta - 1 \right) \leq 0$$

for all $t \in [a, b]$. From the above discussions, we conclude that

$$\max_{a \leq t \leq b} |h(t)| = |h(t_0)| = \frac{\beta}{(\alpha + 2\beta - 1)\Gamma(\alpha + \beta)} \left(\frac{\alpha + \beta - 1}{\alpha + 2\beta - 1} \right)^{\frac{\alpha+\beta-1}{\beta}} (\psi(b) - \psi(a))^{\alpha+\beta-1}. \tag{10}$$

Combining (9) and (10) together with (8), we obtain the result of Step 2.

Finally, we can combine Step 1 and Step 2 to obtain the desired result of Proposition. \square

Proposition 3.4. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let $\psi \in C_+^1[a, b]$ and the Green function G be defined as in Lemma 3.1. Then, for any $a \leq t_1 \leq t_2 \leq b$, we have*

$$|G(\tau, t_2) - G(\tau, t_1)| \leq \frac{2}{\Gamma(\alpha + \beta)} (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1}.$$

Proof. In the process of the proof of Proposition, we use the fact that

$$|v^p - w^p| \leq |v - w|^p \tag{11}$$

for any $v, w \geq 0$ and $0 \leq p \leq 1$.

We will consider three cases.

The first case: $a \leq t_1 \leq t_2 \leq \tau \leq b$. We firstly have $(\psi(t_2) - \psi(t_1))^\beta = (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1} (\psi(t_2) - \psi(t_1))^{1-\alpha} \leq (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1} (\psi(b) - \psi(a))^{1-\alpha}$. Hence, we obtain from (11) that

$$\begin{aligned} |G(\tau, t_2) - G(\tau, t_1)| &\leq C(\psi(b) - \psi(\tau))^{\alpha+\beta-1} \left| (\psi(t_2) - \psi(a))^\beta - (\psi(t_1) - \psi(a))^\beta \right| \\ &\leq C(\psi(b) - \psi(a))^{\alpha+\beta-1} (\psi(t_2) - \psi(t_1))^\beta \\ &\leq C(\psi(b) - \psi(a))^\beta (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1} \\ &= \frac{1}{\Gamma(\alpha + \beta)} (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1} \end{aligned}$$

due to $C(\psi(b) - \psi(a))^\beta = 1/\Gamma(\alpha + \beta)$.

The second case: $a \leq \tau \leq t_1 \leq t_2 \leq b$. Similar to the first case, we have

$$\begin{aligned} |G(\tau, t_2) - G(\tau, t_1)| &\leq C(\psi(b) - \psi(\tau))^{\alpha+\beta-1} \left| (\psi(t_2) - \psi(a))^\beta - (\psi(t_1) - \psi(a))^\beta \right| \\ &\quad + D \left| (\psi(t_2) - \psi(\tau))^{\alpha+\beta-1} - (\psi(t_1) - \psi(\tau))^{\alpha+\beta-1} \right| \\ &\leq \frac{2}{\Gamma(\alpha + \beta)} (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1} \end{aligned}$$

due to $D = 1/\Gamma(\alpha + \beta)$.

The third case: $a \leq t_1 \leq \tau \leq t_2 \leq b$. Using the same methods to that have used to prove the first case, we have

$$\begin{aligned} |G(\tau, t_2) - G(\tau, t_1)| &\leq C(\psi(b) - \psi(\tau))^{\alpha+\beta-1} \left| (\psi(t_2) - \psi(a))^\beta - (\psi(t_1) - \psi(a))^\beta \right| + D(\psi(t_2) - \psi(\tau))^{\alpha+\beta-1} \\ &\leq \frac{2}{\Gamma(\alpha + \beta)} (\psi(t_2) - \psi(t_1))^{\alpha+\beta-1} \end{aligned}$$

due to $\psi(t_2) - \psi(\tau) \leq \psi(t_2) - \psi(t_1)$. Combining three cases above, we obtain the desired result of Proposition. \square

4. Lyapunov-type inequality and existence results

In this section, we investigate a new Lyapunov-type inequality and obtain a lower bound for the possible eigenvalues of our problem. We also present some existence results for our problem.

To obtain Lyapunov-type inequality, the following assumptions will be posed.

- **Assumption (A1):** There exist a positive function $q : (a, b) \rightarrow \mathbb{R}_+$ and a positive, non-decreasing and concave function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(t, x)| \leq q(t)|\phi(x)|$$

for any $t \in (a, b)$ and $x \in \mathbb{R}$.

- **Assumption (A2):** There exists $\kappa \in (0, 1)$ such that

$$|\mathcal{G}(x)| \leq \kappa|x|$$

for any $x \in \mathbb{R}$.

In this section, we also use the concept of mild solutions as follows.

Definition 4.1. The function $x \in C[a, b]$ satisfying the Eq. (4) is called mild solution of the problem (1) and (2).

We now present a Lyapunov-type inequality for our problem.

Theorem 4.2. Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let $\psi \in C_+^1[a, b]$. Suppose that Assumptions (A1) and (A2) are satisfied. If $\psi'(\cdot)q(\cdot) \in L^1(a, b)$ and the problem (1)-(2) has a nontrivial mild solution, then

$$\int_a^b \psi'(\tau)q(\tau) \, d\tau \geq \frac{1 - \kappa}{G_{\max}} \frac{\|x\|}{\phi(\|x\|)},$$

where

$$G_{\max} = \frac{1}{\Gamma(\alpha + \beta)} (\psi(b) - \psi(a))^{\alpha+\beta-1} \max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha+\beta-1}, \frac{\beta(\alpha + \beta - 1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha + 2\beta - 1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\}.$$

Remark 4.3. Lyapunov-type inequality obtained in Theorem 4.2 seems to be new, which is still not proposed in previous papers.

Proof. Using the fact that $\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \leq 1$ for all $t \in [a, b]$, Lemma 2.4 and Lemma 3.3, we obtain from (4) that

$$\begin{aligned} |x(t)| &\leq |\mathcal{G}(x)| + G_{\max} \int_a^b \psi'(\tau) |f(\tau, x(\tau))| \, d\tau \\ &\leq \kappa \|x\| + G_{\max} \int_a^b \psi'(\tau) q(\tau) |\phi(x(\tau))| \, d\tau \\ &\leq \kappa \|x\| + G_{\max} \|\psi'(\cdot)q(\cdot)\|_{L^1(a,b)} \phi \left(\int_a^b \frac{|\psi'(\tau)q(\tau)|}{\|\psi'(\cdot)q(\cdot)\|_{L^1(a,b)}} |x(\tau)| \, d\tau \right) \\ &\leq \kappa \|x\| + G_{\max} \|\psi'(\cdot)q(\cdot)\|_{L^1(a,b)} \phi(\|x\|), \end{aligned}$$

where

$$G_{\max} = \frac{1}{\Gamma(\alpha + \beta)} (\psi(b) - \psi(a))^{\alpha+\beta-1} \max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha+\beta-1}, \frac{\beta(\alpha + \beta - 1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha + 2\beta - 1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\}.$$

From the latter inequality, we obtain

$$\int_a^b \psi'(\tau)q(\tau) \, d\tau \geq \frac{1 - \kappa}{G_{\max}} \frac{\|x\|}{\phi(\|x\|)}.$$

This completes the proof of Theorem. \square

From the result of Theorem 4.2, we immediately obtain Lyapunov-type inequalities for sequential fractional boundary value problems involving Caputo and Hadamard fractional derivatives.

Corollary 4.4. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$. Suppose that Assumption $(\mathcal{A}2)$ holds. Suppose further that there exists $q : (a, b) \rightarrow \mathbb{R}$ such that*

$$f(t, x) = q(t)x$$

for all $t \in (a, b)$. For $\psi(t) = t$, if $q(\cdot) \in L^1(a, b)$ and the problem (1) and (2) has a non-trivial mild solution, then

$$\int_a^b |q(\tau)| \, d\tau \geq \frac{1 - \kappa}{G_1},$$

where

$$G_1 = \frac{1}{\Gamma(\alpha + \beta)} (b - a)^{\alpha+\beta-1} \max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha+\beta-1}, \frac{\beta(\alpha + \beta - 1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha + 2\beta - 1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\}.$$

Corollary 4.5. *Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$. Suppose that Assumption $(\mathcal{A}2)$ holds. Suppose further that there exists $q : (a, b) \rightarrow \mathbb{R}$ ($a > 0$) such that*

$$f(t, x) = q(t)x$$

for all $t \in (a, b)$. For $\psi(t) = \ln t$, if

$$\int_a^b \frac{|q(\tau)|}{\tau} \, d\tau < +\infty$$

and the problem (1) and (2) has a non-trivial mild solution, then

$$\int_a^b \frac{|q(\tau)|}{\tau} \, d\tau \geq \frac{1 - \kappa}{G_2},$$

where

$$G_2 = \frac{1}{\Gamma(\alpha + \beta)} \left(\ln \frac{b}{a} \right)^{\alpha+\beta-1} \max \left\{ \beta^\beta (\alpha + \beta - 1)^{\alpha+\beta-1}, \frac{\beta(\alpha + \beta - 1)^{\frac{\alpha+\beta-1}{\beta}}}{(\alpha + 2\beta - 1)^{\frac{\alpha+2\beta-1}{\beta}}} \right\}.$$

We can also use the obtained Lyapunov-type inequality to derive a lower bound for possible eigenvalues of the sequential fractional value problem. In fact, we have the following result.

Corollary 4.6. Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and $\psi \in C_+^1[a, b]$. Suppose that Assumption (A2) holds. Suppose further that λ is an eigenvalue of the following problem

$$\begin{cases} ({}^C D_{a+}^{\alpha, \psi} {}^C D_{a+}^{\beta, \psi} x)(t) = \lambda x(t), & a < t < b, \\ x(a) = 0, \quad x(b) = G(x). \end{cases}$$

Then

$$|\lambda| \geq \frac{1 - \kappa}{G_{\max}(\psi(b) - \psi(a))},$$

where G_{\max} defined in Proposition 3.3.

Next, we present some existence results for our problem. To this aim, we make the following assumptions.

- **Assumption (A3):** There exist positive functions $\kappa_1, \kappa_2 : (a, b) \rightarrow \mathbb{R}_+$ and a positive non-decreasing function $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\begin{aligned} |f(t, x)| &\leq \kappa_1(t)\Psi(|x|) \text{ for any } t \in (a, b), x \in \mathbb{R}, \\ |f(t, x) - f(t, y)| &\leq \kappa_2(t)|\Phi(x, y)| \text{ for any } t \in (a, b), x, y \in \mathbb{R}, \end{aligned}$$

where $\Phi \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Phi(x, y) \rightarrow 0$ as $|x - y| \rightarrow 0$.

- **Assumption (A4):** There exists a positive function $\kappa_3 : (a, b) \rightarrow \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \leq \kappa_3(t)|x - y| \text{ for any } t \in (a, b), x, y \in \mathbb{R}$$

- **Assumption (A5):** There exists $\kappa_4 \in (0, 1)$ such that

$$|\mathcal{G}(x) - \mathcal{G}(y)| \leq \kappa_4|x - y| \text{ for all } x, y \in \mathbb{R}.$$

In the following theorem, we introduce an existence result for our problem.

Theorem 4.7. Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let $\psi \in C_+^1[a, b]$. Suppose that Assumptions (A2) and (A3) hold. Suppose further that $\mathcal{G} \in C(\mathbb{R}, \mathbb{R})$ and $\psi'(\cdot)\kappa_i(\cdot) \in L^1(a, b)$ for $i = 1, 2$. If there exists $M > 0$ such that

$$M > \frac{1}{1 - \kappa} \Psi(M)G_{\max} \|\psi'(\cdot)\kappa_1(\cdot)\|_{L^1(a, b)}$$

with G_{\max} defined in Theorem 4.2, then the problem (1) and (2) has at least one mild solution.

Proof. Let us define the operator $S : C[a, b] \rightarrow C[a, b]$ by

$$Sx(t) = \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \mathcal{G}(x) + \int_a^b G(\tau, t)\psi'(\tau)f(\tau, x(\tau)) \, d\tau, \tag{12}$$

where G defined in Lemma 3.1. We also denote

$$B_r = \{x \in C[a, b] : \|x\| \leq r\} \text{ for some } r > 0.$$

We firstly show that S is completely continuous operator. We divide the process of proof into three steps.

Step 1. We verify that S is continuous operator. Assume that $x, y \in B_r$ for some $r > 0$, we obtain from Proposition 3.3 and Assumption $(\mathcal{A}3)$ that

$$\begin{aligned} \|Sx - Sy\| &\leq \sup_{a \leq t \leq b} \left| \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta (\mathcal{G}(x) - \mathcal{G}(y)) \right| + \sup_{a \leq t \leq b} \left| \int_a^b G(\tau, t) \psi'(\tau) (f(\tau, x(\tau)) - f(\tau, y(\tau))) \, d\tau \right| \\ &\leq |\mathcal{G}(x) - \mathcal{G}(y)| + G_{\max} \int_a^b \psi'(\tau) \kappa_2(\tau) |\Phi(x(\tau), y(\tau))| \, d\tau \\ &\leq |\mathcal{G}(x) - \mathcal{G}(y)| + G_{\max} \|\Phi(x(\cdot), y(\cdot))\| \|\psi'(\cdot) \kappa_2(\cdot)\|_{L^1(a,b)} \\ &\rightarrow 0 \text{ as } x \rightarrow y \end{aligned}$$

due to $\mathcal{G} \in C(\mathbb{R}, \mathbb{R})$ and $\|\Phi(x(\cdot), y(\cdot))\| \rightarrow 0$ as $x \rightarrow y$ in $C[a, b]$. This completes the proof of Step 1.

Step 2. We prove that S maps bound sets into bound sets in $C[a, b]$. Suppose that $x \in B_r$ for some $r > 0$. By virtue of Proposition 3.3 and Assumption $(\mathcal{A}3)$, one has

$$\begin{aligned} \|Sx\| &\leq \sup_{a \leq t \leq b} \left| \left(\frac{\psi(t) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \mathcal{G}(x) + \int_a^b G(\tau, t) \psi'(\tau) f(\tau, x(\tau)) \, d\tau \right| \\ &\leq |\mathcal{G}(x)| + G_{\max} \int_a^b \psi'(\tau) \kappa_2(\tau) |\Psi(x(\tau))| \, d\tau \\ &\leq \kappa r + \Psi(r) \|\psi'(\cdot) \kappa_1(\cdot)\|_{L^1(a,b)}. \end{aligned}$$

The proof of Step 2 is completed.

Step 3. We verify that S maps bound sets into equicontinuous sets of $C[a, b]$. We can assume that $x \in B_r$ for some $r > 0$. For $a \leq t_1 \leq t_2 \leq b$, using (11), Lemma 3.4 and Assumption $(\mathcal{A}3)$, we have

$$\begin{aligned} |Sx(t_2) - Sx(t_1)| &\leq \left| \left(\frac{\psi(t_2) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta - \left(\frac{\psi(t_1) - \psi(a)}{\psi(b) - \psi(a)} \right)^\beta \right| |\mathcal{G}(x)| + \left| \int_a^b |G(\tau, t_2) - G(\tau, t_1)| \psi'(\tau) \kappa_1(\tau) |\Psi(x(\tau))| \, d\tau \right| \\ &\leq \kappa r \left(\frac{\psi(t_2) - \psi(t_1)}{\psi(b) - \psi(a)} \right)^\beta + \frac{2}{\Gamma(\alpha + \beta)} (\psi(t_2) - \psi(t_1))^{\alpha + \beta - 1} \Psi(r) \|\psi'(\cdot) \kappa_1(\cdot)\|_{L^1(a,b)} \\ &\rightarrow 0 \text{ uniformly as } t_2 \rightarrow t_1. \end{aligned}$$

The proof of Step 3 is completed.

We now prove the result of this theorem. Using Assumption $(\mathcal{A}3)$, we have $|f(t, x(t))| \leq \Psi(M) \kappa_2(t)$ for any $x \in B_M$ and $t \in (a, b)$. By the same method to that used to prove Step 2, we get

$$\|Sx\| \leq \kappa M + \Psi(M) G_{\max} \|\psi'(\cdot) \kappa_1(\cdot)\|_{L^1(a,b)},$$

where G_{\max} defined in Theorem 4.2. If there exist $k \in (0, 1)$ and $x \in \partial B_M$ such that $x = kSx$, then from the latter inequality, we get

$$M \leq \kappa M + \Psi(M) G_{\max} \|\psi'(\cdot) \kappa_1(\cdot)\|_{L^1(a,b)}.$$

Or,

$$M \leq \frac{1}{1 - \kappa} \Psi(M) G_{\max} \|\psi'(\cdot) \kappa_1(\cdot)\|_{L^1(a,b)}.$$

We obtain a contradiction with the hypothesis of Theorem. Hence, using Lemma 2.5, we deduce that S has a fixed point in B_M , which is a mild solution of our problem. The proof of Theorem is done. \square

Remark 4.8. We emphasize that the obtained result in Theorem 4.7 works with some source functions including singularities. For example, in Assumption (A3), we consider the cases: $\kappa_i(t) = k_i(\psi(b) - \psi(t))^{-\gamma_i}(\psi(t) - \psi(a))^{-\mu_i}$ for some $k_i > 0$ ($i = 1, 2$). If $\gamma_i, \mu_i < 1$ then we have $\psi'(\cdot)\kappa_i(\cdot) \in L^1(a, b)$ for $i = 1, 2$. Moreover, if k_1 sufficiently close to zero, then we can find $M > 0$ such that

$$M > \frac{1}{1 - \kappa} \Psi(M) G_{\max} \|\psi'(\cdot)\kappa_1(\cdot)\|_{L^1(a,b)}.$$

To verify the above statements, we may use the fact that

$$\int_a^b \psi'(\tau)(\psi(b) - \psi(\tau))^{-\gamma}(\psi(\tau) - \psi(a))^{-\mu} d\tau = (\psi(b) - \psi(a))^{1-\gamma-\mu} B(1 - \gamma, 1 - \mu),$$

where $\gamma, \mu < 1$ and $B(\cdot, \cdot)$ is the Beta function.

Finally, we give an existence and uniqueness result for problem.

Theorem 4.9. Let $0 < \alpha, \beta \leq 1$ with $\alpha + \beta > 1$, and let $\psi \in C^1_+[a, b]$. Suppose that Assumptions (A4) and (A5) hold. Suppose further that $\psi'(\cdot)\kappa_3(\cdot) \in L^1(a, b)$ and $\psi'(\cdot)f(\cdot, 0) \in L^1(a, b)$. If

$$\kappa_4 + G_{\max} \|\psi'(\cdot)\kappa_3(\cdot)\|_{L^1(a,b)} < 1$$

then the problem (1) and (2) has a unique mild solution.

Proof. Let us consider the operator S defined in (12). We can find from Assumption (A4) that $|f(t, x)| \leq \kappa_3(t)|x| + |f(t, 0)|$. Therefore, we can use the same method to that used to prove Step 3 in the proof of Theorem 4.7 to verify that S is well-defined.

Next, we show that S is a contraction. Indeed, for $x, y \in C[a, b]$, using Assumptions (A4) and (A5), similar to Step 1 in the proof of Theorem 4.7, we obtain

$$\begin{aligned} \|Sx - Sy\| &\leq |\mathcal{G}(x) - \mathcal{G}(y)| + G_{\max} \|\psi'(\cdot)\kappa_3(\cdot)\|_{L^1(a,b)} \|x - y\| \\ &\leq \left(\kappa_4 + G_{\max} \|\psi'(\cdot)\kappa_3(\cdot)\|_{L^1(a,b)} \right) \|x - y\|. \end{aligned}$$

This shows that S is a contraction due to $\kappa_4 + G_{\max} \|\psi'(\cdot)\kappa_3(\cdot)\|_{L^1(a,b)} < 1$. Hence, we conclude that S has a unique fixed point in $C[a, b]$, which is the mild solution of problem (1) and (2). This completes the proof of Theorem. \square

Remark 4.10. If $k_3(t) = k_3(\psi(b) - \psi(t))^{-\gamma_3}(\psi(t) - \psi(a))^{-\mu_3}$ and $|f(t, 0)| \leq k_4(\psi(b) - \psi(t))^{-\gamma_4}(\psi(t) - \psi(a))^{-\mu_4}$ for some $k_i > 0$ and $\gamma_i, \mu_i < 1$ for $i = 3, 4$, similar to Remark 4.8, we can verify that $\psi'(\cdot)\kappa_3(\cdot) \in L^1(a, b)$ and $\psi'(\cdot)f(\cdot, 0) \in L^1(a, b)$. Moreover, if k_3 sufficiently close to zero, then $\kappa_4 + G_{\max} \|\psi'(\cdot)\kappa_3(\cdot)\|_{L^1(a,b)} < 1$. Hence, we can conclude that our result in Theorem 4.9 holds for some source functions including singularities.

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