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# Zero-Divisor Graph of the Rings $C_{\mathscr{P}}(X)$ and $C_\infty^{\mathscr{P}}(X)$

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**Abstract.** In this article we introduce the zero-divisor graphs $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_\infty^{\mathscr{P}}(X)$ of the two rings $C_{\mathscr{P}}(X)$ and $C_\infty^{\mathscr{P}}(X)$; here $\mathscr{P}$ is an ideal of closed sets in $X$ and $C_{\mathscr{P}}(X)$ is the aggregate of those functions in $C(X)$, whose support lie on $\mathscr{P}$. $C_\infty^{\mathscr{P}}(X)$ is the $\mathscr{P}$ analogue of the ring $C_\infty(X)$. We determine when the weakly zero-divisor graph $W\Gamma_{\mathscr{P}}(X)$ of $C_{\mathscr{P}}(X)$ coincides with $\Gamma_{\mathscr{P}}(X)$. We find out conditions on the topology on $X$, under-which $\Gamma_{\mathscr{P}}(X)$ (respectively, $\Gamma_\infty^{\mathscr{P}}(X)$) becomes triangulated/ hypertriangulated. We realize that $\Gamma_{\mathscr{P}}(X)$ (respectively, $\Gamma_\infty^{\mathscr{P}}(X)$) is a complemented graph if and only if the space of minimal prime ideals in $C_{\mathscr{P}}(X)$ (respectively $\Gamma_\infty^{\mathscr{P}}(X)$) is compact. This places a special case of this result with the choice $\mathscr{P} \equiv$ the ideals of closed sets in $X$, obtained by Azarpanah and Motamedi in [8] on a wider setting. We also give an example of a non-locally finite graph having finite chromatic number. Finally it is established with some special choices of the ideals $\mathscr{P}$ and $\mathscr{Q}$ on $X$ and $Y$ respectively that the rings $C_{\mathscr{P}}(X)$ and $C_{\mathscr{Q}}(Y)$ are isomorphic if and only if $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_{\mathscr{Q}}(Y)$ are isomorphic.

## 1. Introduction

In what follows $X$ stands for a Tychonoff space. Let $\mathscr{P}$ be an ideal of closed sets in $X$ in the following sense: if $A \in \mathscr{P}$ and $B \in \mathscr{P}$, then $A \cup B \in \mathscr{P}$ and if $A \in \mathscr{P}$ and $C \subset A$ with $C$, closed in $X$, then $C \in \mathscr{P}$. Suppose $C_{\mathscr{P}}(X)$ is the family of all those functions $f$ in $C(X)$ whose support $cl_X(X \setminus Z(f)) \in \mathscr{P}$, here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of $f$. Suppose $C_\infty^{\mathscr{P}}(X) = \{f \in C(X) : \text{for each } \epsilon > 0, \{x \in X : |f(x)| \geq \epsilon\} \in \mathscr{P}\}$. It turns out that $C_{\mathscr{P}}(X)$ and $C_\infty^{\mathscr{P}}(X)$ are both commutative rings, possibly without identity and $C_{\mathscr{P}}(X) \subset C_\infty^{\mathscr{P}}(X)$. Let $\Gamma(C_{\mathscr{P}}(X)) \equiv \Gamma_{\mathscr{P}}(X)$ be the graph, whose vertices are non-zero divisors of zero in $C_{\mathscr{P}}(X)$ and $\Gamma_\infty^{\mathscr{P}}(X)$ be the analogous graph associated with $C_\infty^{\mathscr{P}}(X)$. Two distinct vertices $f$ and $g$ in $\Gamma_{\mathscr{P}}(X)$ (respectively in $\Gamma_\infty^{\mathscr{P}}(X)$) are said to be connected by an edge, in which case they are called adjacent vertices if and only if $f.g = 0$. Our intention to write this article is to establish a number of facts which highlight possible interaction between graph properties of $\Gamma_{\mathscr{P}}(X)$ (respectively $\Gamma_\infty^{\mathscr{P}}(X)$) and ring properties of $C_{\mathscr{P}}(X)$ (respectively $C_\infty^{\mathscr{P}}(X)$) leading to further interaction between these two properties and the topological properties on $X$. It is easy to see that on choosing $\mathscr{P}$ to be the ideal of all closed sets in $X$, $C_{\mathscr{P}}(X)$ becomes identical to $C(X)$. We realize that some of the results related to zero-divisor graph of $C(X)$ obtained in [8] are special cases of facts obtained in the present paper. There have already appeared in the literature some new papers which generalize the concept of the zero-divisor graph, viz. weakly zero-divisor graph, extended zero-divisor graph, generalized

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zero-divisor graph etc. See the articles [18], [12], [10] in this context. In [18], Nikmehr et al. introduced the weakly zero-divisor graph as a supergraph of zero-divisor graph of a commutative ring whose set of vertices is the same as that of the zero-divisor graph and two distinct vertices $x, y$ are adjacent in this graph if and only if there exist non-zero $r \in ann(x)$ and $s \in ann(y)$ such that $r.s = 0$. In [12], the extended zero-divisor graph was introduced with same vertex set as of zero-divisor graph and two distinct vertices $x, y$ are adjacent if $x^m.y^n = 0$ for some natural numbers $m, n$ with $x^m \neq 0$ and $y^n \neq 0$. Like all of the above graphs the vertex set of the generalized zero-divisor graph, introduced in [10], is also the same as this graphs and two distinct vertices $x, y$ are defined to be adjacent here if the ideal $ann(x) + ann(y)$ is an essential ideal. We have made a comparison of some of the results in our paper with the corresponding relevant results of the above mentioned graphs [see Theorem 2.7, Theorem 2.8 and Remark 3.5]. We would like to mention in this context as far as we dig into literature that there are only four papers on graphs having their vertices lying in $C(X)$. See the articles [4], [8], [9] and [13] in this context. In the technical Section 2 of this paper we introduce several well-known parameters related to the graph $\Gamma_{\mathscr{P}}(X)$. These include distance between distinct vertices, diameter, radius of a graph, eccentricity of a vertex. We show that, the distance between any two vertices of the graph $\Gamma_{\mathscr{P}}(X)$ is at most 3. This leads to necessary and sufficient condition for $\Gamma_{\mathscr{P}}(X)$ to be triangulated (respectively, hypertriangulated).

In Section 3, we compute the lengths of various possible cycles and determine the girth of the graphs in some cases. In Section 4, we find out a few relations interconnecting the dominating number and clique number of $\Gamma_{\mathscr{P}}(X)$ and the cellularity of $X$. Furthermore we determine when does $\Gamma_{\mathscr{P}}(X)$ become a complemented graph.

In Section 5, we calculate several parameters related to the graph $\Gamma_{\infty}^{\mathscr{P}}(X)$. These are mostly parallel to their $\Gamma_{\mathscr{P}}(X)$ analogues obtained in the section 2 and 3. However we proved that the chromatic number of $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_{\infty}^{\mathscr{P}}(X)$ are identical.

If two rings $C_{\mathscr{P}}(X)$ and $C_{\mathscr{Q}}(Y)$ are isomorphic, then it is easy to see that their zero-divisor graphs $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_{\mathscr{Q}}(Y)$ are isomorphic in the following sense: there is a bijection between the set of vertices of these two graphs which preserve the adjacency relation. However the converse problem to find out any possible isomorphism between the rings $C_{\mathscr{P}}(X)$ and $C_{\mathscr{Q}}(Y)$ on the basis of the hypothesis that there is a graph isomorphism between $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_{\mathscr{Q}}(Y)$, in general appears to be too wild to venture into. Nevertheless, by making some special choices of ideals $\mathscr{P}$ and $\mathscr{Q}$ on $X$ and $Y$ respectively, we make some breakthrough in this matter. We establish that if $\mathscr{P}$ is the ideal of all finite subsets of $X$ and $\mathscr{Q}$, the ideal of all finite subsets of $Y$, then the rings $C_{\mathscr{P}}(X)$ and $C_{\mathscr{Q}}(Y)$ are isomorphic if and only if $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_{\mathscr{Q}}(Y)$ are isomorphic [Theorem 6.8]. This is the final result in Section 6.

For more information on the rings $C_{\mathscr{P}}(X)$ and $C_{\infty}^{\mathscr{P}}(X)$, the reader is referred to see the articles [1] and [2]. For graph theoretic information, the reader is referred to the book [14].

## 2. Technical notations related to $\Gamma_{\mathscr{P}}(X)$

The distance between two distinct vertices $f$ and $g$ in $\Gamma_{\mathscr{P}}(X)$, denoted by $d(f, g)$, is the length of the shortest path from $f$ to $g$. We wish to denoted by $V_{\mathscr{P}}(X)$, the set of vertices of the graph $\Gamma_{\mathscr{P}}(X)$. The diameter of the graph $\Gamma_{\mathscr{P}}(X)$ is defined by: $diam(\Gamma_{\mathscr{P}}(X)) = Max\{d(f, g) : f, g \in V_{\mathscr{P}}(X)\}$. The eccentricity $e(f)$ of an $f \in V_{\mathscr{P}}(X)$ is defined by: $e(f) = Max\{d(f, g) : g \in V_{\mathscr{P}}(X)\}$. An $f \in V_{\mathscr{P}}(X)$ is called a center of $\Gamma_{\mathscr{P}}(X)$ if $e(f) \leq e(g)$ holds for each $g \in V_{\mathscr{P}}(X)$ and in this case $e(f)$ is called the radius of the graph. The girth of $\Gamma_{\mathscr{P}}(X)$, denoted by $gr(\Gamma_{\mathscr{P}}(X))$, is the length of the smallest cycle in this graph. Like any graph $\Gamma_{\mathscr{P}}(X)$ is called triangulated (respectively hypertriangulated) if each vertex (respectively each edge) of this graph is a vertex (respectively is an edge) of a triangle. The smallest length of a cycle containing two distinct vertices $f$ and $g$ in $\Gamma_{\mathscr{P}}(X)$ will be denoted by $c(f, g)$.

A subset $D$ of $V_{\mathscr{P}}(X)$ is called a dominating set in $\Gamma_{\mathscr{P}}(X)$ if for each $f \in V_{\mathscr{P}}(X) \setminus D$, there exists $g \in D$ such that $f$ and $g$ are adjacent. The dominating number of $\Gamma_{\mathscr{P}}(X)$ is defined as follows: $dt(\Gamma_{\mathscr{P}}(X)) = min\{|D| : D \text{ is a dominating set in } \Gamma_{\mathscr{P}}(X)\}$. A coloring of a graph is a labeling of the vertices of the graph with colors such that no two adjacent vertices have the same color. More precisely, for a cardinal number $\alpha$ (finite or infinite), an $\alpha$-coloring of $\Gamma_{\mathscr{P}}(X)$ is a map $\psi : V_{\mathscr{P}}(X) \to [0, \alpha)$ with the following condition:

whenever $f, g \in V_{\mathscr{P}}(X)$ and $f.g = 0$, $\psi(f) \neq \psi(g)$. The chromatic number of $\Gamma_{\mathscr{P}}(X)$ is defined as follows: $\chi(\Gamma_{\mathscr{P}}(X)) = min\{\alpha :$ there exists a $\alpha$-coloring of $\Gamma_{\mathscr{P}}(X)\}$.

Let $G$ be a graph. A complete subgraph of $G$ is any subset $H$ of $G$ such that each pair of distinct vertices in $H$ are adjacent. The clique number of $G$ is defined as follows: $\omega(G) = sup\{|H| : H$ is a complete subgraph of $G\}$. $G$ is said to be a weakly perfect graph if $\omega(G) = \chi(G)$. A subset $S$ of the vertex set of $G$ is called a stable set if no two vertices in $S$ are adjacent. $G$ is said to be an $r-$partite graph if $G$ has $r$ many stable sets. An $r-$partite graph is called complete $r-$partite graph if any two vertices from different stable sets are adjacent. An $r-$partite (respectively, complete $r-$partite) is called bipartite (resp., complete bipartite) if $r = 2$.

A collection $\mathscr{B}$ of non-empty open sets in $X$ is called a cellular family if any two distinct members of $\mathscr{B}$ are disjoint. The cellularity of a space $X$ is defined as follows: $c(X) = sup\{|\mathscr{B}| : \mathscr{B}$ is a cellular family of open sets in $X\}$.

**Definition 2.1.** $X$ is called locally $\mathscr{P}$ at a point $x \in X$, if there exists an open neighbourhood $V$ of $x$ in $X$ such that $cl_X V \in \mathscr{P}$. $X$ is said to be locally $\mathscr{P}$ if it is locally $\mathscr{P}$ at each point on it.

Let $X_{\mathscr{P}} = \{x \in X : X$ is locally $\mathscr{P}$ at $x\}$. Then it is easy to prove that $X_{\mathscr{P}}$ is an open set in $X$. Also, $X$ is locally $\mathscr{P}$ if and only if $X_{\mathscr{P}} = X$.

**Lemma 2.2.** *Given $x \in X_{\mathscr{P}}$ and an open neighbourhood $G$ of $x$, there exists $f \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(f) \subset cl_X(X \setminus Z(f)) \subset G$.*

*Proof.* Since $x \in X_{\mathscr{P}}$ there exists an open neighbourhood $U$ of $x$ in $X$ such that $cl_X U \in \mathscr{P}$. Consider the open neighbourhood $U \cap G$ of $x$. Then by complete regularity of $X$, there exists $f \in C(X)$ such that $x \in X \setminus Z(f) \subset cl_X(X \setminus Z(f)) \subset U \cap G$. Since $cl_X(X \setminus Z(f)) \subset U \subset cl_X U \in \mathscr{P}$, it follows that $cl_X(X \setminus Z(f)) \in \mathscr{P}$, i.e., $f \in C_{\mathscr{P}}(X)$. $\square$

The following result decides which non-zero elements in $C_{\mathscr{P}}(X)$ are vertices in the graph $\Gamma_{\mathscr{P}}(X)$.

**Theorem 2.3.** *For any $f \in C_{\mathscr{P}}(X) \setminus \{0\}$, the following three statements are equivalent:*

1. $f \in V_{\mathscr{P}}(X)$
2. $X_{\mathscr{P}} - cl_X(X \setminus Z(f)) = X_{\mathscr{P}} \cap int_X(Z(f)) \neq \emptyset$
3. $cl_X(X_{\mathscr{P}}) \cap int_X Z(f) \neq \emptyset$

*Proof.* (1) $\implies$ (2): Let (1) hold. Then there exists $g \in V_{\mathscr{P}}(X)$ such that $f.g = 0$. This implies that $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$. Choose a point $x \in X \setminus Z(g)$. Then $x \in X_{\mathscr{P}}$ because $X \setminus Z(g) \subset X_{\mathscr{P}}$. On the other hand $x \notin cl_X(X \setminus Z(f))$, i.e., $x \in int_X Z(f)$. Thus $x \in X_{\mathscr{P}} \cap int_X Z(f)$.

(2) $\implies$ (3): This is trivial.

(3) $\implies$ (1): Let (3) be true. Choose a point $p \in cl_X(X_{\mathscr{P}}) \cap int_X Z(f)$. Clearly then $X_{\mathscr{P}} \cap int_X Z(f) \neq \emptyset$. Choose a point $q \in X_{\mathscr{P}} \cap int_X Z(f)$. Then by Lemma 2.2, there exists $g \in C_{\mathscr{P}}(X)$ such that $q \in X \setminus Z(g) \subset cl_X(X \setminus Z(g)) \subset int_X Z(f)$. Consequently $f.g = 0$ and $g \in C_{\mathscr{P}}(X)$ is a vertex in $\Gamma_{\mathscr{P}}(X)$. Thus $f \in V_{\mathscr{P}}(X)$. $\square$

**Remark 2.4.** On choosing $\mathscr{P} \equiv$ ideal of all closed sets in $X$, Theorem 2.3 reads: $f \in C(X) \setminus \{0\}$ is a vertex of the zero-divisor graph $\Gamma(C(X))$ of $C(X)$ [considered in [8]] if and only if $int_X Z(f) \neq \emptyset$. This is indeed a special case of Sublemma 1.1 with the choice $f = g$ in [8].

Incidentally, the vertices of each of the three recently investigated graphs viz. weakly zero-divisor graph, generalized zero-divisor graph and extended zero-divisor graph, considered in [18], [10] and [12] constructed over the ring $R = C_{\mathscr{P}}(X)$ with the choice $\mathscr{P} \equiv$ ideal of all closed sets in $X$ are all divisors of zero in $C_{\mathscr{P}}(X)$.

**Corollary 2.5.** *Let $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$. Then each non-zero element $f$ of $C_{\mathscr{P}}(X)$ is a vertex of $\Gamma_{\mathscr{P}}(X)$.*

*Proof.* If possible let there exist $f \in C_{\mathscr{P}}$, $f \neq 0$ such that $f \notin V_{\mathscr{P}}(X)$. Then $cl_X(X_{\mathscr{P}}) \cap int_X Z(f) = \emptyset$ and consequently $cl_X(X_{\mathscr{P}}) \subset X \setminus int_X Z(f) = cl_X(X \setminus Z(f))$. Since $cl_X(X \setminus Z(f)) \in \mathscr{P}$, this implies that $cl_X(X_{\mathscr{P}}) \in \mathscr{P}$, a contradiction. □

The converse of the last corollary is not true. The following is a simple counterexample.

**Example 2.6.** Let $X = \mathbb{Q} \equiv$ the space of all rational numbers. Suppose $\mathscr{P} \equiv$ the ideal of all compact subsets of $X$. Since $\mathbb{Q}$ is nowhere locally compact, it follows that $X_{\mathscr{P}} = \emptyset$. Consequently, $cl_X X_{\mathscr{P}} = \emptyset \in \mathscr{P}$. But since $C_{\mathscr{P}}(X) = C_K(\mathbb{Q}) = \{0\}$ [see $4D2$, [15]]. The condition that each non-zero element in $C_{\mathscr{P}}(X)$ is a member of $\Gamma_{\mathscr{P}}(X)$ is vacuously satisfied.

We shall show in Section 6 that by an appropriate choice of the space $X$ and the ideal $\mathscr{P}$ of closed sets in $X$, the converse of Corollary 2.5 is true.

We now compare a few basic facts related to the zero-divisor graph $\Gamma_{\mathscr{P}}(X)$ with the corresponding results enjoyed by some new supergraphs introduced in [12], [10] and [18].

Since $C_{\mathscr{P}}(X)$ is a reduced ring, it follows by Corollary 2.7 in [12] that the extended zero-divisor graph $\overline{\Gamma}(C_{\mathscr{P}}(X))$ of $C_{\mathscr{P}}(X)$ is the same as $\Gamma(C_{\mathscr{P}}(X))$.

The generalized zero-divisor graph $\Gamma_g(R)$ of a commutative ring $R$ with identity may well be a proper supergraph of the zero-divisor graph of $\Gamma(R)$ of $R$ [see example 3.1(1) in [10]]. However it is proved in Theorem 3.3 in [10] that for a reduced ring $R$, $\Gamma_g(R) = \Gamma(R)$ when and only when for nonzero vertices $x, y$, $x.y \neq 0$ implies $ann_R(ann_R(x) + ann_R(y)) \neq (0)$. We use this fact to prove the following Theorem.

**Theorem 2.7.** $\Gamma_g(C_{\mathscr{P}}(X)) = \Gamma(C_{\mathscr{P}}(X))$

*Proof.* Let $f, g \in V_{\mathscr{P}}$ such that $f.g \neq 0$. Then $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$, let $x \in X \setminus Z(f) \cap X \setminus Z(g)$. Then by Lemma 2.2, there exists $h \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(h) \subset cl_X(X \setminus Z(h)) = X \setminus intZ(h) \subset X \setminus Z(f) \cap X \setminus Z(g)$. Clearly $h \neq 0$ and $Z(f) \cup Z(g) \subset intZ(h)$. Now for all $f_1 \in ann(f)$ and $g_1 \in ann(g)$, $f_1.f = 0 = g_1.g \implies X \setminus Z(f_1) \subset Z(f)$ and $X \setminus Z(g_1) \subset Z(g)$ and therefore $X \setminus Z(f_1) \cup X \setminus Z(g_1) \subset intZ(h) \implies h.f_1 = 0 = h.g_1$ i.e., $h(f_1 + g_1) = 0 \implies h \in ann(ann(f) + ann(g))$. Thus $ann(ann(f) + ann(g)) \neq (0)$ and hence by Theorem 3.3, [10], $\Gamma(C_{\mathscr{P}}(X) = \Gamma_g(C_{\mathscr{P}}(X))$. □

As in [18], let $W\Gamma(C_{\mathscr{P}}(X))$ denote the weakly zero-divisor graph of $C_{\mathscr{P}}(X)$. Unlike the graphs $\overline{\Gamma}(C_{\mathscr{P}}(X))$ and $\Gamma_g(C_{\mathscr{P}}(X))$, $W\Gamma(C_{\mathscr{P}}(X))$ is not always equal to $\Gamma(C_{\mathscr{P}}(X))$. The next result shows when these last two graphs are equal.

**Theorem 2.8.** *$W\Gamma(C_{\mathscr{P}}(X)) = \Gamma(C_{\mathscr{P}}(X))$ if and only if $|X_{\mathscr{P}}| = 2$.*

*Proof.* First let $X_{\mathscr{P}} = \{x, y\}$. Then as $X_{\mathscr{P}}$ is an open subset of $X$, $x$ and $y$ are isolated points in $X$. Let $f \in V_{\mathscr{P}}(X)$. Then $\emptyset \neq X \setminus Z(f) \subset X_{\mathscr{P}}$. It follows from Theorem 2.3 that $f$ vanishes at either $x$ or $y$ but not at both of $x$ and $y$, i.e., $f$ is equal to either $r.1_x$ or $r.1_y$ for some non-zero real number $r$, here $1_x$ stands for the characteristic function of the set $\{x\}$, i.e., $1_x(z) = \begin{cases} 1 & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$. Thus $V_{\mathscr{P}}(X) = \{r.1_x : r \in \mathbb{R} \setminus \{0\}\} \cup \{r.1_y : r \in \mathbb{R} \setminus \{0\}\}$ and hence $\Gamma(C_{\mathscr{P}}(X))$ is a complete bipartite graph. Let $f, g$ be non-adjacent vertices in $\Gamma(C_{\mathscr{P}}(X))$. Then $f, g$ lie on the same stable set. Without loss of generality, let $f, g \in \{r.1_x : r \in \mathbb{R} \setminus \{0\}\}$. Then $ann(f) = \{r.1_y : r \in \mathbb{R} \setminus \{0\}\} = ann(g)$. This implies there does not exist any $h_1 \in ann(f)$ and $h_2 \in ann(g)$ such that $h_1.h_2 = 0$ and hence $f, g$ are not adjacent in $W\Gamma(C_{\mathscr{P}}(X))$. Therefore $W\Gamma(C_{\mathscr{P}}(X)) = \Gamma(C_{\mathscr{P}}(X))$.

Let $|X_{\mathscr{P}}| \geq 3$ and $x, y, z$ be distinct points in $X_{\mathscr{P}}$. By using the complete regularity of $X$, we can find an $f_0 \in C(X)$ such that $x \in intZ(f_0)$ and $\{y, z\} \subset X \setminus Z(f_0)$. Since $y, z \in X_{\mathscr{P}}$, on using Lemma 2.2, we can produce $h_1, h_2 \in C_{\mathscr{P}}(X)$ such that $y \in X \setminus Z(h_1) \subset X \setminus Z(f_0)$ and $z \in X \setminus Z(h_2) \subset X \setminus Z(f_0)$. Let $f = (h_1)^2 + (h_2)^2$. Then $f \in C_{\mathscr{P}}(X)$ and $X \setminus Z(f) \subset X \setminus Z(h_1) \cup X \setminus Z(h_2) \subset X \setminus Z(f_0)$. This implies that $Z(f_0) \subset Z(f)$. Hence $x \in intZ(f)$ and $\{y, z\} \subset X \setminus Z(f)$. Analogously, there exists $g \in C_{\mathscr{P}}(X)$ such that $y \in intZ(g)$ and $\{x, z\} \subset X \setminus Z(g)$. Then $z \in X \setminus Z(f) \cap X \setminus Z(g) \implies f.g \neq 0$. Hence $f, g$ are not adjacent vertices in $\Gamma(C_{\mathscr{P}}(X))$. Now $x \in intZ(f) \cap (X \setminus Z(g)) \implies$ by Lemma 2.2 there exists $h_x \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(h_x) \subset cl_X(X \setminus Z(h_x)) \subset$

$intZ(f) \cap X \setminus Z(g)$. Then $h_x \neq 0$, $f.h_x = 0$ and $Z(g) \subset Z(h_x)$. Again $y \in intZ(g) \implies$ by Lemma 2.2 there exists $h_y \in C_{\mathscr{P}}(X)$ such that $y \in X \setminus Z(h_y) \subset cl_X(X \setminus Z(h_y)) \subset intZ(g) \subset Z(h_x)$. It follows that $h_y.g = 0$ and $h_y.h_x = 0$. Thus $h_x \in ann(f)$ and $h_y \in ann(g)$ such that $h_x.h_y = 0$. Therefore $f, g$ are adjacent in $W\Gamma(C_{\mathscr{P}}(X))$ and hence $W\Gamma(C_{\mathscr{P}}(X)) \neq \Gamma(C_{\mathscr{P}}(X))$. □

The following proposition decides, when a given pair of vertices in $\Gamma_{\mathscr{P}}(X)$ admit of a third vertex adjacent to both of them.

**Theorem 2.9.** *Let $f, g \in V_{\mathscr{P}}(X)$. Then there exists a vertex $h$, adjacent to both $f$ and $g$ if and only if $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) \neq \emptyset$.*

*Proof.* First let $h \in V_{\mathscr{P}}(X)$ be adjacent to $f$ and $g$. Then $h \in C_{\mathscr{P}}(X)$ implies that $X \setminus Z(h) \subset X_{\mathscr{P}}$. On the other hand $h.f = 0 = g.h$ implies that $\emptyset \neq X \setminus Z(h) \subset Z(f) \cap Z(g)$. Thus $\emptyset \neq X \setminus Z(h) \subset X_{\mathscr{P}} \cap Z(f) \cap Z(g)$. Hence $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) \neq \emptyset$.

Conversely let $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) \neq \emptyset$. So there exists a non-empty open set $W$ contained in $Z(f) \cap Z(g) \cap X_{\mathscr{P}}$. Choose a point $x \in W$. Then by Lemma 2.2, there exists $h \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(h) \subset cl_X(X \setminus Z(h)) \subset W \subset Z(f) \cap Z(g) \cap X_{\mathscr{P}}$. It is clear that $h \neq 0$ and $h.f = h.g = 0$. Then $h \in V_{\mathscr{P}}(X)$ and is adjacent to both $f$ and $g$. □

**Remark 2.10.** Sublemma 1.1 in [8] is a special case of Theorem 2.9 with the choice $\mathscr{P} \equiv$ ideal of all closed sets in $X$. The reason is, in that case $X_{\mathscr{P}} = X$.

In the next theorem, we compute the possible distance between pairs of distinct vertices.

**Theorem 2.11.** *Let $f, g \in V_{\mathscr{P}}(X)$. Then:*

1. *$d(f, g) = 1$ if and only if $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$.*
2. *$d(f, g) = 2$ if and only if $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) \neq \emptyset$.*
3. *$d(f, g) = 3$ if and only if $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) = \emptyset$.*

*Proof.* 1. Trivial.

2. Follows immediately from part (1) of this Theorem and Theorem 2.9.

3. Suppose $d(f, g) = 3$. Then it follows from part (1) and (2) of the present Theorem that $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) = \emptyset$.

Conversely let $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) = \emptyset$. Then it follows from part (1) and (2) that $d(f, g) > 2$. Since $f, g \in V_{\mathscr{P}}(X)$, there exist $f_1, g_1 \in V_{\mathscr{P}}(X)$ such that $f.f_1 = 0 = g.g_1$. To ascertain that $d(f, g) = 3$, it suffices to show that $f_1.g_1 = 0$. Indeed $f.f_1 = 0$ implies that $X \setminus Z(f_1) \subset X_{\mathscr{P}} - cl_X(X \setminus Z(f)) = X_{\mathscr{P}} \cap int_X Z(f)$. Analogously $X \setminus Z(g_1) \subset X_{\mathscr{P}} \cap int_X Z(g)$. It follows that: $X \setminus Z(f_1) \cap X \setminus Z(g_1) \subset X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(g) = int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g))$. The hypothesis $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) = \emptyset$, therefore implies that $X \setminus Z(f_1) \cap X \setminus Z(g_1) = \emptyset$. Hence $f_1.g_1 = 0$. □

**Corollary 2.12.** $2 \leq diam(\Gamma_{\mathscr{P}}(X)) \leq 3$. *Moreover, if $diam(\Gamma_{\mathscr{P}}(X)) = 3$, then $cl_X(X_{\mathscr{P}}) \in \mathscr{P}$.*

*Proof.* $2 \leq diam(\Gamma_{\mathscr{P}}(X)) \leq 3$ follows directly from the previous Theorem. Let $diam(\Gamma_{\mathscr{P}}(X)) = 3$. So there exists a pair of vertices $f, g \in V_{\mathscr{P}}(X)$ such that $d(f, g) = 3$. Then from Theorem 2.11(3) it follows that, $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$. Now $X_{\mathscr{P}} - (cl_X(X \setminus Z(f)) \cup cl_X(X \setminus Z(g))) = X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(g) = int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$, i.e., $X_{\mathscr{P}} \subset cl_X(X \setminus Z(f)) \cup cl_X(X \setminus Z(g))$ implies $cl_X(X_{\mathscr{P}}) \subset cl_X(X \setminus Z(f)) \cup cl_X(X \setminus Z(g)) \in \mathscr{P}$ and hence $cl_X(X_{\mathscr{P}}) \in \mathscr{P}$. □

We would like to mention in this context, the following result proved in [8], Corollary 1.3 on choosing $\mathscr{P} \equiv$ the ideal of all closed sets in $X$.

**Theorem 2.13.** *Whenever $X$ has at least three points, then the diameter of the zero-divisor graph of $C(X)$ is* 3.

Thus converse of the Corollary 2.12 is not true. Consider the following example

**Example 2.14.** Let $X$ be a Tychonoff space with $|X| = 2$ and $\mathscr{P} \equiv$ the ideal of all closed sets in $X$. Then $C_{\mathscr{P}}(X) = C(X)$ and $X_{\mathscr{P}} = X$. It can be easily proved that the zero-divisor graph of $C(X)$ is complete bipartite and hence $diam(\Gamma_{\mathscr{P}}(X)) = 2$. But $cl_X(X_{\mathscr{P}}) = X \in \mathscr{P}$.

**Remark 2.15.** We would like to mention that for any commutative ring $R$, $diam(W\Gamma(R)) \leq 2$. But in contrast, for some choice of $\mathscr{P}$, $diam(\Gamma(C_{\mathscr{P}}(X)))$ may be equal to 3. We justify it by the following example. Let $X = \{1, 2, 3\} \cup [4, 8]$ with the subspace topology of $\mathbb{R}$ and $\mathscr{P}$ be the ideal of all finite sets of $X$. Then $X_{\mathscr{P}} = \{1, 2, 3\}$. Let $f = 1_{\{1,2\}}$, the characteristic function of $\{1, 2\}$ and $g = 1_{\{2,3\}}$, the characteristic function of $\{2, 3\}$. Then $f, g \in V_{\mathscr{P}}(X)$ and $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $X_{\mathscr{P}} \cap Z(f) \cap Z(g) = \emptyset$. It follows from Theorem 2.11(3) that $d(f, g) = 3$ and hence $diam(\Gamma_{\mathscr{P}}(X)) = 3$.

Before proceeding further we have to rule out the cases where $V_{\mathscr{P}}(X) = \emptyset$, i.e., where the graph is empty.

**Theorem 2.16.** *For an ideal $\mathscr{P}$ of closed sets in $X$, the following two statements are equivalent:*

1. $V_{\mathscr{P}}(X) \neq \emptyset$.
2. $|X_{\mathscr{P}}| \geq 2$.

*Proof.* If $X_{\mathscr{P}} = \emptyset$, i.e., $X$ is nowhere locally $\mathscr{P}$, then from Theorem 2.3, it follows that no non-zero element in $C_{\mathscr{P}}(X)$ can be a vertex, in other words $V_{\mathscr{P}}(X) = \emptyset$. On the other hand, if $X$ is locally $\mathscr{P}$ just at a single point $p$ on $X$, then $p$ is an isolated point of $X$ and $\{p\} \in \mathscr{P}$. Now if $f$ is a non-zero function in $C_{\mathscr{P}}(X)$ and $f \in V_{\mathscr{P}}(X)$, then from Theorem 2.3, it follows that $p \in int_X Z(f)$. Consequently, $cl_X(X \setminus Z(f))(\subset X \setminus \{p\}) \in \mathscr{P}$ and any point on $X \setminus Z(f)$ is a member of $X_{\mathscr{P}}$ – a contradiction to the initial assumption that $X$ is locally $\mathscr{P}$ only at the point $p$. Thus $f$ can not be a non-zero function in $C_{\mathscr{P}}(X)$, in other words $f \equiv 0$. Hence $V_{\mathscr{P}}(X) = \emptyset$. Then (1) $\implies$ (2) is proved.

(2) $\implies$ (1) : Assume that (2) is true. So we can find out a pair of distinct points $p, q$ from $X_{\mathscr{P}}$. Then there exists a co-zero set neighbourhood $C_p$ of $p$ in $X$ such that $cl_X(C_p) \in \mathscr{P}$ and also there is a co-zero set neighbourhood $C_q$ of $q$ in $X$ with $cl_X(C_q) \in \mathscr{P}$. By using the complete regularity of $X$, we can find out co-zero set neighbourhoods $C^*_p$ and $C^*_q$ of $p$ and $q$ in $X$ respectively such that $C^*_p \cap C^*_q = \emptyset$. Let $\widehat{C_p} = C^*_p \cap C_p$ and $\widehat{C_q} = C^*_q \cap C_q$. Then $\widehat{C_p}$ and $\widehat{C_q}$ are disjoint co-zero set neighbourhoods of $p$ and $q$ respectively with $cl_X(\widehat{C_p}) \in \mathscr{P}$ and $cl_X(\widehat{C_q}) \in \mathscr{P}$. We can write $\widehat{C_p} = X \setminus Z(f)$ for some $f \in C(X)$. Then $f \in C_{\mathscr{P}}(X)$ as $cl_X(X \setminus Z(f)) \in \mathscr{P}$ and also $f \neq 0$. Furthermore $\widehat{C_q} \subset Z(f)$ with $q \in X_{\mathscr{P}}$. This shows that $q \in int_X Z(f)$. Thus $X_{\mathscr{P}} \cap int_X Z(f) \neq \emptyset$. Hence from Theorem 2.3, we get that $f \in V_{\mathscr{P}}(X)$. Therefore, $V_{\mathscr{P}}(X) \neq \emptyset$. □

**Remark 2.17.** Remark 1.4, particularly the third sentence in this remark in [8] follows as a special case of Theorem 2.16 on choosing $\mathscr{P} \equiv$ ideal of the entire family of closed sets in $X$.

**Convention 2.18.** In what follows, we shall assume that $|X_{\mathscr{P}}| \geq 2$ and this will ensure that the graph $\Gamma_{\mathscr{P}}(X)$ will be non-void.

The following result shows that for some choice of $\mathscr{P}$, each vertex of $\Gamma_{\mathscr{P}}(X)$ will be a center, i.e., $\Gamma_{\mathscr{P}}(X)$ is a self-centric graph. [A graph is said to be a self-centric graph if every vertex in the graph is a center.]

**Theorem 2.19.** *Let $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$. Then for an arbitrary $f \in V_{\mathscr{P}}(X)$, $e(f) = 2$.*

*Proof.* From Corollary 2.5, we get that each non-zero function in $C_{\mathscr{P}}(X)$ is a vertex of $\Gamma_{\mathscr{P}}(X)$. So if $g \in C_{\mathscr{P}}(X)$ , then $d(f, g) = 1$ if $f$ and $g$ are adjacent. Suppose $f$ and $g$ are not adjacent. Then since $f^2 + g^2$ is a vertex of $\Gamma_{\mathscr{P}}(X)$, it follows from Theorem 2.3 that $X_{\mathscr{P}} \cap int_X Z(f^2 + g^2) \neq \emptyset$, in other words: $X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(g) \neq \emptyset$, i.e., $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$, it follows from Theorem 2.11 that $d(f, g) = 2$. Now it is clear that for $f \in V_{\mathscr{P}}(X)$, $f$ and $2f$ are not adjacent, hence $d(f, 2f) = 2$. Thus $e(f) = 2$. □

**Corollary 2.20.** *For a locally compact non-compact space $X$, each vertex of the zero-divisor graph of the ring $C_K(X)$ of all continuous functions with compact support is a center of it and the radius of the graph is* 2.

**Remark 2.21.** We show that the conclusion of Theorem 2.19 may not be valid if the hypothesis $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$ is dropped. Let $X = \mathbb{R}$ and $\mathscr{P} \equiv$ the ideal of closed sets in $\mathbb{R}$. Then $X_{\mathscr{P}} = X = \mathbb{R}$ and therefore $cl_X(X_{\mathscr{P}}) = \mathbb{R} \in \mathscr{P}$. In this case $C_{\mathscr{P}}(X) = C(X) = C(\mathbb{R})$. Let $f$ be any vertex of the zero-divisor graph of $C(\mathbb{R})$. Then $int_{\mathbb{R}}Z(f) \neq \emptyset$. Choose a point $x \in \mathbb{R}\backslash Z(f)$. Then there exists a zero set neighbourhood $Z(g)$ of $x$ in $\mathbb{R}$ such that $Z(g) \cap Z(f) = \emptyset$. We see that $g$ is a vertex of this graph and $int_{\mathbb{R}}(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = int_{\mathbb{R}}(Z(f) \cap Z(g)) = \emptyset$. Furthermore $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$. It follows from Theorem 2.11(3) that $d(f, g) = 3$. Then $e(f) = 3$. We note that each vertex in this graph is a center but its radius is 3.

The following result is a key one to characterize triangulated graph of the form $\Gamma_{\mathscr{P}}(X)$.

**Theorem 2.22.** *Let $f \in V_{\mathscr{P}}(X)$. Then $f$ is a vertex of a triangle if and only if $|X_{\mathscr{P}} \cap int_X Z(f)| \geq 2$.*

*Proof.* First let $f$ be a vertex of a triangle. Then there exist $g, h \in V_{\mathscr{P}}(X)$ such that $f.g = g.h = h.f = 0$. Choose $x \in X \setminus Z(g)$ and $y \in X \setminus Z(h)$. Then $\{x, y\} \subset int_X Z(f) \cap X_{\mathscr{P}}$. Since $X \setminus Z(g) \cap X \setminus Z(h) = \emptyset$, it follows that $x \neq y$. Thus $|X_{\mathscr{P}} \cap int_X Z(f)| \geq 2$.

Conversely let $|X_{\mathscr{P}} \cap int_X Z(f)| \geq 2$. Choose $x, y \in X_{\mathscr{P}} \cap int_X Z(f)$, $x \neq y$. Then $X_{\mathscr{P}} - (cl_X(X \setminus Z(f)) \cup \{y\})$ is an open neighbourhood of $x$ in $X$. By Lemma 2.2, there exists $g \in V_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(g) \subset cl_X(X \setminus Z(g)) \subset X_{\mathscr{P}} - (cl_X(X \setminus Z(f)) \cup \{y\})$. Clearly $g.f = 0$. Furthermore $y \notin X \setminus Z(g)$ i.e., $y \in int_X Z(g)$. Thus $y \in X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(g) = int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g))$. Thus the set $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g))$ is non-empty. It follows from Theorem 2.9 that there exists a vertex $h \in V_{\mathscr{P}}(X)$, adjacent to both of $f$ and $g$. We have already obtained that $f$ and $g$ are adjacent. Hence $f - g - h - f$ is a triangle. □

**Theorem 2.23.** *The graph $\Gamma(C_{\mathscr{P}}(X)) \equiv \Gamma_{\mathscr{P}}(X)$ is triangulated if and only if for each vertex $f$, $|X_{\mathscr{P}} \cap int_X Z(f)| \geq 2$.*

[Immediate consequence of Theorem 2.22.]

With the special choice $\mathscr{P} \equiv$ the ideal of all closed sets in $X$, we get the following particular case of Theorem 2.23.

**Theorem 2.24.** *The zero-divisor graph of $C(X)$ is triangulated if and only if $X$ does not contain any isolated point.*

Theorem 2.24 was proved independently in [8] [Proposition 2.1(*ii*)].

We exploit Theorem 2.23, to prove the following sufficient condition for $\Gamma_{\mathscr{P}}(X)$ to be triangulated.

**Theorem 2.25.** *Suppose $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$ and each one-pointic set is a member of $\mathscr{P}$. Then $\Gamma_{\mathscr{P}}(X)$ is triangulated.*

*Proof.* It is easy to check that for any $f \in C_{\mathscr{P}}(X)$, $cl_X(X_{\mathscr{P}}) \cap int_X Z(f) \subset cl_X(X_{\mathscr{P}} \cap int_X Z(f))$. In view of Theorem 2.23, it suffices to check that $X_{\mathscr{P}} \cap int_X Z(f)$ is an infinite set. If possible let $X_{\mathscr{P}} \cap int_X Z(f) = \{x_1, x_2, ..., x_n\}$, a finite set. Then it follows from the above inclusion relation that $cl_X(X_{\mathscr{P}}) \cap int_X Z(f) = \{x_1, x_2, ..., x_m\}$ for some $m \leq n$ [abusing notation]. We can write now: $cl_X(X_{\mathscr{P}}) = cl_X(X \setminus Z(f)) \cup \{x_1, x_2, ..., x_m\}$. Since $f \in C_{\mathscr{P}}(X)$, $cl_X(X \setminus Z(f)) \in \mathscr{P}$ and since each one-pointic set is a member of $\mathscr{P}$, it follows that each finite set is a member of $\mathscr{P}$. Consequently then $cl_X(X_{\mathscr{P}}) \in \mathscr{P}$, a contradiction. □

We want to record the following two special cases of the last theorem.

**Theorem 2.26.** *If $X$ is locally compact and non-compact, then the zero-divisor graph of $C_K(X)$ is triangulated.*

**Theorem 2.27.** *Suppose $X$ is locally pseudocompact and non-pseudocompact. Then the zero-divisor graph of the ring $C_\psi(X)$ of all continuous functions with pseudocompact support is triangulated.*

We shall now determine, when $\Gamma_{\mathscr{P}}(X)$ becomes hypertriangulated. The following result is a straightforward consequence of Theorem 2.9.

**Theorem 2.28.** *$\Gamma_{\mathscr{P}}(X)$ is hypertriangulated if and only if for any edge $f - g$, $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$.*

The next theorem is a sufficient condition for the hypertriangulatedness of $\Gamma_{\mathscr{P}}(X)$.

**Theorem 2.29.** *If $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$, then $\Gamma_{\mathscr{P}}(X)$ is hypertriangulated.*

*Proof.* It follows from Corollary 2.5 that each non-zero element of $C_{\mathscr{P}}(X)$ is a vertex of $\Gamma_{\mathscr{P}}(X)$. Therefore if $f - g$ is an edge in this graph, then $f \neq 0$ and $g \neq 0 \implies f^2 + g^2 \neq 0$. Hence $f^2 + g^2$ is a vertex. Consequently by Theorem 2.3, $int_X Z(f^2 + g^2) \cap X_{\mathscr{P}} \neq \emptyset$. This means that $int_X(Z(f) \cap Z(g) \cap X_{\mathscr{P}}) \neq \emptyset$. Hence by Theorem 2.28, $\Gamma_{\mathscr{P}}(X)$ becomes hypertriangulated. □

We would like to mention in this context, the following result proved in [8], Proposition 2.1(*iii*).

**Theorem 2.30.** *If $|X| > 1$, then the zero-divisor graph of $C(X)$ is hypertriangulated if and only if $X$ is a connected middle $P$-space.*

Since $\mathbb{R}$ is not a middle $P$-space, it follows from the last Theorem that the zero-divisor graph of $C(\mathbb{R})$ is not hypertriangulated. On putting $\mathscr{P} \equiv$ the ideal of all closed sets in $\mathbb{R}$, this reads: the zero-divisor graph of $C_{\mathscr{P}}(\mathbb{R})$ is not hypertriangulated. We note that with this special choice of $\mathscr{P}$, $cl_{\mathbb{R}}(\mathbb{R}_{\mathscr{P}}) = \mathbb{R} \in \mathscr{P}$. Thus the condition of the Theorem 2.29 may not hold good without the hypothesis $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$.

## 3. Cycles in $\Gamma_{\mathscr{P}}(X)$

For any graph $G$, obviously the length of the smallest cycle in $G$, i.e., $gr(G) \geq 3$.

**Theorem 3.1.** $3 \leq gr(\Gamma_{\mathscr{P}}(X)) \leq 4$.

*Proof.* Let $f \in V_{\mathscr{P}}(X)$. Then there exists $g \in V_{\mathscr{P}}(X)$ such that $f.g = 0$. Thus we always have a square in $\Gamma_{\mathscr{P}}(X)$: $f - g - 2f - 2g - f$. Therefore, $gr(\Gamma_{\mathscr{P}}(X)) \leq 4$. □

**Theorem 3.2.** *If $X_{\mathscr{P}}$ contains atleast three points, then $gr(\Gamma_{\mathscr{P}}(X)) = 3$.*

*Proof.* We have to find an $f \in V_{\mathscr{P}}(X)$ such that $|X_{\mathscr{P}} \cap int_X Z(f)| \geq 2$ and the rest follows from Theorem 2.22. Let $x, y, z$ be distinct points in $X_{\mathscr{P}}$. Then by complete regularity of $X$, there exists $g \in C(X)$ such that $x \in int_X Z(g)$ and $y, z \in X \setminus Z(g)$. From Lemma 2.2, there exists $f \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(f) \subset cl_X(X \setminus Z(f)) \subset int_X Z(g)$. Since $y, z \in X \setminus Z(g)$, then $y, z \in X \setminus cl_X(X \setminus Z(f)) = int_X Z(f)$, i.e., $y, z \in X_{\mathscr{P}} \cap int_X Z(f)$ and so $f \in V_{\mathscr{P}}(X)$. □

With the special choice $\mathscr{P} \equiv$ the ideal of all closed sets in $X$, we get the following particular case of Theorem 3.2.

**Theorem 3.3.** *Whenever $X$ has at least three points, the girth of the zero-divisor graph of $C(X)$ is 3.*

Theorem 3.3 was proved independently in [8] [Corollary 1.3].

**Remark 3.4.** Let $|X_{\mathscr{P}}| \geq 3$. Then by Theorem 3.2, $\Gamma(C_{\mathscr{P}}(X))$ contains a triangle and hence the supergraph $W\Gamma(C_{\mathscr{P}}(X))$ also contain a triangle. Thus $gr(W\Gamma(C_{\mathscr{P}}(X))) = 3$.

**Remark 3.5.** By Theorem 2.8 and the Remark 3.4, $W\Gamma_{\mathscr{P}}(X) \neq \Gamma_{\mathscr{P}}(X)$ and $gr(W\Gamma_{\mathscr{P}}(X)) = 3$ if and only if $|X_{\mathscr{P}}| \geq 3$ but $Z(C_{\mathscr{P}}(X)) \equiv V_{\mathscr{P}}(X) \cup \{0\}$ may not be an ideal of $C_{\mathscr{P}}(X)$. For example consider a disconnected space $X$ with $|X| \geq 3$ and $\mathscr{P}$ as the ideal of all closed sets. Then $C_{\mathscr{P}}(X) = C(X)$ and $X_{\mathscr{P}} = X$. Consider any non-empty proper clopen set $K$ in $X$. Then $1_K$ and $1_{X \setminus K}$ are both in $Z(C_{\mathscr{P}}(X))$, but $1_K + 1_{X \setminus K} = 1 \implies 1_K + 1_{X \setminus K} \notin Z(C_{\mathscr{P}}(X))$ and so $Z(C_{\mathscr{P}}(X))$ is not an ideal of $C_{\mathscr{P}}(X)$.

We mention in this context that for a reduced ring $R$ for which $Z(R)$ is an ideal of $R$, $W\Gamma(R) \neq \Gamma(R)$ and $gr(W\Gamma(R)) = 3$ [Theorem 2.4[18]]. Therefore the condition of $Z(R)$ being an ideal is sufficient but not necessary for Theorem 2.4 in [18].

The following theorem contains an exhaustive list of the lengths of all possible smallest cycles joining two distinct vertices.

**Theorem 3.6.** *Let* $f, g \in V_{\mathscr{P}}(X)$. *Then:*

1. $c(f,g) = 3$ *if and only if* $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$ *and* $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$.
2. $c(f,g) = 4$ *if and only if either* $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$ *and* $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$ *or* $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ *and* $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$.
3. $c(f,g) = 6$ *if and only if* $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ *and* $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$.

*Proof.* 1. It follows from Theorem 2.9 and Theorem 2.11(1).

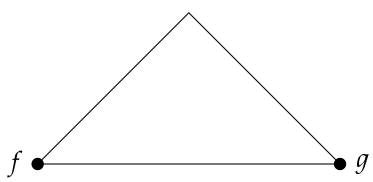
2. Let $c(f,g) = 4$. If $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$, then it follows from part (1) of this Theorem that $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$. On the other hand if $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$, then clearly $f$ and $g$ are non-adjacent. But since $c(f,g) = 4$, there exists a square of the form: $f - h - g - k - f$. Thus $f$ and $g$ have a common adjacent vertex ($h$ or $k$). It follows from Theorem 2.9 that $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$.

To prove the converse let the condition hold. Then it follows from part (1) of this Theorem that $c(f,g) > 3$. Now if $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$, then $f$ and $g$ are adjacent vertices, in which case $f - g - 2f - 2g - f$ is a 4-cycle containing $f$ and $g$. Hence $c(f,g) = 4$ in this case. On the other hand if $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$, then this yields in view of Theorem 2.9, there exists a vertex $h$, adjacent to both $f$ and $g$ while $f$ and $g$ are non-adjacent. These result the 4-cycle $f - h - g - 2h - f$. Hence $c(f,g) = 4$ in this case also.

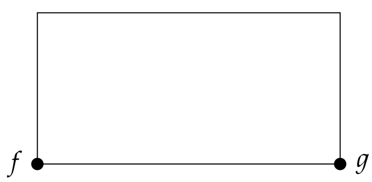
3. First assume that $c(f,g) = 6$. Then it follows from (1) and (2) of this Theorem that $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$.

Conversely let the conditions hold. Then it follows from (1) and (2) of this Theorem that $c(f,g) \neq 3$ and $c(f,g) \neq 4$, i.e., $c(f,g) > 4$. Now the assumed conditions imply in view of Theorem 2.11(3) that $d(f,g) = 3$. So there exists a path $f - l - k - g$ of length 3 joining $f$ and $g$. Surely then $f - l - k - g - 2k - 2l - f$ is a 6-cycle containing $f$ and $g$. To complete the proof it remains therefore to show that there does not exist any 5-cycle in this graph joining $f$ and $g$. We argue by contradiction. If possible let there exist a 5-cycle which is either of the form: $f - l - k - g - h - f$, taking care of $d(f,g) = 3$ or of the form: $f - l - k - h - g - f$, when there is a path of length 4 joining $f$ and $g$. The first possibility contradicts Theorem 2.9 while the second contradicts the observation that $d(f,g) = 3$. $\square$

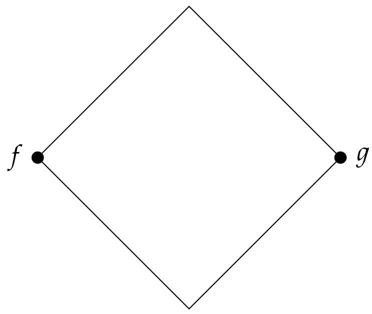
The following diagrams are the graphical representations of the above theorem.



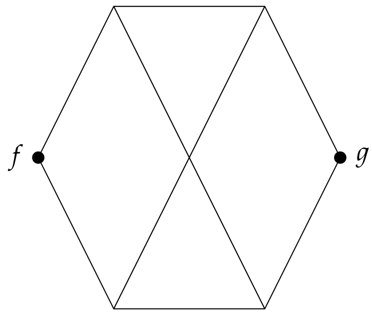

$X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$
$int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$




$X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$
$int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$




$X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$
$int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset$




$X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$
$int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$

The next Corollary directly follows from these diagrams.

**Corollary 3.7.**

1. *Each chord-less cycle in* $\Gamma_{\mathscr{P}}(X)$ *is of length* 3 *or* 4.
2. *Every edge in* $\Gamma_{\mathscr{P}}(X)$ *is either an edge of a triangle or an edge of a square.*

## 4. Relations between dominating number, chromatic number and clique number of $\Gamma_{\mathscr{P}}(X)$

We start observing the fact which relates the clique number of $\Gamma_{\mathscr{P}}(X)$ and the cellularity of the space $X_{\mathscr{P}}$ of $X$.

**Theorem 4.1.** $\omega(\Gamma_{\mathscr{P}}(X)) = c(X_{\mathscr{P}})$.

*Proof.* We shall first show that for an arbitrary complete subgraph $H$ of $\Gamma_{\mathscr{P}}(X)$, $|H| \leq c(X_{\mathscr{P}})$ and this will imply that $\omega(\Gamma_{\mathscr{P}}(X)) \leq c(X_{\mathscr{P}})$. If $V(H)$ is the set of all vertices in $H$, then for each $f \in V(H)$, $X \setminus Z(f)$ is a non-empty open set contained in $X_{\mathscr{P}}$. Consequently by Theorem 2.11(1), $\{X \setminus Z(f) : f \in V(H)\} \equiv \mathscr{B}$ becomes a cellular family in $X_{\mathscr{P}}$. Hence $|H| = |\mathscr{B}| \leq c(X_{\mathscr{P}})$. To prove the reverse inequality, $c(X_{\mathscr{P}}) \leq \omega(\Gamma_{\mathscr{P}}(X))$, it suffices to show for an arbitrary selected cellular family $\mathscr{B}$ in $X_{\mathscr{P}}$ that $|\mathscr{B}| \leq \omega(\Gamma_{\mathscr{P}}(X))$. Indeed for each set $B$ in the family $\mathscr{B}$, choose a point $x_B \in B$. Then by using Lemma 2.2 we can find out an $f_B \in C_{\mathscr{P}}(X)$ such that $x_B \in X \setminus Z(f_B) \subset cl_X(X \setminus Z(f_B)) \subset B$. If $H$ is a subgraph of $\Gamma_{\mathscr{P}}(X)$ where the set of vertices is $\{f_B : B \in \mathscr{B}\}$, then the cellularity of $\mathscr{B}$ conjoined with Theorem 2.11(1), therefore ensures that $H$ is a complete subgraph of $\Gamma_{\mathscr{P}}(X)$. This implies that $|\mathscr{B}| = |H| \leq \omega(\Gamma_{\mathscr{P}}(X))$. □

**Corollary 4.2.** *The clique number of the zero-divisor graph of* $C(X)$ *and the cellularity of* $X$ *are identical.*

*Proof.* This follows on choosing $\mathscr{P} \equiv$ the ideal of all closed sets in $X$. □

[This result is proved independently in [8], Proposition 3.1.]

Since the chromatic number of any graph is not less than its clique number, the following proposition is immediate:

**Theorem 4.3.** $\chi(\Gamma_{\mathscr{P}}(X)) \geq c(X_{\mathscr{P}})$.

The weight of a topological space $X$, denoted by $w(X)$, is the smallest of the cardinal numbers of the open bases for $X$.

**Theorem 4.4.** $dt(\Gamma_{\mathscr{P}}(X)) \leq w(X_{\mathscr{P}})$.

*Proof.* Let $\mathscr{B}$ be an open base for the subspace $X_{\mathscr{P}}$. It suffices to find out a dominating set $D$ in $\Gamma_{\mathscr{P}}(X)$ with $|D| \leq |\mathscr{B}|$. Let $B \in \mathscr{B}$ such that $B \neq \emptyset$, $B \neq X_{\mathscr{P}}$. Fix $x_B \in B$. Then by Lemma 2.2, there exists $f_B \in C_{\mathscr{P}}(X)$ such that $x_B \in X \setminus Z(f_B) \subset cl_X(X \setminus Z(f_B)) \subset B$. Since $B \subsetneq X_{\mathscr{P}}$, there exists a point $y \in X_{\mathscr{P}}$ such that $y \notin B$. It follows from the last inclusion relation that $y \in X - cl_X(X \setminus Z(f_B)) = int_X Z(f_B)$. Consequently, $y \in X_{\mathscr{P}} \cap int_X Z(f_B)$. Hence from Theorem 2.3, $f_B \in V_{\mathscr{P}}(X)$. Let $D = \{f_B : B \in \mathscr{B}\}$. We claim that $D$ is a dominating set in $\Gamma_{\mathscr{P}}(X)$. Towards that claim choose $f \in V_{\mathscr{P}}(X)$. Then from Theorem 2.3, $X_{\mathscr{P}} - cl_X(X \setminus Z(f))$ is a non-empty open set in $X_{\mathscr{P}}$. Therefore, there exists $B \in \mathscr{B}$ such that $B \subset X_{\mathscr{P}} - cl_X(X \setminus Z(f))$. Consequently $f_B.f = 0$. It is clear that $|D| \leq |\mathscr{B}|$, since the map: $\left.\begin{array}{ll} \mathscr{B} & \to D \\ B & \mapsto f_B \end{array}\right\}$ is onto $D$. □

**Definition 4.5.** In a graph $G$, two distinct vertices $u$ and $v$ are called orthogonal if $u$ and $v$ are adjacent and there is no third vertex adjacent to $u$ and $v$ both. In this case, we write $u \perp v$. $G$ is called complemented if given a vertex $u$ in $G$, there exists a vertex $v$ in $G$ such that $u \perp v$.

The following result is a consequence of Theorem 2.9 and Theorem 2.11.

**Theorem 4.6.** *$\Gamma_{\mathscr{P}}(X)$ is complemented if and only if given $f \in V_{\mathscr{P}}$, there exists $g \in V_{\mathscr{P}}$ such that $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$ and $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset$.*

It follows from Theorem 2.29 that if $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$, then $\Gamma_{\mathscr{P}}(X)$ is not complemented. This means that for a complemented graph $\Gamma_{\mathscr{P}}(X)$, $cl_X(X_{\mathscr{P}}) \in \mathscr{P}$. On choice of $\mathscr{P} \equiv$ the ideal of all closed sets in $X$, we observe that $cl_X(X_{\mathscr{P}}) \in \mathscr{P}$ and $C_{\mathscr{P}}(X) = C(X)$. Therefore the zero-divisor graph of $C(X)$ is a candidate for a complemented graph. Indeed the following fact is proved in [8], Corollary 2.5.

**Theorem 4.7.** *The zero-divisor graph of $C(X)$ is complemented if and only if the space of minimal prime ideals in $C(X)$ is compact.*

We are now going to establish that this Theorem can be deduced as a special case of the more general Theorem which says that for any ideal $\mathscr{P}$ of closed sets in $X$, $\Gamma_{\mathscr{P}}(X)$ is complemented if and only if the space of minimal prime ideals of the ring $C_{\mathscr{P}}(X)$ is compact. We need a little bit of technicalities to arrive at this result.

We reproduce from [16] the following basic information related to the space of minimal prime ideals of a commutative ring $A$ (possibly without identity), which is further reduced in the sense that there does not exist any non-zero nilpotent member of $A$. A prime ideal $P$ of $A$ is called a minimal prime ideal if there does not exist any prime ideal $Q$ of $A$ such that $Q \subsetneq P$. It is easy to check on using Zorn's Lemma in a straight forward manner that if $P$ is a prime ideal in $A$, then there is a minimal prime ideal $Q$ in $A$ such that $Q \subset P$. Let $\mathcal{P}(A)$ be the set of all minimal prime ideals in $A$. For any subset $S$ of $A$, the hull of $S$, denoted by $h(S)$, is defined as $h(S) = \{P \in \mathcal{P}(A) : S \subset P\}$. If $S$ is a single point $= \{s\}$, then we write $h(s)$ instead of $h(\{s\})$. It turns out that the family $\{h(a) : a \in A\}$ is a base for the closed sets for some topology on $\mathcal{P}(A)$. $\mathcal{P}(A)$ equipped with this topology is often called the space of minimal prime ideals in $A$. For any subset $S$ of $A$, the set $\mathcal{A}(S) = \{b \in A : bS = 0\}$ is called the annihilator of $S$. We just state the following results which are already proved in [16].

**Theorem 4.8.** *For each member $a$ of $A$, $h(\mathcal{A}(a)) = \mathcal{P}(A) \setminus h(a)$. In particular therefore $h(a)$ is a clopen set in $\mathcal{P}(A)$. Consequently $\mathcal{P}(A)$ becomes a zero-dimensional space and it is easy to prove that $\mathcal{P}(A)$ is also Hausdorff.*

**Theorem 4.9.** *For any subset $S$ of $A$, $\mathcal{A}(S) =$ the intersection of all minimal prime ideals in $A$, which contains $S$.*

**Theorem 4.10.** *For any two points $x, y$ in $A$, $\mathcal{A}(\mathcal{A}(x)) = \mathcal{A}(y)$ if and only if $h(x) = h(\mathcal{A}(y))$.*

$A$ is said to satisfy the annihilator condition or is called an a.c. ring if for $x, y \in A$, there exists $z \in A$ such that $\mathcal{A}(z) = \mathcal{A}(x) \cap \mathcal{A}(y)$. Given $x \in A$, an element $x' \in A$ is called a complement of $x$ if $\mathcal{A}(\mathcal{A}(x')) = \mathcal{A}(x)$. It is easy to see that if $x'$ is a complement of $x$, then $x$ is a complement of $x'$.

The following result relates the existence of complement of each element of $A$ with the compactness of the space of minimal prime ideals of $A$.

**Theorem 4.11.** *The following statements are equivalent for the ring $A$.*

1. *The space $\mathcal{P}(A)$ is compact and $A$ is an a.c. ring.*
2. *Each member of $A$ has a complement.*

We reproduce the following results which appeared as Lemma 1.2 and Theorem 0.1 in [3] and also in [17], Example 10, 4.9, page 66.

**Theorem 4.12.** *If $M$ is an ideal of $A$ such that the quotient ring $A/M$ is a field, then $M$ is a maximal ideal in $A$.*

**Theorem 4.13.** *If $A = A^2 \equiv \{\sum_{i=1}^{n} a_i.b_i : a_i, b_i \in A, n \in \mathbb{N}\} \equiv$ the internal direct product of $A$ with itself, then every maximal ideal in $A$ is prime.*

With the convention $|X_{\mathscr{P}}| \geq 2$ made in 2.18, we first show that the space $\mathcal{P}(C_{\mathscr{P}}(X))$ of minimal prime ideals of the ring $C_{\mathscr{P}}(X)$ is non-empty. For that purpose, choose a point $x \in X_{\mathscr{P}}$.

**Theorem 4.14.** *The ideal $M_x^{\mathscr{P}} = \{f \in C_{\mathscr{P}}(X) : f(x) = 0\}$ is a maximal ideal in $C_{\mathscr{P}}(X)$.*

*Proof.* Let $t : C_{\mathscr{P}}(X) \to \mathbb{R}$ be the map defined by $t(f) = f(x)$. Then $t$ is a ring homomorphism. We assert that $t$ is onto $\mathbb{R}$; indeed by Lemma 2.2, there exists $f \in C_{\mathscr{P}}(X)$ such that $f(x) \neq 0$. Consequently given $r \in \mathbb{R}$, the function $\frac{r}{f(x)}.f \in C_{\mathscr{P}}(X)$ and $t(\frac{r}{f(x)}.f) = r$. Therefore the residue class ring of $C_{\mathscr{P}}(X)$ modulo the kernel of $t$ becomes isomorphic to $\mathbb{R}$. It follows that $C_{\mathscr{P}}(X)/ker(t)$ is a field. Hence by Theorem 4.12, $M_x^{\mathscr{P}} = ker(t)$, is a maximal ideal in $C_{\mathscr{P}}(X)$. □

It can be easily checked by using the notion of maximality of ideals in a ring that each fixed maximal ideal $M$ in $C_{\mathscr{P}}(X)$ is of the form $M_x^{\mathscr{P}}$ for some point $x \in X$ [a maximal ideal $M$ is called a fixed maximal ideal in $C_{\mathscr{P}}(X)$ if there exists a point $y \in X$ such that $g(y) = 0$ for all $g \in M$].

Since for any $g \in C_{\mathscr{P}}(X)$, $g^{\frac{1}{3}}$ and $g^{\frac{2}{3}}$ also belong to $C_{\mathscr{P}}(X)$ and $g = g^{\frac{1}{3}}.g^{\frac{2}{3}}$, it follows that $C_{\mathscr{P}}(X)$ is identical with the internal direct product with itself. Hence in view of Theorem 4.12, we can make the following comment.

**Remark 4.15.** $M_x^{\mathscr{P}}$ is a prime ideal in $C_{\mathscr{P}}(X)$. Consequently, $\mathcal{P}(C_{\mathscr{P}}(X)) \neq \emptyset$.

Before proceeding further, we make the simple observation that $C_{\mathscr{P}}(X)$ is an a.c. ring, because for $f, g \in C_{\mathscr{P}}(X)$, $\mathcal{A}(f) \cap \mathcal{A}(g) = \mathcal{A}(f^2 + g^2)$.

The following subsidiary result will be helpful to us towards proving the main result of this section.

**Theorem 4.16.** *Let $f, g \in C_{\mathscr{P}}(X)$. Then:*

1. *$h(g) \supset h(\mathcal{A}(f))$ if and only if $f.g = 0$.*
2. *$h(g) \subset h(\mathcal{A}(f))$ if and only if $X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(g) = \emptyset$.*

[Lemma 5.4 in [16] is a special case of this Theorem on choosing $\mathscr{P} \equiv$ the ideal of all closed sets in $X$.]

*Proof.* 1. If $f.g = 0$, then it is clear that $g \in \mathcal{A}(f)$ and consequently $h(g) \supset h(\mathcal{A}(f))$. Conversely let $h(g) \supset h(\mathcal{A}(f))$. Then we claim in view of Theorem 4.9 that $g \in \mathcal{A}(f)$ and hence $f.g = 0$.

2. From Theorem 4.8, we have $h(\mathcal{A}(f)) = \mathcal{P}(C_{\mathscr{P}}(X)) \setminus h(f)$. Therefore $h(g) \subset h(\mathcal{A}(f))$ if and only if $h(g) \subset \mathcal{P}(C_{\mathscr{P}}(X)) \setminus h(f)$, this holds if and only if $h(g) \cap h(f) = \emptyset$ and this is the case when and only when $h(f^2 + g^2) = \emptyset$ meaning that $\mathcal{P}(C_{\mathscr{P}}(X)) \setminus h(f^2 + g^2) = \mathcal{P}(C_{\mathscr{P}}(X))$, which is the same thing in view of Theorem 4.8 as $h(\mathcal{A}(f^2 + g^2)) = \mathcal{P}(C_{\mathscr{P}}(X))$. This means that $\mathcal{A}(f^2 + g^2) \subset P$ for each $P \in \mathcal{P}(C_{\mathscr{P}}(X))$, equivalently $\mathcal{A}(f^2 + g^2) = \{0\}$, because $C_{\mathscr{P}}(X)$ is a reduced ring where the intersection of all minimal prime ideals is the zero ideal. Now $\mathcal{A}(f^2 + g^2) = \{0\}$ if and only if $f^2 + g^2$ is not a divisor of zero in $C_{\mathscr{P}}(X)$. This happens in view of Theorem 2.3, when and only when $int_X Z(f^2 + g^2) \cap X_{\mathscr{P}} = \emptyset$ meaning $X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(g) = \emptyset$. □

A consequence of this theorem is as follows:

**Theorem 4.17.** *Given $f \in C_{\mathscr{P}}(X)$, a function $f' \in C_{\mathscr{P}}(X)$ is a complement of $f$ in this ring if and only if $f.f' = 0$ and $X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(f') = \emptyset$.*

*Proof.* $f'$ is a complement of $C_{\mathscr{P}}(X)$ if and only if $\mathcal{A}(\mathcal{A}(f')) = \mathcal{A}(f)$. In view of Theorem 4.10, this is equivalent to the statement that $h(f') = h(\mathcal{A}(f))$. If we now apply the result of Theorem 4.16, then we see that the last equality is equivalent to the statements that: $f.f' = 0$ and $X_{\mathscr{P}} \cap int_X Z(f) \cap int_X Z(f') = \emptyset$. □

**Theorem 4.18.** *The zero-divisor graph $\Gamma_{\mathscr{P}}(X)$ of $C_{\mathscr{P}}(X)$ is complemented if and only if the space $\mathcal{P}(C_{\mathscr{P}}(X))$ of all minimal prime ideals of $C_{\mathscr{P}}(X)$ is compact.*

*Proof.* It follows by combining Theorem 4.6 and Theorem 4.17 that $\Gamma_{\mathscr{P}}(X)$ is complemented if and only if each function $f$ in the ring $C_{\mathscr{P}}(X)$ has a complement. We now apply Theorem 4.11(1) to ensure that this last statement holds if and only if $\mathcal{P}(C_{\mathscr{P}}(X))$ is compact. □

An additional observation.

**Remark 4.19.** It follows from Theorem 2.29 that if $X$ is a locally compact non-compact space (respectively a locally pseudocompact non-pseudocompact space), then the zero-divisor of $C_K(X)$ (respectively $C_\psi(X)$) is hypertriangulated and therefore not complemented. It follows from Theorem 4.18 that the space $\mathcal{P}(C_K(X))$ of minimal prime ideals of such a $C_K(X)$ (respectively, the space $\mathcal{P}(C_\psi(X))$ of minimal prime ideals of $C_\psi(X)$) is non-compact. This is an instance of how a graph theoretic result leads to a result in topology.

A complemented graph $G$ is called uniquely complemented, whenever $u \perp v$ and $u \perp w$ for any three vertices $u, v, w$ in $G$, then $v$ and $w$ are adjacent to exactly the same vertices.

**Theorem 4.20.** *If $\Gamma_{\mathscr{P}}(X)$ is complemented, then it is uniquely complemented.*

*Proof.* Let $\Gamma_{\mathscr{P}}(X)$ be complemented and $f \perp g$, $f \perp h$ for some $f, g, h \in V_{\mathscr{P}}(X)$. We claim that $\mathcal{A}(g) = \mathcal{A}(h)$, where $\mathcal{A}(g) = \{l \in C_{\mathscr{P}}(X) : l.g = 0\}$. If we establish our claim, then $g$ and $h$ are adjacent to exactly the same vertices and hence $\Gamma_{\mathscr{P}}(X)$ is uniquely complemented. Since $f \perp g$, by Theorem 4.6, $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset \implies X \setminus Z(g) \cap cl_X(X \setminus Z(f)) = \emptyset$ and $int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset \implies X_{\mathscr{P}} - (cl_X(X \setminus Z(f)) \cup cl_X(X \setminus Z(g))) = \emptyset \implies X_{\mathscr{P}} \subset cl_X(X \setminus Z(f)) \cup cl_X(X \setminus Z(g))$. Similarly $X \setminus Z(h) \cap cl_X(X \setminus Z(f)) = \emptyset$ and $X_{\mathscr{P}} \subset cl_X(X \setminus Z(f)) \cup cl_X(X \setminus Z(g))$. Now let $l \in \mathcal{A}(g)$, then $l.g = 0 \implies X \setminus Z(l) \cap cl_X(X \setminus Z(g)) = \emptyset \implies X \setminus Z(l) \subset cl_X(X \setminus Z(f)) \implies X \setminus Z(l) \cap X \setminus Z(h) = \emptyset \implies l.h = 0 \implies l \in \mathcal{A}(h)$, i.e., $\mathcal{A}(g) \subset \mathcal{A}(h)$. Similarly $\mathcal{A}(h) \subset \mathcal{A}(g)$ and hence $\mathcal{A}(g) = \mathcal{A}(h)$. □

The next result directly follows from Theorem 4.18 and Theorem 4.20.

**Corollary 4.21.** *The zero-divisor graph $\Gamma_{\mathscr{P}}(X)$ of $C_{\mathscr{P}}(X)$ is uniquely complemented if and only if the space $\mathcal{P}(C_{\mathscr{P}}(X))$ of all minimal prime ideals of $C_{\mathscr{P}}(X)$ is compact.*

## 5. The zero-divisor graph of $C_\infty^{\mathscr{P}}(X)$

Let $\Gamma_\infty^{\mathscr{P}}(X)$ stand for the zero-divisor graph of $C_\infty^{\mathscr{P}}(X)$ whose set of vertices is the aggregate of all non-zero zero divisors of this ring and two distinct vertices $f$ and $g$ are adjacent if and only if $f.g = 0$. Let $V_\infty^{\mathscr{P}}(X)$ be the set of vertices of $\Gamma_\infty^{\mathscr{P}}(X)$. It can be realized without any difficulty that most of the results related to the zero-divisor graph $\Gamma_{\mathscr{P}}(X)$ of $C_{\mathscr{P}}(X)$ have their analogs for the zero-divisor graph $\Gamma_\infty^{\mathscr{P}}(X)$ of $C_\infty^{\mathscr{P}}(X)$. Since the proof of these later results are also parallel to the proof of their corresponding counterparts involving $\Gamma_{\mathscr{P}}(X)$, as obtained in Section 2, 3 and 4, we simply omit them. However we state all these parallel facts for $\Gamma_\infty^{\mathscr{P}}(X)$, for our convenience.

**Theorem 5.1.** *For each $f \in C_\infty^{\mathscr{P}}(X)$, $X \setminus Z(f) \subset X_{\mathscr{P}}$ and an $f \in C_\infty^{\mathscr{P}}(X)$ is a member of $V_\infty^{\mathscr{P}}(X)$ if and only if $X_{\mathscr{P}} \cap int_X Z(f) \neq \emptyset$.*

**Remark 5.2.** Analogously, we can show that the extended zero-divisor graph, the generalized zero-divisor graph and the zero-divisor graph of $C_\infty^{\mathscr{P}}(X)$ all are equal. Similarly, the weakly zero-divisor graph and the zero-divisor graph of $C_\infty^{\mathscr{P}}(X)$ are equal if and only if $|X_{\mathscr{P}}| = 2$.

**Theorem 5.3.** *An $f \in V_\infty^{\mathscr{P}}(X)$ is a vertex of a triangle in $\Gamma_\infty^{\mathscr{P}}(X)$ if and only if $|X_{\mathscr{P}} \cap int_X Z(f)| \geq 2$.*

**Theorem 5.4.** *If $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$ and every finite set in $X$ is a member of $\mathscr{P}$, then $\Gamma_\infty^{\mathscr{P}}(X)$ is triangulated.*

**Theorem 5.5.** *If $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$, then $\Gamma_\infty^{\mathscr{P}}(X)$ is hypertriangulated.*

**Theorem 5.6.** *Let $f, g \in V_\infty^{\mathscr{P}}(X)$. Then:*

$$d(f,g) = \begin{cases} 1 & \text{if } f.g = 0 \\ 2 & \text{if } f.g \neq 0 \text{ and } int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) \neq \emptyset \\ 3 & \text{if } f.g \neq 0 \text{ and } int_X(X_{\mathscr{P}} \cap Z(f) \cap Z(g)) = \emptyset \end{cases}$$

**Theorem 5.7.** $3 \leq gr(\Gamma_\infty^{\mathscr{P}}(X)) \leq 4$.

**Theorem 5.8.** *For $f, g \in V_\infty^{\mathscr{P}}(X)$, $c(f,g)$ is either 3 or 4 or 6.*

**Theorem 5.9.** *$C_\infty^{\mathscr{P}}(X)$ is an a.c. ring and $\mathcal{P}(C_\infty^{\mathscr{P}}(X)) \neq \emptyset$. Furthermore, $\Gamma_\infty^{\mathscr{P}}(X)$ is uniquely complemented if and only if $\mathcal{P}(C_\infty^{\mathscr{P}}(X))$ is a compact space.*

**Theorem 5.10.** $\omega(\Gamma_\infty^{\mathscr{P}}(X)) = \omega(\Gamma_{\mathscr{P}}(X)) = c(X_{\mathscr{P}})$.

**Theorem 5.11.** $dt(\Gamma_\infty^{\mathscr{P}}(X)) \leq w(X_{\mathscr{P}})$.

We now state and establish two new results in this section which connect $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_\infty^{\mathscr{P}}(X)$.

**Theorem 5.12.** *$V_{\mathscr{P}}(X)$ is a dominating set in the graph $\Gamma_\infty^{\mathscr{P}}(X)$.*

*Proof.* Let $f \in V_\infty^{\mathscr{P}}(X)$. Then from Theorem 5.1, we can choose a point $x \in X_{\mathscr{P}} \cap int_X Z(f)$. From Lemma 2.2, we can have a $g \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(g) \subset cl_X(X \setminus Z(g)) \subset X_{\mathscr{P}} \cap int_X Z(f)$. It follows that $g \in V_{\mathscr{P}}(X)$ and $g.f = 0$, thus $g$ is adjacent to $f$. □

**Theorem 5.13.** $\chi(\Gamma_\infty^{\mathscr{P}}(X)) = \chi(\Gamma_{\mathscr{P}}(X))$.

*Proof.* Since $\Gamma_{\mathscr{P}}(X)$ is a subgraph of $\Gamma_\infty^{\mathscr{P}}(X)$, it is plain that $\chi(\Gamma_\infty^{\mathscr{P}}(X)) \geq \chi(\Gamma_{\mathscr{P}}(X))$. To prove the reverse inequality let $f \in V_\infty^{\mathscr{P}}(X) \setminus V_{\mathscr{P}}(X)$. Then $X \setminus Z(f) \neq \emptyset$. Choose $x \in X \setminus Z(f)$. As $X \setminus Z(f) \subset X_{\mathscr{P}}$, it is clear that $x \in X_{\mathscr{P}}$. By Lemma 2.2, there exists $g \in C_{\mathscr{P}}(X)$ such that $x \in X \setminus Z(g) \subset cl_X(X \setminus Z(g)) \subset X \setminus Z(f)$. Let $A_f = \{g \in V_{\mathscr{P}}(X) : X \setminus Z(g) \subset cl_X(X \setminus Z(g)) \subset X \setminus Z(f)\}$. Then $A_f \neq \emptyset$ as observed above and $g \in A_f$ implies that $f$ and $g$ are non-adjacent as $f.g \neq 0$ in this case. Now there already exists a coloring of the vertices $V_{\mathscr{P}}(X)$ of $\Gamma_{\mathscr{P}}(X)$ by $\chi(\Gamma_{\mathscr{P}}(X))$ many colors. We want to extend this coloring to color the entire set of vertices $V_\infty^{\mathscr{P}}(X)$ in $\Gamma_\infty^{\mathscr{P}}(X)$ in a consistent manner. Indeed for any $f \in V_\infty^{\mathscr{P}}(X) \setminus V_{\mathscr{P}}(X)$, we color $f$, by the coloring of any chosen member of $A_f$. Once this assignment of colors to the members of $V_\infty^{\mathscr{P}}(X)$ is proved to be consistent, it will follow that the set of vertices $V_\infty^{\mathscr{P}}(X)$ in $\Gamma_\infty^{\mathscr{P}}(X)$ can be colored by the already existing colors needed to color $V_{\mathscr{P}}(X)$ and hence $\chi(\Gamma_\infty^{\mathscr{P}}(X)) \leq \chi(\Gamma_{\mathscr{P}}(X))$. Towards the proof of the consistency of the above method of coloring, suppose $h \in V_\infty^{\mathscr{P}}(X)$ and $f \in V_\infty^{\mathscr{P}}(X) \setminus V_{\mathscr{P}}(X)$ have the same color as that of $g \in V_{\mathscr{P}}(X)$. It suffices to show that $h$ and $f$ are non-adjacent. For that purpose we need to show first that $h$ and $g$ are non-adjacent. If $h \in V_{\mathscr{P}}(X)$, then surely $h$ and $g$ are non-adjacent because there is already a (consistent) coloring of $V_{\mathscr{P}}(X)$. On the other hand if $h \in V_\infty^{\mathscr{P}}(X) \setminus V_{\mathscr{P}}(X)$, then $g \in A_g$ and hence $g$ and $h$ are non-adjacent. Therefore, $X \setminus Z(g.h) \neq \emptyset$. Also $g \in A_f$ because $f$ is colored by the color of $g$. Hence $X \setminus Z(g) \subset X \setminus Z(f)$. Consequently, $\emptyset \neq X \setminus Z(g.h) = X \setminus Z(g) \cap X \setminus Z(h) \subset X \setminus Z(f) \cap X \setminus Z(h) = X \setminus Z(f.h)$. Thus $f.h \neq 0$ and hence $f$ and $h$ are non-adjacent. □

## 6. $\Gamma_{\mathscr{P}}(X)$ for a special choice of $\mathscr{P}$

The main result in this concluding section of the present article is to prove a Banach-Stone like theorem, which tells that for appropriate choices of $\mathscr{P}$ and $\mathscr{Q}$, a graph isomorphism between $\Gamma_{\mathscr{P}}(X)$ and $\Gamma_{\mathscr{Q}}(Y)$ will lead to an isomorphism between the rings $C_{\mathscr{P}}(X)$ and $C_{\mathscr{Q}}(Y)$. Indeed we let $\mathscr{P}$ to be the ideal of all finite sets in $X$ and write $C_{\mathscr{P}}(X) = C_F(X) = \{f \in C(X) : X \setminus Z(f) \text{ is a finite set in } X\}$. Let $\Gamma_F(X)$ denote the zero-divisor graph of $C_F(X)$ and $V_F(X)$, the set of vertices of this graph.

Essentially we shall show that the ring structure of $C_F(X)$ is uniquely determined by the graph structure of $\Gamma_F(X)$. We see that with this special choice of $\mathscr{P}$, $X_{\mathscr{P}} = K_X \equiv$ the set of all isolated points in $X$ with the Convention 2.18, therefore $|K_X| \geq 2$. Since for each $f \in C_F(X)$, $Z(f)$ is a clopen subset of $X$ and therefore, $X \setminus Z(f) \subset K_X$. The following special cases of Theorem 2.3 and Theorem 2.11 are recorded below for our immediate need.

**Theorem 6.1.** 1. *An $f(\neq 0) \in C_F(X)$ is a member of $V_F(X)$ if and only if $K_X \cap Z(f) \neq \emptyset$. Thus for $f \in C_F(X)$, $\emptyset \neq X \setminus Z(f) \subsetneq K_X$ if and only if $f \in V_F(X)$.*

2. *Let $f, g \in V_F(X)$. Then:*
   a) *$d(f,g) = 1$ if and only if $X \setminus Z(f) \cap X \setminus Z(g) = \emptyset$.*
   b) *$d(f,g) = 2$ if and only if $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $K_X \cap Z(f) \cap Z(g) \neq \emptyset$.*
   c) *$d(f,g) = 3$ if and only if $X \setminus Z(f) \cap X \setminus Z(g) \neq \emptyset$ and $K_X \cap Z(f) \cap Z(g) = \emptyset$.*

The following properties determine in some cases the centers of the graph $\Gamma_F(X)$.

**Theorem 6.2.** *Let $K_X$ be finite and $f \in V_F(X)$. Then $e(f) = 2$ if and only if $X \setminus Z(f)$ is one-pointic set. Therefore $c(\Gamma_F(X)) \equiv$ the center of $\Gamma_F(X) = \{r1_x : x \in K_X, r \in \mathbb{R} \setminus \{0\}\}$.*

*Proof.* We prove only the first part of this Theorem, because the second part follows immediately from the first part. Assume that $X \setminus Z(f) = \{x\}$ for some $x \in X$. Let $g \in V_F(X)$. If $x \in Z(g)$, Then $f.g = 0$ so that from Theorem 6.1(2a), $d(f,g) = 1$. Suppose that $x \notin Z(g)$. Then $x \in X \setminus Z(g) \cap X \setminus Z(f)$. On the other hand we get from Theorem 6.1(1) that $X \setminus Z(g) \subsetneq K_X$. Hence $X \setminus Z(f) \cup X \setminus Z(g) = X \setminus Z(g) \subsetneq K_X$. This implies that $K_X \cap Z(f) \cap Z(g) \neq \emptyset$. It follows from Theorem 6.1(2b) that $d(f,g) = 2$. Thus $e(f) = 2$.

To prove the converse part suppose that $X \setminus Z(f)$ contains atleast two points $x, y (x \neq y)$. We shall find out a $g \in V_F(X)$ such that $d(f,g) = 3$ and that finishes the Theorem. Indeed take $K_1 = (Z(f) \cap K_X) \cup \{x\}$. Then since $y \in X \setminus Z(f) \subset K_X$, it follows that $y \notin K_1$, thus $K_1 \subsetneq K_X$. It is easy to verify that $K_X = K_1 \cup (X \setminus Z(f))$. Take $g = 1_{K_1} \equiv$ the characteristic function of the set $K_1$. Then $g$ is a continuous function on $X$ as $K_X$ is a finite subset and hence $g \in V_F(X)$. So $g(x) \neq 0$ and $f(x) \neq 0$ imply that $X \setminus Z(g) \cap X \setminus Z(f) \neq \emptyset$. Furthermore $Z(f) \cap Z(g) \cap K_X = \emptyset$. It follows from Theorem 6.1(2c) that $d(f,g) = 3$. □

Earlier we mention in Corollary 2.5 that whenever $cl_X(X_{\mathscr{P}}) \notin \mathscr{P}$, then every non-zero element of $C_{\mathscr{P}}(X)$ is a vertex of $\Gamma_{\mathscr{P}}(X)$. On choosing $\mathscr{P} \equiv$ the ideal of all finite sets in $X$, the converse is also true.

**Theorem 6.3.** *Every non-zero element of $C_F(X)$ is a vertex of $\Gamma_F(X)$ if and only if $cl_X(K_X) \notin \mathscr{P}$. In other words, every non-zero element of $C_F(X)$ is a vertex of $\Gamma_F(X)$ if and only if $K_X$ is infinite.*

*Proof.* If $K_X$ is infinite, so is also $cl_X(K_X)$ and hence $cl_X(K_X) \notin \mathscr{P}$. So by Corollary 2.5, every non-zero element of $C_F(X)$ is a vertex of $\Gamma_F(X)$. For the converse part, it suffices to show that whenever $K_X$ is finite, then we can find a non-zero element in $C_F(X)$ which is not a vertex of $\Gamma_F(X)$. Let $K_X$ be finite. Then $K_X$ is clopen. Consider $f = 1_{K_X} \in C(X)$. Then $f \neq 0$ and $X \setminus Z(f) = K_X$, which implies that, $f \in C_F(X)$. From Theorem 6.1(1), it follows that $f \notin V_F(X)$. □

The next result follows from the Theorem 4.1 and the fact that the cellularity of a discrete space is its cardinality.

**Theorem 6.4.** $\omega(\Gamma_F(X)) = c(K_X) = |K_X|$.

The next proposition sharpens the result in Theorem 4.3.

**Theorem 6.5.** $\chi(\Gamma_F(X)) = |K_X|$

*Proof.* It follows from Theorem 4.3 on choosing $\mathscr{P} \equiv$ the ideal of all finite sets in $X$ that $\chi(\Gamma_F(X)) \geq |K_X|$. So it remains to prove only the reversed inequality. We first observe that, because of the Convention $|K_X| \geq 2$, for any point $x \in K_X$, $1_x \in V_F(X)$. This follows directly from Theorem 6.1(1). Let $H$ be the subgraph of $\Gamma_F(X)$ with its set of vertices $V(H) = \{1_x : x \in K_X\}$. Any pair of distinct vertices in $V(H)$ are adjacent because $x, y \in K_X, x \neq y \implies 1_x.1_y = 0$. So there is essentially one coloring of this graph $H$, which assigns different colors to different vertices. Let us denote the color assigned to the vertex $1_x$ by notation $x$, $x \in K_X$. We now extend the coloring on $V(H)$, to a coloring on the whole graph $\Gamma_F(X)$ without breaking the consistency with the following property that, the colors $\{x : x \in K\}$, which already exist to color $V(H)$ are adequate enough to color the vertices of $\Gamma_F(X)$. Once such an extension is done, the Theorem finishes thereon. For that purpose choose $f \in V_F(X)$. So $X \setminus Z(f)$ is a non-void finite set, say $X \setminus Z(f) = \{x_1, x_2, ..., x_n\}$. Let us color $f$ be any $x_i$ $(1 \leq i \leq n)$ (Surely each $x_j \in K_X$, $1 \leq j \leq n$, by Theorem 6.1(1)). We claim that we have already defined a consistent coloring on $V_F(X)$. So to prove this claim choose any $g \in V_F(X)$ such that $f$ and $g$ are adjacent. This means that $f.g = 0$ and hence $g(x_i) = 0$ because $f(x_i) \neq 0$. Thus $x_i \notin X \setminus Z(g)$ and therefore by the above mode of coloring, $g$ is colored by some element $x$ of $K_X$ with $x \neq x_i$ for each $i \in \{1, 2, ...n\}$. Hence the above coloring on $V_F(X)$ is consistent. □

**Remark 6.6.** A graph is said to be locally finite if every vertex of the graph is adjacent with only finitely many vertices. Clearly, the graph $\Gamma_F(X)$ is not locally finite, because if $f \in V_F(X)$, then there exists $g \in V_F(X)$ such that $f.g = 0$ and this imply that $f.(rg) = 0$ for each $r \in \mathbb{R} \setminus \{0\}$. Now if we consider the topological space $X$ such that $K_X$ is finite with $|K_X| \geq 2$, then $\Gamma_F(X)$ is an example of an infinite graph which is not locally finite but $\Gamma_F(X)$ is finitely colorable.

Combining the Theorems 6.4 and 6.5, we get the following result.

**Corollary 6.7.** $\omega(\Gamma_F(X)) = |K_X| = \chi(\Gamma_F(X))$, *i.e.,* $\Gamma_F(X)$ *is a weakly perfect graph.*

It follows that the infinite ring $C_F(X)$ satisfies the well-known Beck conjecture viz., for a commutative ring $R$, $\chi(R) = \omega(R)$ [Conjecture 1, [11]].

We are now ready to prove the main result of this section.

**Theorem 6.8.** *Let $X$ and $Y$ be two Tychonoff spaces with $|K_X| \geq 2$ and $|K_Y| \geq 2$. Then the ring $C_F(X)$ is isomorphic to the ring $C_F(Y)$ if and only if the graph $\Gamma_F(X)$ is isomorphic to the graph $\Gamma_F(Y)$.*

*Proof.* If $C_F(X)$ is isomorphic to $C_F(Y)$, then it is trivial that $\Gamma_F(X)$ is isomorphic to $\Gamma_F(Y)$. Assume that there exists a graph isomorphism $\psi : \Gamma_F(X) \to \Gamma_F(Y)$ (onto $\Gamma_F(Y)$). Now if $f \in C_F(X)$, then $X \setminus Z_X(f)$ is a finite subset of $K_X$, say $X \setminus Z_X(f) = \{x_1, x_2, ..., x_n\}$. So we can write $f = \sum_{i=1}^{n} f(x_i).1_{x_i}$. Formulation of an isomorphism from $C_F(X)$ onto $C_F(Y)$ is therefore guided by the following procedure: we first define a bijection on the set $K_X$ onto the set $K_Y$ and extend it linearly to a bijective map on $C_F(X)$ onto $C_F(Y)$ so that it ultimately becomes a ring isomorphism. However, we must assert before proceeding further that $|K_X| = |K_Y|$. Indeed the chromatic number of a graph is invariant under graph isomorphism, it follows that $\chi(\Gamma_F(X)) = \chi(\Gamma_F(Y))$. Hence from Theorem 6.5, $|K_X| = |K_Y|$. Now choose $x \in K_X$, then $1_x \in V_F(X)$, as observed in the beginning of proof of Theorem 6.5. Consequently $\psi(1_x) \in V_F(Y)$ and let $g = \psi(1_x)$. It follows from Theorem 6.1(1) that $Y \setminus Z_Y(g) \subsetneq K_Y$. We claim that $Y \setminus Z_Y(g)$ is a singleton.

If $K_X$ is a finite set, $K_Y$ is also a finite set and from Theorem 6.2, we get that $e(1_x) = 2$. Since $\psi$ is a graph isomorphism it follows that $e(\psi(1_x)) = 2$, i.e., $e(g) = 2$. We apply once again Theorem 6.2, to ascertain that $Y \setminus Z_Y(g)$ is a singleton.

Assume therefore that $K_X$ is an infinite set (and therefore $K_Y$ is an infinite set). If possible let there exist two distinct points $y_1, y_2 \in Y \setminus Z_Y(g)$. Since $Y \setminus Z_Y(g) \subsetneq K_Y$, so we can choose a point $y \in K_Y \cap Z_Y(g)$. Then $g - 1_y - 1_{y_1} - 1_{y_2} - 2.1_y - g$ ia a 5-cycle in $\Gamma_F(Y)$. As $\psi^{-1} : \Gamma_F(Y) \to \Gamma_F(X)$ is a graph isomorphism, it follows that $1_x = \psi^{-1}(g)$ is a vertex of a 5-cycle in $\Gamma_F(X)$, say: $1_x - f_1 - f_2 - f_3 - f_4 - 1_x$. Since in this cycle $f_2$ and $1_x$ are not adjacent, it follows that $f_2.1_x \neq 0$ and hence $f_2(x) \neq 0$. By an identical reasoning $f_3(x) \neq 0$. Consequently, $f_2.f_3 \neq 0$, i.e., $f_2$ and $f_3$ are not adjacent – this contradicts the adjacency of $f_2$ and $f_3$ in the last cycle.

Thus we realize that for the chosen $x \in K_X$, $Y \setminus Z_Y(\psi(1_x))$ is a one-pointic set, say $Y \setminus Z_Y(\psi(1_x)) = \{y\}$, for some $y \in Y$ (eventually $y \in K_Y$, as $Y \setminus Z_Y(\psi(1_x)) \subset K_Y$). This implies that $\psi(1_x) = c.1_y$ for some non-zero $c \in \mathbb{R}$. We now set $\phi(x) = y$. The map $\phi : K_X \to K_Y$ thus defined without ambiguity is certainly one-to-one and onto $K_Y$. Indeed if $y \in K_Y$ then by following the arguments adopted above and taking care of the fact that $\psi^{-1} : \Gamma_F(Y) \to \Gamma_F(X)$ is a graph isomorphism, we can show that $\psi^{-1}(1_y) = d.1_z$ for some $z \in K_X$ and $d \neq 0$ in $\mathbb{R}$. It is easy to check that, $\phi(z) = y$.

Finally, define the map $\Phi : C_F(X) \to C_F(Y)$ by the following rule as contemplated earlier: if $f \in C_F(X)$, then $f = \sum_{i=1}^{n} f(x_i).1_{x_i}$, where $X \setminus Z_X(f) = \{x_1, x_2, ..., x_n\}$. We set $\Phi(f) = \sum_{i=1}^{n} f(x_i).1_{\phi(x_i)}$, if $f \neq 0$; and $\Phi(0) = 0$. It can be proved by some routine calculations that for $f, g \in C_F(X)$, $\Phi(f+g) = \Phi(f)+\Phi(g)$ and $\Phi(f.g) = \Phi(f).\Phi(g)$. Thus $\Phi$ is a ring homomorphism which is one-to-one because for $f = \sum_{i=1}^{n} f(x_i).1_{x_i} \in C_F(X)$, $\Phi(f) = 0$ implies that $f(x_i) = 0$ for each $i = 1, 2, ..., n$ and hence $f = 0$. Finally for $h = \sum_{i=1}^{n} h(y_i).1_{y_i} \in C_F(Y)$ where $Y \setminus Z_Y(h) = \{y_1.y_2, ..., y_n\}$. We see that $f = \sum_{i=1}^{n} h(y_i).1_{\phi^{-1}(y_i)} \in C_F(X)$ and $\Phi(f) = h$. Thus $\Phi$ is an isomorphism from $C_F(X)$ onto $C_F(Y)$. □

Note that the conclusion of the above theorem may not be true if the conditions $|K_X| \geq 2$ and $|K_Y| \geq 2$ are dropped. Consider the following example:

**Example 6.9.** Consider $X = [0, 1]$ and $Y = [0, 1] \cup \{2\}$. Then $X$ has no isolated point whereas 2 is the only isolated point of $Y$. So $C_F(X) = \{0\}$ and $C_F(Y) = \{r.1_2 : r \in \mathbb{R}\}$ and hence $C_F(X)$ and $C_F(Y)$ are not isomorphic but the zero divisor graph of both the rings $C_F(X)$ and $C_F(Y)$ are empty and hence isomorphic.

However the following example suggests that the conclusion of the Theorem 6.8 may still be valid without the hypothesis $|K_X| \geq 2$ and $|K_Y| \geq 2$.

**Example 6.10.** We know that a normed linear space $X$ is path connected, in particular connected. It follows that the normed linear space $X$ with the weak topology is also connected and hence it does not contain any isolated point. Therefore in this case $C_F(X) = \{0\}$. Consequently the zero-divisor graph $\Gamma(C_F(X))$ of $C_F(X)$ becomes empty. So, if $X$ and $Y$ are both normed linear spaces each with weak topology, then the conclusion of the Theorem 6.8 also holds for these choices.

**Remark 6.11.** In Theorem 6.8 we prove that two infinite rings $C_F(X)$ and $C_F(Y)$ are isomorphic if and only if $\Gamma(C_F(X))$ and $\Gamma(C_F(Y))$ are isomorphic as graph (where $|K_X|, |K_Y| \geq 2$). This shows that the "finiteness" hypothesis is not a necessary condition for Theorem 4.1 in [5], which states that two finite commutative reduced rings which are not fields are isomorphic if and only if their zero-divisor graphs are isomorphic.

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