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The Method of Lower and Upper Solutions for Sobolev Type Hilfer Fractional Evolution Equations

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Abstract. The purpose of this paper is concerned with the existence of extremal mild solutions for Sobolev type Hilfer fractional evolution equations with nonlocal conditions in an ordered Banach spaces E. By using monotone iterative technique coupled with the method of lower and upper solutions, with the help of the theory of propagation family as well as the theory of the measure of noncompactness and Sadovskii's fixed point theorem, we obtain some existence results of extremal mild solutions for Hilfer fractional evolution equations. Finally, an example is provided to show the feasibility of the theory discussed in this paper.

1. Introduction

Nonlinear fractional differential equations can be studied in many areas such as population dynamics, heat condition in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics, and so on, see [33-42]. Fractional differential equations provide an excellent instrument for the description of memory and hereditary properties of various materials and processes and there has been a significant development in fractional differential equations theory. Especially, in recent years, the numerical solution of fractional differential equation (fractional Schrödinger equations) and its application in partial differential equation are concerned by many authors, we refer to monographs [38-42].

Hilfer [5] proposed a generalized Riemann-Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. This operator appeared in the theoretical simulation of dielectric relaxation in glass forming materials. In recent years, many authors began to study Hilfer fractional differential equations, we refer the reader to [5,8,11,12,13,6,19]. Presently, Hilfer fractional evolution equations has also been favored by many scholars. Gu and Trujillo [8] investigated a class of evolution equations involving Hilfer fractional derivatives, the definition of mild solutions to such problems is given. Furati et al. [10] considered an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative.

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Over the past year, many authors have studied the existence of mild solution for Hilfer fractional evolution equations with nonlocal conditions. In [11], Min Yang et al. studied the existence and uniqueness of mild solutions to the following Hilfer fractional evolution equations

$$\begin{cases} D_{0+}^{\nu,\mu}[u(t)-h(t,u(t))]=Au(t)+f(t,u(t)), \quad t\in J'=(0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}[u(0)-h(0,u(0))]-g(u)=u_0, \end{cases}$$

with the associated C_0-semigroup being compact or not, where $D_{0+}^{\nu,\mu}$ denotes the Hilfer fractional derivative of order μ and type ν which will be given in next section, $0\le\nu\le 1, 0<\mu<1$. In [13], Hamdy M. Ahmed et al. studied the existence of mild solutions of Hilfer fractional stochastic integro-differential equations of the form

$$\begin{cases} D_{0+}^{\nu,\mu}[u(t)+F(t,v(t))]+Au(t)=\int_0^t G(s,\eta(s))d\omega(s), \quad t\in J:=(0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}u(0)-g(u)=u_0, \end{cases}$$

where $(t,v(t))=(t,u(t),u(b_1(t))),\dots,u(b_m(t)))$ and $(t,\eta(t))=(t,u(t),u(a_1(t))),\dots,u(a_n(t)))$, $D_{0+}^{\nu,\mu}$ denotes the Hilfer fractional derivative $0\le\nu\le1, 0<\mu<1$, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t), t\ge 0$, on a separable Hilbert space H.

Moreover, Sobolev type fractional differential equations admit more adequate abstract representation to the partial differential equations arising in numerous applications for example in control theory of dynamical systems, flow of fluid through fissured rocks [44], propagation of long waves of small amplitude, shear in second order fluids [45], thermodynamics [46] etc. In particular, Sobolev type fractional differential equations serve abstract formulation in the form of implicit operator differential equations when an operator coefficient multiplying by the highest derivative [47]. For more literature on Sobolev type differential equations, see [9,48-50] and references therein.

On the other hand, by employing the method of lower and upper to study the existence of extremal mild solution for fractional evolution equation is an interesting issue, which has been attention in [7,17,28,31,32]. In [28], Chen and Li used monotone iterative method and lower and upper solutions to discuss the existence and uniqueness of mild solutions for a class of semilinear evolution equations with nonlocal conditions in an ordered Banach space E:

$$\begin{cases} u'(t)+Au(t)=f(t,u(t)), \quad t\in J=[0,b], \\ u(0)=\sum_{k=1}^{p} c_k u(t_k)+u_0, \end{cases}$$

where $A: D(A)\subset E\to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t\ge 0)$ on E, $f\in C(J\times E,E), J=[0,b], b>0$ is a constant, $0<t_1<t_2<\cdots<t_b, p\in\mathbb{N}$, c_k are real numbers, $c_k\neq 0, k=1,2,\dots,p, u_0\in E$.

In [23], Vikram Singh et al. investigated the existence and uniqueness of mild solutions for Sobolev type fractional impulsive differential systems with nonlocal conditions

$$\begin{cases} {}^cD^{\beta}[Bu(t)]=Au(t)+f(t,u(t),\int_0^t K(t,s,u(s))ds), \ t\in J=[0,a], t\neq t_j, \\ \Delta u|_{t=t_j}=I_j(u(t_j)), \quad j=1,2,\dots,m, m\in\mathbb{N}, \\ {}^LD^{1-\beta}[Tu(0)]=u_0+g(u(t)), \end{cases}$$

where ${}^cD^q, {}^LD^q$ denote Caputo and Riemann-Liouville fractional order derivatives of order $q\in(0,1)$, respectively. By applying monotone iterative technique coupled with the method of lower and upper solutions.

However, so far we have not seen relevant papers that study Sobolev type Hilfer fractional evolution equations with nonlocal problems by applying the monotone iterative technique and the method of lower and upper solutions. In this paper, we use the method of lower and upper solutions combined with monotone iterative technique to discuss the existence of extremal mild solutions for Hilfer fractional evolution equations of Sobolev type with nonlocal conditions

$$\begin{cases} D_{0+}^{\nu,\mu}Bu(t)+Au(t)=Bf(t,u(t),Gu(t)), \quad t\in(0,b], \\ I_{0+}^{1-\gamma}Bu(0)=B[u_0+\sum_{i=1}^{m}\lambda_i u(\tau_i)], \quad \tau_i\in(0,b], \end{cases} \tag{1.1}$$

where the two parameter family of fractional derivative $D_{0+}^{\nu,\mu}$ denote Hilfer fractional derivative of order μ and type $\nu(0 \leq \nu \leq 1)$, which is a interpolator between Riemann-Liouville and Caputo fractional derivatives, the operator $I_{0+}^{1-\gamma}$ is generalized fractional derivative of order $1-\gamma = (1-\nu)(1-\mu)(\gamma = \nu + \mu - \nu\mu, 0 < \mu < 1)$, A and B are closed (unbounded) linear operator with domains contained in E, the pair $(-A, B)$ generate a propagation family $\{T(t)\}_{t\geq 0}$. $J = [0, b](b > 0), J' = (0, b], f : J' \times E \times E \to D(B) \subset E$ is given functions satisfying some assumptions, $u_0 \in E$ and $\tau_i(i = 1, 2, \ldots, m)$ are prefixed points satisfying $0 < \tau_1 \leq \cdots \leq \tau_m < b$ and λ_i are real numbers. Here nonlocal condition $I_{0+}^{1-\gamma}u(0) = u_0 + \sum_{i=1}^{m} \lambda_i u(\tau_i)$ can be applied in physical problem yields better effect than the initial conditions $I_{0+}^{1-\gamma}u(0) = u_0$. The operator G is given by

$$Gu(t) = \int_0^t K(t, s, u(s))ds, \tag{1.2}$$

where $K \in C(\nabla \times E, E), \nabla = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b\}$.

As far as we know, the nonlocal condition can be better effect than the initial condition $u(0) = u_0$ in physics application. In this article, the nonlocal function $g(u)$ can be given by $g(u) = \sum_{i=1}^{m} \lambda_i u(\tau_i)$, we only assume that $\lambda_i(i = 1, 2, \ldots, m)$ satisfy the condition $(F1)$ (see in Section 2) without the compactness of nonlocal function. Firstly, we introduce the definition of mild solutions of the problem (1.1), and then we prove the existence of extremal mild solutions of the problem (1.1) by employing the Sadovskii's fixed point theorem. What's more, an existence result without using noncompactness measure condition is obtained in order and weakly sequentially complete Banach spaces, which is very useful in Application. More importantly, our method is different from that in paper [23]. Particularly, in this work, we do not assume that the solution operators generated by linear systems are compact. In this paper, we study (1.1) without assuming B has bounded (or compact) inverse as well as without any assumption on the relation between $D(A)$ and $D(B)$. This work is based on the theory of propagation family $\{T(t)\}_{t\geq 0}$ (an operator family generated by the operator pair (A, B)) introduced by Jin Liang and Ti-Jun Xiao [43], and a special measure of noncompactness which ensure us to do not assume the nonlinear term f satisfies a Lipschitz type condition. Actually, our result is new when in the case of $B = I$ (the identity operator on E).

The rest of this paper is organized as follows: In Section 2, we review some essential facts and introduce some notations. In Section 3, we state and prove the existence of mild solutions for Hilfer fractional differential system (1.1). Finally, in Section 4, an example is given to illustrate the effectiveness of the abstract results.

2. Preliminaries

Throughout this paper, by $C(J, E)$ and $C(J', E)$, we denote the spaces of all continuous functions from J to E and J' to E, respectively. Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E : x \geq \theta\}$ is normal with normal constant N.

Define $C_{1-\gamma}(J, E) = \left\{u \in C(J', E) : t^{1-\gamma}u(t) \in C(J, E)\right\}$. Clearly, $C_{1-\gamma}(J, E)$ is a Banach space with the norm $\|u\|_\gamma = \sup_{t\in J'} |t^{1-\gamma}u(t)|$. And $C_{1-\gamma}(J, E)$ is also an ordered Banach space with the partial order $\leq$ induced by the positive cone $P' = \{u \in C_{1-\gamma}(J, E) | u(t) \geq \theta, t \in J\}$ which is also normal with the same normal constant N.

For the convenience of discussion, we recall some definitions and basic results on fractional calculus, for more details see [8-12].

Definition 2.1. *The Riemann-Liouville fractional integral of order α of a function $f : [0, \infty) \to R$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \ t > 0, \alpha > 0,$$

provided the right side is point-wise defined on $[0, \infty)$.

Definition 2.2. *The Riemann-Liouville derivative of order α with the lower limit zero for a function $f:[0,\infty)\to R$ can be written as*

$$D_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\alpha+1-n}}ds,\ \ t>0,n-1<\alpha<n.$$

Definition 2.3. *The Caputo fractional derivative of order α for a function $f:[0,\infty)\to R$ can be written as*

$${}^cD_{0^+}^{\alpha}f(t)=D_{0^+}^{\alpha}\Big[f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\Big],\ t>0,\ n-1<\alpha<n,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.4. *(Hilfer fractional derivative see [5]). The generazlied Riemann-Liouville fractional derivative of order $0\le\nu\le1$ and $0<\mu<1$ with lower limit a is defined as*

$$D_{a+}^{\nu,\mu}f(t)=I_{a+}^{\nu(1-\mu)}\frac{d}{dt}I_{a+}^{(1-\nu)(1-\mu)}f(t).$$

for functions such that the expression on the right hand side exists.

Remark 2.1. *(i) If $\nu=0,0<\mu<1$ and $a=0$, the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative:*

$$D_{0+}^{0,\mu}f(t)=\frac{d}{dt}I_{0+}^{1-\mu}f(t)=D_{0+}^{\mu}f(t).$$

(ii) If If $\nu=1,0<\mu<1$ and $a=0$, the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:

$$D_{0+}^{1,\mu}f(t)=I_{0+}^{1-\mu}\frac{d}{dt}f(t)={}^cD_{0^+}^{\mu}f(t).$$

Remark 2.2. *The Hilfer fractional derivative is considered as an interpolator between the Riemann-Liouville and Caputo derivative.*

Remark 2.3. *For $0<\mu<1$, the Laplace transformation of Hilfer fractional derivatives is given by*

$$\mathcal{L}[D_{0+}^{\mu,\nu}f(x)](\lambda)=\lambda^{\mu}\mathcal{L}[f(x)](\lambda)-\lambda^{\nu(\mu-1)}(I_{0+}^{(1-\nu)(1-\mu)}f)(0+),$$

where $(I_{0+}^{(1-\nu)(1-\mu)}f)(0+)$ is the Riemann-Liouville fractional integral of order $(1-\nu)(1-\mu)$ in the limits as $t\to0+$, and

$$\mathcal{L}[f(x)](\lambda)=\int_0^{\infty}e^{-\lambda x}f(x)dx.$$

The symbol $\alpha(\cdot)$ is the Kuratowski noncompactness measure defined on bounded subset Ω of E. For any $\Omega\subset C(J,E)$ and $t\in J$, set $\Omega(t)=\{u(t):u\in B\}\subset E$. If B is bounded in $C(J,E)$, then $\Omega(t)$ is bounded in E, and $\alpha(\Omega(t))\le\alpha(\Omega)$. As is well known, the Kuratowski measure of noncompactness has the following properties.

Lemma 2.1. [6] *Let $B\subset C(J,E)$ be bounded and equicontinuous, then $\overline{co}B\subset C(J,E)$ is also bounded and equicontinuous.*

Lemma 2.2. [2] *Let E be a Banach space, and let $D\subset E$ be bounded. Then there exists a countable set $D_0\subset D$, such that $\alpha(D)\le2\alpha(D_0)$.*

Lemma 2.3. [3] *Let E be a Banach space, and let $\Omega\subset C(J,E)$ is equicontinuous and bounded, then $\alpha(\Omega(t))$ is continuous on J, and $\alpha(\Omega)=\max_{t\in J}\alpha(\Omega(t))$.*

Lemma 2.4. [4] *Let* $\Omega = \{u_n\}_{n=1}^{\infty} \subset C(J, E)$ *be a bounded and countable set and there exists a function* $m \in L^1(J, R^+)$ *such that for every* $n \in N$,

$$\|u_n(t)\| \leq m(t), \ \ a.e. t \in J.$$

Then $\alpha(\Omega(t))$ *is Lebesgue integral on* J, *and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_J \alpha(\Omega(t))dt.$$

We recall the abstract degenerate Cauchy problem as follows [43]:

$$\begin{cases} \frac{d}{dt}Bu(t) = Au(t), \ \ t \in J, \\ Bu(0) = Bu_0. \end{cases} \tag{2.1}$$

Definition 2.5. (*See*[12, *Definition* 1.4].) *A strongly continuous operator family* $\{T(t)\}_{t\geq 0}$ *of* $D(B)$ *to a Banach space* E, *satisfying that* $\{T(t)\}_{t\geq 0}$ *is exponentially bounded, which means that for any* $u \in D(B)$ *there exist* $a > 0, M > 0$ *such that*

$$\|T(t)u\| \leq Me^{at}\|u\|, \ \ t \geq 0,$$

is called an exponentially bounded propagation family for (2.1) if for $\lambda > a$,

$$(\lambda B - A)^{-1}Bu = \int_0^{\infty} e^{-\lambda t}T(t)udt, \ \ u \in D(B).$$

In this case, we also say that (2.1) has an exponentially bounded propagation family $\{T(t)\}_{t\geq 0}$.

Based on the Lemma 2.12 in [19], we give the following the lemma.

Lemma 2.5. *Assume that* A *and* B *are closed (unbounded) linear operator and the pair* $(-A, B)$ *generate a propagation family* $\{T(t)\}_{t\geq 0}$. *If* $f \in C_{1-\gamma}(J, E)$, *for any* $u \in C_{1-\gamma}(J, E)$, *a function* u *is a solution of the equation*

$$\begin{cases} D_{0+}^{\nu,\mu}Bu(t) + Au(t) = Bf(t, u(t), Gu(t)), \ t \in J', \\ I_{0+}^{1-\gamma}Bu(0) = Bu_0, \end{cases} \tag{2.2}$$

if and only if u *satisfies the following integral equation:*

$$u(t) = S_{\nu,\mu}(t)u_0 + \int_0^t K_\mu(t-s)f(s, u(s), Gu(s))ds,$$

where

$$S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)}K_\mu(t), \ \ K_\mu(t) = \mu\int_0^{\infty} \sigma t^{\mu-1}\xi_\mu(\sigma)T(t^\mu\sigma)u_0 d\sigma, \tag{2.3}$$

the function ξ_μ *is a probability density function defined on* $(0, \infty)$ *such that*

$$\xi_\mu(\sigma) = \frac{1}{\mu}\sigma^{-1-\frac{1}{\mu}}\varpi_\mu(\sigma^{-\frac{1}{\mu}}) \geq 0$$

and the one sided stable probability density in [20] as follows:

$$\varpi_\mu(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-\mu n-1}\frac{\Gamma(n\mu+1)}{n!}\sin(n\pi\mu), \ \sigma \in (0, \infty).$$

Lemma 2.6. [19] *Assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a propagation family $\{T(t)\}_{t\geq 0}$ and $T(t)$ is continuous in the uniform operator topology for $t > 0$. That is, there exists $M \geq 1$ such that $\sup_{t\in[0,+\infty)} \|T(t)\| \leq M$. Then the operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ have the following properties.*

(i) For any fixed $t \geq 0$, $\{S_{\nu,\mu}(t)\}_{t>0}$ and $\{K_\mu(t)\}_{t>0}$ are linear operators, and for any $u \in E$,

$$\|S_{\nu,\mu}(t)u\| \leq \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|u\|, \quad \|K_\mu(t)u\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|u\|.$$

(ii) The operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ are strongly continuous for all $t \geq 0$.

(iii) If $T(t)(t \geq 0)$ is an equicontinuous semigroup, then $S_{\nu,\mu}(t)$ and $K_\mu(t)$ are equicontinuous in E for $t > 0$.

In view of [19], from Lemma 2.6, we adopt the following definition of mild solution of the system (2.2).

Definition 2.6. *A function $u \in C_{1-\gamma}(J, E)$ is said to be a mild solution of (2.2) if $u_0 \in E$ the integral equation*

$$u(t) = S_{\nu,\mu}(t)u_0 + \int_0^t K_\mu(t-s)f(s, u(s), Gu(s))ds,$$

is satisfied, for all $t \in J'$.

Next, we present useful lemma which plays an important role.

Lemma 2.7. *Assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a propagation family $\{T(t)\}_{t\geq 0}$, for $0 \leq \nu \leq 1, 0 < \mu < 1$, then*

$$D_{0+}^{\nu,\mu}\Big(BS_{\nu,\mu}(t)u_0\Big) = -A\Big(S_{\nu,\mu}(t)u_0\Big),$$

and

$$D_{0+}^{\nu,\mu}\Big(\int_0^t K_\mu(t-s)Bf(s, u(s), Gu(s))ds\Big) = -A\int_0^t K_\mu(t-s)f(s, u(s), Gu(s))ds + Bf(t, u(t), Gu(t)). \tag{2.4}$$

Proof. Let $\lambda > 0$, we consider the one sided stable probability density in [20] as follows:

$$\varpi_\mu(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-\mu n-1}\frac{\Gamma(n\mu+1)}{n!}\sin(n\pi\mu),\ \sigma \in (0, \infty),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\sigma}\varpi_\mu(\sigma)d\sigma = e^{-\lambda^\mu},\quad \mu \in (0, 1). \tag{2.5}$$

Then, using (2.5) and Definition 2.5, we have

$$\begin{aligned}
(\lambda^\mu B + A)^{-1}Bu &= \int_0^\infty e^{-\lambda^\mu s}T(s)uds = \int_0^\infty \mu t^{\mu-1}e^{-(\lambda t)^\mu}T(t^\mu)udt \\
&= \int_0^\infty\int_0^\infty e^{-(\lambda t\sigma)}\mu t^{\mu-1}\varpi_\mu(\sigma)T(t^\mu)ud\sigma dt \\
&= \mu\int_0^\infty\int_0^\infty e^{-\lambda\theta}\frac{\theta^{\mu-1}}{\sigma^\mu}\varpi_\mu(\sigma)T\Big(\frac{\theta^\mu}{\sigma^\mu}\Big)ud\theta d\sigma \\
&= \int_0^\infty e^{-\lambda\tau}\Big[\mu\int_0^\infty \frac{\tau^{\mu-1}}{\sigma^\mu}\varpi_\mu(\sigma)T\Big(\frac{\tau^\mu}{\sigma^\mu}\Big)ud\sigma\Big]d\tau \\
&= \int_0^\infty e^{-\lambda t}\Big[\mu\int_0^\infty \frac{t^{\mu-1}}{\sigma^\mu}\varpi_\mu(\sigma)T\Big(\frac{t^\mu}{\sigma^\mu}\Big)ud\sigma\Big]dt \\
&= \int_0^\infty e^{-\lambda t}\Big[\mu\int_0^\infty \sigma t^{\mu-1}\xi_\mu(\sigma)T(t^\mu\sigma)ud\sigma\Big]dt \\
&= \int_0^\infty e^{-\lambda t}K_\mu(t)udt,
\end{aligned} \tag{2.6}$$

where ξ_μ is a probability density function defined on $(0,\infty)$ such that

$$\xi_\mu(\sigma) = \frac{1}{\mu}\sigma^{-1-\frac{1}{\mu}}\varpi_\mu(\sigma^{-\frac{1}{\mu}}) \geq 0.$$

Since the Laplace inverse transform of $\lambda^{\nu(\mu-1)}$ is

$$\mathcal{L}^{-1}(\lambda^{\nu(\mu-1)}) = \begin{cases} \frac{t^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))}, & 0 < \nu \leq 1, \\ \delta(t), & \nu = 0, \end{cases} \tag{2.7}$$

where $\delta(t)$ is the Delta function.

It follows from (2.6), (2.7) and Laplace transform, it is obvious to see that

$$\begin{aligned} \mathcal{L}(BS_{\nu,\mu}(t)u_0) &= \mathcal{L}(I_{0+}^{\nu(1-\mu)}BK_\mu(t)u_0) \\ &= \mathcal{L}\Big(\frac{t^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))} * BK_\mu(t)u_0\Big) \\ &= \mathcal{L}\Big(\mathcal{L}^{-1}(\lambda^{\nu(\mu-1)}) * BK_\mu(t)u_0\Big) \\ &= \lambda^{\nu(\mu-1)}B(\lambda^\mu B + A)^{-1}Bu_0, \end{aligned} \tag{2.8}$$

where the symbol $*$ is convolution symbol. By Remark 2.3, we obtain

$$\begin{aligned} \mathcal{L}(D_{0+}^{\nu,\mu}[BS_{\nu,\mu}(t)u_0]) &= \lambda^\mu \mathcal{L}(BS_{\nu,\mu}(t)u_0) - \lambda^{\nu(\mu-1)}Bu_0 \\ &= \lambda^\mu B\Big[\lambda^{\nu(\mu-1)}(\lambda^\mu B + A)^{-1}B\Big]u_0 - \lambda^{\nu(\mu-1)}Bu_0 \\ &= \lambda^{\nu(\mu-1)}(\lambda^\mu B + A)^{-1}B\Big[\lambda^\mu B - (\lambda^\mu B + A)\Big]u_0 \\ &= \lambda^{\nu(\mu-1)}(\lambda^\mu B + A)^{-1}B\Big[\lambda^\mu B - \lambda^\mu B - A\Big]u_0 \\ &= -\lambda^{\nu(\mu-1)}(\lambda^\mu B + A)^{-1}BAu_0 \\ &= -A\lambda^{\nu(\mu-1)}(\lambda^\mu B + A)^{-1}Bu_0. \end{aligned} \tag{2.9}$$

Combing (2.8) and (2.9) yields

$$D_{0+}^{\nu,\mu}[BS_{\nu,\mu}(t)u_0] = -A[S_{\nu,\mu}(t)u_0].$$

Similarly, we have

$$\mathcal{L}\Big(\int_0^t K_\mu(t-s)Bf(s,u(s),Gu(s))ds\Big) = \mathcal{L}(K_\mu(t)) \cdot \mathcal{L}\Big(Bf(t,u(t),Gu(t))\Big) \tag{2.10}$$

and

$$\begin{aligned} &\mathcal{L}\Big(D_{0+}^{\nu,\mu}\Big[\int_0^t K_\mu(t-s)Bf(s,u(s),Gu(s))ds\Big]\Big) \\ &= \lambda^\mu \mathcal{L}\Big(\int_0^t K_\mu(t-s)Bf(s,u(s),Gu(s))ds\Big) - \lambda^{\nu(\mu-1)} \cdot 0 \\ &= \lambda^\mu \mathcal{L}(K_\mu(t)) \cdot \mathcal{L}\Big(Bf(t,u(t),Gu(t))\Big) \\ &= \lambda^\mu(\lambda^\mu B + A)^{-1}B \cdot \mathcal{L}\Big(Bf(t,u(t),Gu(t))\Big) \\ &= (\lambda^\mu B + A - A)(\lambda^\mu B + A)^{-1}B \cdot \mathcal{L}\Big(f(t,u(t),Gu(t))\Big) \\ &= -A(\lambda^\mu B + A)^{-1}B \cdot \mathcal{L}\Big(f(t,u(t),Gu(t))\Big) + B \cdot \mathcal{L}\Big(f(t,u(t),Gu(t))\Big). \end{aligned} \tag{2.11}$$

Thus, it follows from (2.10) and (2.11) that

$$D_{0+}^{\nu,\mu}\Big[\int_0^t K_\mu(t-s)Bf(s,u(s),Gu(s))ds\Big] = -A\int_0^t K_\mu(t-s)f(s,u(s),Gu(s))ds + Bf(t,u(t),Gu(t)). \quad (2.12)$$

For the convenience of discussion, we assume that

(F0) Assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a propagation family $\{T(t)\}_{t\geq 0}$ in E and $T(t)$ is continuous in the uniform operator topology for $t > 0$. That is, there exists $M \geq 1$ such that $\sup_{t\in[0,+\infty)} \|T(t)\| \leq M$.

(F1) $\lambda_i > 0 (i = 1, 2, \ldots, m)$ and $\sum\limits_{i=1}^{m} \lambda_i < \frac{\Gamma(\gamma)}{Mb^{\gamma-1}}$.

In view of [28] and [30], we present the following lemma.

Lemma 2.8. *Assume that (F0) and (F1) holds. For any $u \in C_{1-\gamma}(J)$ such that $f(\cdot, u, Gu) \in C_{1-\gamma}(J)$, then the problem (1.1) has mild solution $u \in C_{1-\gamma}(J)$ given by*

$$u(t) = S_{\nu,\mu}(t)\overline{\Theta}u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),Gu(s))ds$$
$$+ \int_0^t K_\mu(t-s)f(s,u(s),Gu(s))ds, \quad (2.13)$$

where $\overline{\Theta} = \left[I - \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(\tau_i)\right]^{-1}$.

Proof. By assumption (F1), we have

$$\Big\|\sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\Big\| \leq \sum_{i=1}^{m} |\lambda_i| \cdot \Big\|S_{\nu,\mu}(t)\Big\| \leq \sum_{i=1}^{m} |\lambda_i| \frac{Mb^{\gamma-1}}{\Gamma(\gamma)} < 1.$$

By operator spectrum theorem, the operator $\overline{\Theta} := \left(I - \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(\tau_i))\right)^{-1}$ exists and is bounded. Furthermore, by Neumann expression, we obtain

$$\|\overline{\Theta}\| \leq \sum_{i=0}^{\infty} \|\sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(\tau_i)\|^n = \frac{1}{1 - \|\sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(\tau_i)\|} \leq \frac{1}{1 - \frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\sum_{i=1}^{m} \lambda_i}.$$

According to Definition 2.6, a solution of system (2.2) can be expressed by

$$u(t) = S_{\nu,\mu}(t) I_{0+}^{1-\gamma} u(0) + \int_0^t K_\mu(t-s)f(s,u(s),Gu(s))ds. \quad (2.14)$$

Next, we substitute $t = \tau_i$ into (2.14) and by multiplying λ_i to both side of (2.14), we have

$$\lambda_i u(\tau_i) = \lambda_i S_{\nu,\mu}(\tau_i) I_{0+}^{1-\gamma} u(0) + \lambda_i \int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),Gu(s))ds. \quad (2.15)$$

Thus, we have

$$I_{0+}^{1-\gamma} u(0) = u_0 + \sum_{i=1}^{m} \lambda_i u(\tau_i)$$
$$= u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(\tau_i) I_{0+}^{1-\gamma} u(0) + \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),Gu(s))ds.$$

Since $I-\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)$ has a bounded inverse operator $\overline{\Theta}$, which implies

$$\begin{aligned}I_{0+}^{1-\gamma}u(0)&=\Big[I-\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\Big]^{-1}\Big(u_0+\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),Gu(s))ds\Big)\\&=\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}\overline{\Theta}K_\mu(\tau_i-s)f(s,u(s),Gu(s))ds.\end{aligned}\tag{2.16}$$

Submitting (2.16) to (2.14), we derive that (2.13). It is probative that u is also a solution of the integral of Eq.(2.13) when u is a solution of system (2.2).

The necessity has been already proved, next, we are read to prove its sufficiency. Applying $I_{0+}^{1-\gamma}$ to both side of (2.13), and by Lemma 2.7, we have

$$\begin{aligned}I_{0+}^{1-\gamma}Bu(t)=I_{0+}^{1-\gamma}\Big(&S_{\nu,\mu}(t)\overline{\Theta}Bu_0+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds\\&+\int_0^{t}K_\mu(t-s)Bf(s,u(s),Gu(s))ds\Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\lim_{t\to0}I_{0+}^{1-\gamma}Bu(t)&=\lim_{t\to0}I_{0+}^{1-\gamma}S_{\nu,\mu}(t)\overline{\Theta}Bu_0+\sum_{i=1}^{m}\lambda_i\lim_{t\to0}I_{0+}^{1-\gamma}S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds\\&=I_{0+}^{1-\gamma}(\lim_{t\to0}S_{\nu,\mu}(t)(\overline{\Theta}Bu_0)+I_{0+}^{1-\gamma}\lim_{t\to0}S_{\nu,\mu}(t)\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds\\&=I_{0+}^{1-\gamma}\Big(\frac{\overline{\Theta}Bu_0}{\Gamma(\gamma)}t^{\gamma-1}\Big)+I_{0+}^{1-\gamma}\Big(\frac{\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds}{\Gamma(\gamma)}t^{\gamma-1}\Big)\\&=\overline{\Theta}Bu_0+\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds.\end{aligned}\tag{2.17}$$

Substituting $t=\tau_i$ into (2.13), we have

$$\begin{aligned}u(\tau_i)=S_{\nu,\mu}(\tau_i)\overline{\Theta}u_0&+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),Gu(s))ds\\&+\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),Gu(s))ds.\end{aligned}$$

Then, we derive

$$\begin{aligned}B\Big(u_0+\sum_{i=1}^{m}\lambda_i u(\tau_i)\Big)&=Bu_0+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds\\&+\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds\\&=\Big(I+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}\Big)\Big(Bu_0+\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}K_\mu(\tau_i-s)Bf(s,u(s),Gu(s))ds\Big)\end{aligned}$$

$$= \Big(\overline{\Theta}^{-1} + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(\tau_i)\Big)\Big(\overline{\Theta} Bu_0 + \sum_{i=1}^{m} \lambda_i \overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)Bf(s, u(s), Gu(s))ds\Big)$$

$$= \overline{\Theta} Bu_0 + \sum_{i=1}^{m} \lambda_i \overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)Bf(s, u(s), Gu(s))ds. \tag{2.18}$$

It follows (2.16) and (2.18) that

$$I_{0+}^{1-\gamma} Bu(0) = B\Big(u_0 + \sum_{i=1}^{m} \lambda_i u(\tau_i)\Big).$$

Next, by applying $D_{0+}^{\nu,\mu}$ to both sides of (2.13) and using Lemma 2.7, we have

$$\begin{aligned}
D_{0+}^{\nu,\mu} Bu(t) &= D_{0+}^{\nu,\mu}\Big[S_{\nu,\mu}(t)\overline{\Theta} Bu_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)Bf(s, u(s), Gu(s))ds \\
&\quad + \int_0^t K_\mu(t - s)Bf(s, u(s), Gu(s))ds\Big] \\
&= D_{0+}^{\nu,\mu}\Big[S_{\nu,\mu}(t)\overline{\Theta} Bu_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)Bf(s, u(s), Gu(s))ds\Big] \\
&\quad + D_{0+}^{\nu,\mu}\Big[\int_0^t K_\mu(t - s)Bf(s, u(s), Gu(s))ds\Big] \\
&= \Big[\overline{\Theta} u_0 + \sum_{i=1}^{m} \lambda_i \overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)f(s, u(s), Gu(s))ds\Big] D_{0+}^{\nu,\mu}\Big[BS_{\nu,\mu}(t)\Big] \\
&\quad + D_{0+}^{\nu,\mu}\Big[\int_0^t K_\mu(t - s)Bf(s, u(s), Gu(s))\Big] \\
&= \Big[\overline{\Theta} u_0 + \sum_{i=1}^{m} \lambda_i \overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)f(s, u(s), Gu(s))ds\Big](-AS_{\nu,\mu}(t)) \\
&\quad - A\int_0^t K_\mu(t - s)f(s, u(s), Gu(s))ds + Bf(t, u(t), Gu(t)) \\
&= -A\Big(S_{\nu,\mu}(t)\overline{\Theta} u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)f(s, u(s), Gu(s))ds \\
&\quad + \int_0^t K_\mu(t - s)f(s, u(s), Gu(s))ds\Big) + Bf(t, u(t), Gu(t)) \\
&= -Au(t) + Bf(t, u(t), Gu(t)).
\end{aligned}$$

Hence, it reduces to

$$D_{0+}^{\nu,\mu} Bu(t) + Au(t) = Bf(t, u(t), Gu(t)).$$

The results are proved completely. □

From Lemma 2.8, we adopt the following definition of mild solution of the problem (1.1).

Definition 2.7. *A function $u \in C_{1-\gamma}(J, E)$ is said to be a mild solution of the problem (1.1), if it satisfies the operator equation*

$$\begin{aligned}
u(t) &= S_{\nu,\mu}(t)\overline{\Theta} u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\overline{\Theta} \int_0^{\tau_i} K_\mu(\tau_i - s)f(s, u(s), Gu(s))ds \\
&\quad + \int_0^t K_\mu(t - s)f(s, u(s), Gu(s))ds, \quad t \in J',
\end{aligned} \tag{2.19}$$

where the operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ are given by (2.3).

Definition 2.8. A strongly continuous propagation family $\{T(t)\}_{t\geq 0}$ in E is called to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta, x \in E$ and $t \geq 0$.

To end this section, we state a fixed point theorem, which plays a major role in the proof of our main results.

Lemma 2.9. *(Sadovskii fixed point theorem). Let D ba a convex, closed and bounded subset of a Banach space E and $Q : D \to D$ be a condensing map. Then Q has one fixed point in D.*

Lemma 2.10. [27] *Let $a \geq 0, \mu > 0, c(t)$ and $u(t)$ be the nonnegative locally integrable functions on $0 \leq t < T < +\infty$, such that*

$$u(t) \leq c(t) + a\int_0^t (t-s)^{\mu-1}u(s)ds,$$

then

$$u(t) \leq c(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(a\Gamma(\mu))^n}{\Gamma(n\mu)}(t-s)^{n\mu-1}c(s)\Big]ds, \ \ 0 \leq t < T.$$

3. Main results

In this section, we will discuss the existence of extremal mild solutions for problem (1.1).

Definition 3.1. *An abstract function $u \in C_{1-\gamma}(J, E)$ is called a solution of the problem (1.1) if $u(t)$ satisfies all the equalities of (1.1).*

Definition 3.2. If a function $v_0 \in C_{1-\gamma}(J, E)$ satisfies

$$\begin{cases} D_{0+}^{\nu,\mu}Bv_0(t) + Av_0(t) \leq Bf(t, v_0(t), Gv_0(t)), \quad t \in J, \\ I_{0+}^{1-\gamma}Bv_0(0) \leq B[u_0 + \sum_{i=1}^m \lambda_i v_0(\tau_i)], \end{cases} \tag{3.1}$$

we call it a lower solution of the priblem (1.1); if all the inequalities in (3.1) are reversed, we call it an upper solution of the problem (1.1).

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone P is normal, assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a positive propagation family $\{T(t)\}_{t\geq 0}$ on E, $f \in C(J \times E \times E, E)$ and $u_0 \in E$. If the problem (1.1) has a lower solution $v_0 \in C_{1-\gamma}(J, E)$ and an upper solution $w_0 \in C_{1-\gamma}(J, E)$ with $v_0 \leq w_0$. Suppose also that the conditions (F0), (F1) and the following conditions are satisfied*

(F2) 1. The function $K(t, s, \cdot) : E \to E$ satisfies $K(t, s, u_1) \leq K(t, s, u_2)$, for any $(t, s) \in \Delta, u_1, u_2 \in E$ with $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$.

2. The function $f(t, \cdot, \cdot) : E \times E \to E$ satisfies

$$f(t, u_1, v_1) \leq f(t, u_2, v_2)$$

for $\forall \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Gv_0(t) \leq v_1 \leq v_2 \leq Gw_0(t)$.

(F3) 1. For each bounded set $D \subset E$, there exists an integrable function $\zeta : \nabla \to [0, \infty)$ such that

$$\alpha(\{K(t, s, D)\}) \leq \zeta(t, s)\alpha(D),$$

for a.e. $(t, s) \in \nabla$. For simplication, put $K_0 = \sup_{t\in J}\int_0^t \zeta(t, s)ds$.

2. There exists a constant $L > 0$ such that

$$\alpha(\{f(t, u_n, v_n)\}) \leq L(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

for $\forall t \in J$, and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

(F4) The sequence $v_n(0)$ and $w_n(0)$ are convergent, where $v_n = Qv_{n-1}, w_n = Qw_{n-1}, n = 1, 2, \ldots.$

Then the problem(1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.

Proof. We can define operator $Q : [v_0, w_0] \to C_{1-\gamma}(J, E)$ as follows

$$(Qu)(t) = S_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, u(s), Gu(s))ds \\ + \int_0^t K_\mu(t-s) f(s, u(s), Gu(s))ds, \quad t \in J'. \tag{3.1}$$

Since f is continuous, it is easily see that the map $Q : [v_0, w_0] \to C_{1-\gamma}(J, E)$ is continuous. And by Lemma 2.8, the mild solutions of the problem (1.1) are equivalent to the fixed points of the operator Q. For convenience, we divide the proof in the following steps.

Step 1. We show $Q : [v_0, w_0] \to C_{1-\gamma}(J, E)$ is an increasing monotone operator.

In fact, for $\forall t \in J'$, $v_0(t) \le u \le v \le w_0$, by the assumptions (F2), we have

$$f(s, v_0(s), Gv_0(s)) \le f(s, u(s), Gu(s)) \le f(s, v(s), Gv(s)) \le f(s, w_0(s), Gw_0(s)).$$

So

$$\int_0^t K_\mu(t-s) f(s, u(s), Gu(s))ds \le \int_0^t K_\mu(t-s) f(s, v(s), Gv(s))ds.$$

And by the positive of the operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ for $t \ge 0$, from (3.1) we see that $Qu \le Qv$.

Step 2. We first show $v_0 \le Qv_0$, $Qw_0 \le w_0$. Let $h(t) = D_{0+}^{\nu,\mu} v_0(t) + Av_0(t)$, $h \in C(J, E)$ and $h(t) \le f(t, v_0(t), Gv_0(t)), t \in J'$. By Definition 2.7, 3.2 and positivity of the operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ for $t \ge 0$, we have

$$\begin{aligned} v_0(t) &= S_{\nu,\mu}(t)v_0(0) + \int_0^t K_\mu(t-s)h(s)ds \\ &\le S_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, v_0(s), Gv_0(s))ds \\ &+ \int_0^t K_\mu(t-s) f(s, v_0(s), Gv_0(s))ds \\ &= Qv_0(t), \quad t \in J'. \end{aligned}$$

It implies that $v_0 \le Qv_0$. Similarly, it can be show that $Qw_0 \le w_0$. So $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuous increasing monotone operator.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots. \tag{3.2}$$

Then from the monotonicity of Q, it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.3}$$

Step 3. We prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J'.

For convenience, we denote $B = \{v_n : n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} : n \in \mathbb{N}\}$. Then $B = Q(B_0)$. From $B_0 = B \bigcup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J'$. Let $\varphi(t) := \alpha(B(t)), t \in J'$, we will show that $\varphi(t) \equiv 0$ in J'.

For $t \in J'$, from (3.1), using Lemma 2.2, assumption (F3) and (F4), we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t)) \\
&= \alpha\Big(\Big\{S_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\Theta\int_0^{\tau_i}K_\mu(\tau_i - s)f(s, v_{n-1}(s), Gv_{n-1}(s))ds \\
&+ \int_0^t K_\mu(t-s)f(s, v_{n-1}(s), Gv_{n-1}(s))ds\Big\}\Big) \\
&\le \frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\alpha\Big(\Big\{\Theta u_0 + \sum_{i=1}^{m}\lambda_i\Theta\int_0^{\tau_i}K_\mu(\tau_i - s)f(s, v_{n-1}(s), Gv_{n-1}(s))ds\Big\}\Big) \\
&+ \frac{2Mb^{\mu-1}}{\Gamma(\mu)}\int_0^t \alpha\Big(\Big\{f(s, v_{n-1}(s), Gv_{n-1}(s))\Big\}\Big)ds \\
&\le \frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\alpha\Big(\Big\{v_n(0)\Big\}\Big) + \frac{2Mb^{\mu-1}(L+LK_0)}{\Gamma(\mu)}\int_0^t \alpha\Big(B_0(s)\Big)ds \\
&\le \frac{2Mb^{\mu-1}(L+LK_0)}{\Gamma(\mu)}\int_0^t \varphi(s)ds.
\end{aligned}$$

Hence Lemma 2.10, $\varphi(t) \equiv 0$ in J. So, for any $t \in J, \{v_n(t)\}$ is precompact, and $\{v_n(t)\}$ has a convergent subsequence. Combining this with the monotonicity (3.2), we prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t), t \in J$. Similarly, $\lim_{n\to\infty} w_n(t) = \overline{u}(t), t \in J$.

Evidently, $\{v_n(t)\} \in C_{1-\gamma}(J, E)$, so $\underline{u}(t)$ is bounded integrable on J. For any $t \in J$,

$$v_n(t) = Q(v_{n-1}) = S_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\Theta\int_0^{\tau_i}K_\mu(\tau_i - s)f(s, v_{n-1}(s), Gv_{n-1}(s))ds + \int_0^t K_\mu(t-s)f(s, v_{n-1}(s), Gv_{n-1}(s))ds. \quad (3.4)$$

If $n \to \infty$ in (3.4), by the Lebesgue dominated convergence theorem, we obtain

$$\underline{u}(t) = Q(\underline{u}(t)) = S_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\Theta\int_0^{\tau_i}K_\mu(\tau_i - s)f(s, \underline{u}(s), G\underline{u}(s))ds + \int_0^t K_\mu(t-s)f(s, \underline{u}(s), G\underline{u}(s))ds.$$

Thus, we have $\underline{u}(t) \in C_{1-\gamma}(J, E)$, and $\underline{u} = Q\underline{u}$. In a similar way, we can prove that there exists $\overline{u}(t) \in C_{1-\gamma}(J, E)$ such that $\overline{u} = Q\overline{u}$. Combing this with monotonicity (3.3), we see that $v_0 \le \underline{u} \le \overline{u} \le w_0$, which implies that $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of the problem (1.1) in $[v_0, w_0]$. □

Corollary 3.1. *Let E be an ordered Banach space, whose positive cone P is regular, assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a positive propagation family $\{T(t)\}_{t\ge0}$ on E, $f \in C(J \times E \times E, E)$ and $u_0 \in E$. If the problem (1.1) has a lower solution $v_0 \in C_{1-\gamma}(J, E)$ and an upper solution $w_0 \in C_{1-\gamma}(J, E)$ with $v_0 \le w_0$. Suppose also that the conditions (F0)-(F3) are satisfied. Then the problem(1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. Since P is regular, any ordered monotonic and ordered bounded sequence in E is convergent. For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Gv_0, Gw_0(t)]$ be two increasing or decreasing sequences. By Definition of regular cone and assumption (F2), $\{K(t, s, u_n)\}$ is convergent. Therefore $\alpha(\{K(t, s, u_n)\}) = \alpha(\{u_n\}) = 0$. Similarly, we have

$$\alpha(\{f(t, u_n, v_n)\}) \le \alpha(\{u_n\}) + \alpha(\{v_n\}) = 0.$$

Therefore, (F3) holds. Then, by Theorem 3.1, the proof is complete. □

As a supplement to Theorem 3.1, we further discuss the existence of mild solutions for the problem (1.1) in weakly sequentially complete Banach space, we only need to verify the conditions (F1) and (F2) are satisfied.

Corollary 3.2. *Let E be an ordered and weakly sequentially complete Banach space, whose positive cone P is normal, assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a positive propagation family $\{T(t)\}_{t\geq 0}$ on E, $f \in C(J \times E \times E, E)$ and $u_0 \in E$. If the problem (1.1) has a lower solution $v_0 \in C_{1-\gamma}(J, E)$ and an upper solution $w_0 \in C_{1-\gamma}(J, E)$ with $v_0 \leq w_0$. Suppose also that the conditions (F0)-(F3) are satisfied. Then the problem(1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively..*

Proof. In Theorem 3.1, if E is weakly sequentially complete, the condition (F3) and (F4) holds automatically. In fact, by Theorem 2.2 in [28], any monotonic and order bounded sequence is precompact. By the monotonicity (3.3), we can easily see that $v_n(t)$ and $w_n(t)$ are convergent on J. In particular, $v_n(0)$ and $w_n(0)$ are convergent. Thus, condition (F4) holds. For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Gv_0, Gw_0(t)]$ be two increasing or decreasing sequences. By (F2), $\{f(t, u_n, v_n)\}$ is an ordered monotonic and ordered bounded sequence in E. Then, $\alpha(\{f(t, u_n, v_n)\}) = 0$. Therefore, (F3) holds. Then, by Theorem 3.1, our conclusion is valid. □

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal, assume that A and B are closed (unbounded) linear operator and the pair $(-A, B)$ generate a positive and equicontinuous propagation family $\{T(t)\}_{t\geq 0}$ on E, $f \in C(J \times E \times E, E)$ and $u_0 \in E$. If the problem (1.1) has a lower solution $v_0 \in C_{1-\gamma}(J, E)$ and an upper solution $w_0 \in C_{1-\gamma}(J, E)$ with $v_0 \leq w_0$. Suppose also that the conditions (F0)-(F3) are satisfied and*

(F5) There exists a nonnegative constant L_1 with

$$\frac{2Mb^{\mu}(L + LK_0)}{\Gamma(\mu)}\Big[\frac{(b^{\gamma-1} - \Gamma(\gamma))M\sum_{i=1}^{m}\lambda_i + \Gamma(\gamma)}{\Gamma(\gamma)(1 - M\sum_{i=1}^{m}\lambda_i)}\Big] < 1$$

such that

$$\alpha(\{f(t, u_n, v_n)\}) \leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

for $\forall t \in J$, and equicontinuous countable set $\{u_n\} \subset [v_0(t), w_0(t)]$, $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

Then the problem(1.1) has minimal mild solution $\underline{u}$ and maximal mild solutions $\overline{u}$ in $[v_0, w_0]$, moreover

$$v_n(t) \to \underline{u}(t),\;\; w_n(t) \to \overline{u}(t),\;\; (n \to +\infty), t \in J,$$

where $v_n(t) = Qv_{n-1}(t), w_n(t) = Qw_{n-1}(t)$ which satisfy

$$v_0(t) \leq v_1(t) \leq \cdots v_n(t) \leq \cdots \underline{u}(t) \leq \overline{u}(t) \leq \cdots \leq w_n(t) \leq \cdots w_1(t) \leq w_0(t), \forall t \in J.$$

Proof. From the proof of Theorem 3.1, we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is continuous. First, we will prove that $Q : [v_0, w_0] \to C_{1-\gamma}(J, E)$ is an equicontinuous operator. Since $T(t)(t \geq 0)$ is a equicontinuous propagation family, and by Lemma 2.6, the operators $S_{\nu,\mu}(t)$ and $K_{\mu}(t)$ for $t \geq 0$ are also equicontinuous. By the normality of the cone P, there exists $\overline{M} > 0$ such that

$$\|f(t, u(t), Gu(t))\| \leq \overline{M},\;\; u \in [v_0, w_0].$$

For any $u \in C_{1-\gamma}(J, E)$, let $y(t) = t^{1-\gamma}u(t)$, for $t_1 = 0, 0 < t_2 \leq b$, we get

$$\begin{aligned}\|y(t_2) - y(0)\| &\leq \Big\|t_2^{1-\gamma}S_{\nu,\mu}(t_2)\Big\|(\Theta u_0) + \sum_{i=1}^{m}\lambda_i\Theta\|t_2^{1-\gamma}S_{\nu,\mu}(t_2)\|\int_0^{\tau_i}K_{\mu}(\tau_i - s)f(s, u(s), Gu(s))ds\\ &+ t_2^{1-\gamma}\Big\|\int_0^{t_2}K_{\mu}(t_2 - s)f(s, u(s), Gu(s))ds\Big\|\\ &\leq \Big\|t_2^{1-\gamma}S_{\nu,\mu}(t_2)\Big\|(\Theta u_0) + \overline{M}\sum_{i=1}^{m}\lambda_i\Theta\|t_2^{1-\gamma}S_{\nu,\mu}(t_2)\|\int_0^{\tau_i}K_{\mu}(\tau_i - s)ds + \overline{M}\Big\|\int_0^{t_2}t_2^{1-\gamma}K_{\mu}(t_2 - s)ds\Big\|\\ &\to 0, \quad \text{as } t_2 \to t_1 = 0.\end{aligned}$$

For $0 < t_1 < t_2 \le b$, by (3.1), we get that

$$\begin{aligned}
&\|y(t_2) - y(t_1)\| \le \left\| t_2^{1-\gamma}(Qu)(t_2) - t_1^{1-\gamma}(Qu)(t_1) \right\| \\
&\le \left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_1^{1-\gamma} S^{*}_{\nu,\mu}(t_1) \right\| (\Theta u_0) + \left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_1^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \\
&\times \sum_{i=1}^{m} \lambda_i \Theta \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, u(s), Gu(s)) ds + \int_0^{t_2} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \\
&- \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_1 - s) f(s, u(s), Gu(s)) ds \\
&\le \Big(\left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_2^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \\
&+ \left\| t_2^{1-\gamma} S_{\nu,\mu}(t_1) - t_1^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \Big) (\Theta u_0) + \left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_1^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \\
&\times \sum_{i=1}^{m} \lambda_i \Theta \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, u(s), Gu(s)) ds + \left\| \int_{t_1}^{t_2} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \right\| \\
&+ \left\| \int_0^{t_1} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds - \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \right\| \\
&+ \left\| \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds - \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_1 - s) f(s, u(s), Gu(s)) ds \right\| \\
&= J_1 + J_2 + J_3 + J_4 + J_5 + J_6,
\end{aligned}$$

where

$$J_1 = \Big(\left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_2^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \Big) (\Theta u_0),$$

$$J_2 = \Big(\left\| t_2^{1-\gamma} S_{\nu,\mu}(t_1) - t_1^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \Big) (\Theta u_0),$$

$$J_3 = \left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_1^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \sum_{i=1}^{m} \lambda_i \Theta \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, u(s), Gu(s)) ds,$$

$$J_4 = \left\| \int_{t_1}^{t_2} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \right\|,$$

$$J_5 = \left\| \int_0^{t_1} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds - \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \right\|,$$

$$J_6 = \left\| \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds - \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_1 - s) f(s, u(s), Gu(s)) ds \right\|.$$

Here we calculate

$$\left\| t_2^{1-\gamma}(Qu)(t_2) - t_1^{1-\gamma}(Qu)(t_1) \right\| \le \sum_{i=1}^{6} \|J_i\|.$$

Therefore, it is not difficult to see that $\|J_i\|$ tend to 0, when $t_2 - t_1 \to 0, i = 1, 2, \dots, 6$.

For J_1, by Lemma 2.6, we get

$$J_1 = \Big(\left\| t_2^{1-\gamma} S_{\nu,\mu}(t_2) - t_2^{1-\gamma} S_{\nu,\mu}(t_1) \right\| \Big) (\Theta u_0) \le \left\| t_2^{1-\gamma} \big(S_{\nu,\mu}(t_2) - S_{\nu,\mu}(t_1) \big) \right\| (\Theta u_0) \to 0, \ \text{ as } t_2 \to t_1.$$

For J_2, by Lemma 2.6, we get

$$\begin{aligned} J_2 &= \left(\left\|t_2^{1-\gamma} S_{\nu,\mu}(t_1) - t_1^{1-\gamma} S_{\nu,\mu}(t_1)\right\|\right)(\Theta u_0) \\ &\leq \frac{Mb^{\gamma-1}}{\Gamma(\gamma)} \left\|t_2^{1-\gamma} - t_1^{1-\gamma}\right\| \|\Theta u_0\| \\ &\leq \frac{Mb^{\gamma-1}}{\Gamma(\gamma)} \left\|(t_2 - t_1)^{1-\gamma}\right\| \|\Theta u_0\| \to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

For J_3, by Lemma 2.6, we have

$$\begin{aligned} J_3 &= \sum_{i=1}^{m} \lambda_i \Theta \left\|t_2^{1-\gamma} S_{\nu,\mu}(t_1) - t_1^{1-\gamma} S_{\nu,\mu}(t_1)\right\| \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, u(s), Gu(s)) ds \\ &\leq \frac{\overline{M} \sum_{i=1}^{m} |\lambda_i|}{1 - M \sum_{i=1}^{m} |\lambda_i|} \left\|t_2^{1-\gamma} S_{\nu,\mu}(t_1) - t_1^{1-\gamma} S_{\nu,\mu}(t_1)\right\| \int_0^{\tau_i} K_\mu(\tau_i - s) ds \\ &\to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

For J_4, by Lemma 2.6, we have

$$\begin{aligned} J_4 &= \left\| \int_{t_1}^{t_2} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \right\| \\ &\leq \overline{M} \int_{t_1}^{t_2} t_2^{1-\gamma} K_\mu(t_2 - s) ds \\ &\to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

For J_5, by Lemma 2.6, we have

$$\begin{aligned} J_5 &= \left\| \int_0^{t_1} t_2^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s)) ds \right\| \\ &\leq \frac{2M\overline{M}}{\Gamma(\mu)} \int_0^{t_1} \left[t_2^{1-\gamma}(t_2 - s)^{\mu-1} - t_1^{1-\gamma}(t_1 - s)^{\mu-1}\right] ds. \end{aligned}$$

Noting that

$$\begin{aligned} &\int_0^{t_1} \left[t_2^{1-\gamma}(t_2 - s)^{\mu-1} - t_1^{1-\gamma}(t_1 - s)^{\mu-1}\right] f(s, u(s), Gu(s)) ds \\ &\qquad \leq \int_0^{t_1} t_2^{1-\gamma}(t_2 - s)^{\mu-1} f(s, u(s), Gu(s)) ds. \end{aligned}$$

and

$$\int_0^{t_1} \left[t_2^{1-\gamma}(t_2 - s)^{\mu-1} - t_1^{1-\gamma}(t_1 - s)^{\mu-1}\right] f(s, u(s), Gu(s)) ds$$

exists, then by Lebesgue dominated convergence Theorem, we have

$$\begin{aligned} &\int_0^{t_1} \left[t_2^{1-\gamma}(t_2 - s)^{\mu-1} - t_1^{1-\gamma}(t_1 - s)^{\mu-1}\right] f(s, u(s), Gu(s)) ds \\ &\quad \leq \overline{M} \int_0^{t_1} \left[t_2^{1-\gamma}(t_2 - s)^{\mu-1} - t_1^{1-\gamma}(t_1 - s)^{\mu-1}\right] ds \\ &\quad \to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

It is easy to see that $\lim_{t_2\to t_1} J_5 = 0$.

For J_6, by Lemma 2.6, we have

$$\begin{aligned}
J_6 &= \left\| \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_2 - s) f(s, u(s), Gu(s))ds - \int_0^{t_1} t_1^{1-\gamma} K_\mu(t_1 - s) f(s, u(s), Gu(s))ds \right\| \\
&\le \overline{M} \int_0^{t_1} t_1^{1-\gamma} \left\| K_\mu(t_2 - s) - K_\mu(t_1 - s) \right\| ds \\
&\to 0, \quad \text{as } t_2 \to t_1.
\end{aligned}$$

In conclusion,

$$\|y(t_2) - y(t_1)\| \le \left\| t_2^{1-\gamma}(Qu)(t_2) - t_1^{1-\gamma}(Qu)(t_1) \right\| \to 0,$$

as $t_2 \to t_1$, i.e,

$$\left\| (Qu)(t_2) - (Qu)(t_1) \right\|_{1-\gamma} \to 0, \quad \text{as } t_2 \to t_1,$$

which means that $Q : [v_0, w_0] \to [v_0, w_0]$ is equicontinuous.

So, for any $D \subset [v_0, w_0]$, $Q(D) \subset [v_0, w_0]$ is bounded and equicontinuous. Therefore, by Lemma 2.2, there exists a countable set $D_0 = \{u_n\} \subset D$ such that

$$\alpha(Q(D)) \le 2\alpha(Q(D_0)). \tag{3.5}$$

For $t \in J$, by the definition of the operator Q, we have

$$\begin{aligned}
\alpha(Q(D_0(t))) &= \alpha\Big(\Big\{ S_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^m \lambda_i S_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K_\mu(\tau_i - s) f(s, v_{n-1}(s), Gv_{n-1}(s))ds \\
&+ \int_0^t K_\mu(t - s) f(s, v_{n-1}(s), Gv_{n-1}(s))ds \Big\}\Big) \\
&\le \frac{2M^2 \sum_{i=1}^m \lambda_i b^{\mu+\gamma-2}(L + LK_0)}{\Gamma(\gamma)\Gamma(\mu)(1 - M\sum_{i=1}^m)} \int_0^{\tau_i} \alpha(D_0(s))ds + \frac{2Mb^{\mu-1}(L + LK_0)}{\Gamma(\mu)} \int_0^t \alpha(D_0(s))ds \\
&\le \frac{2M^2 \sum_{i=1}^m \lambda_i b^{\mu+\gamma-1}(L + LK_0)}{\Gamma(\gamma)\Gamma(\mu)(1 - \sum_{i=1}^m \lambda_i)} \alpha(D) + \frac{2Mb^{\mu}(L + LK_0)}{\Gamma(\mu)} \alpha(D) \\
&\le \frac{2Mb^{\mu}(L + LK_0)}{\Gamma(\mu)} \Big[\frac{b^{\gamma-1} M \sum_{i=1}^m \lambda_i}{\Gamma(\gamma)(1 - \sum_{i=1}^m \lambda_i)} + 1 \Big] \alpha(D) \\
&= \frac{2Mb^{\mu}(L + LK_0)}{\Gamma(\mu)} \Big[\frac{(b^{\gamma-1} - \Gamma(\gamma))M \sum_{i=1}^m \lambda_i + \Gamma(\gamma)}{\Gamma(\gamma)(1 - M\sum_{i=1}^m \lambda_i)} \Big] \alpha(D).
\end{aligned}$$

Since $Q(D_0)$ is bounded and equicontinuous, we know from Lemma 2.3 that

$$\alpha(Q(D_0)) = \max_{t\in I} \alpha(Q(D_0)(t)).$$

Combining with (3.5), we have

$$\alpha(Q(D)) \le \eta\alpha(D),$$

where

$$\eta = \frac{2Mb^{\mu}(L + LK_0)}{\Gamma(\mu)} \Big[\frac{(b^{\gamma-1} - \Gamma(\gamma))M \sum_{i=1}^m \lambda_i + \Gamma(\gamma)}{\Gamma(\gamma)(1 - M\sum_{i=1}^m \lambda_i)} \Big] < 1.$$

Thus, $Q : [v_0, w_0] \to [v_0, w_0]$ is a strict set contraction operator. And by Lemma 2.9 that our conclusion valid. □

When $B = I$, then $D(B) = E$. We assume that A generates a norm continuous semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on E, then from the proof of Theorem 3.1, we have the following theorem.

Theorem 3.3. *Assume that (F1)-(F4) are satisfied. Then the following problem*

$$\begin{cases} D_{0+}^{\nu,\mu}u(t) + Au(t) = f(t,u(t),Gu(t)), \quad t \in (0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}u(0) = u_0 + \sum\limits_{i=1}^{m} \lambda_i u(\tau_i), \quad \tau_i \in (0,b], \end{cases}$$

has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.

4. Applications

In this section, we present an example, which illustrate the applicability of our main results.

Example 4.1. *We consider the following fractional partial differential equation*

$$\begin{cases} D_{0+}^{\nu,\mu}q(D_x)u(t,x) + p(D_x)u(t,x) = q(D_x)f(t,x,u(t,x),Gu(t,x)), (t,x) \in J\times\Omega, \\ I_{0+}^{(1-\nu)(1-\mu)}q(D_x)u(0,x) = q(D_x)\Big(u_0 + \sum_{i=1}^{m}\lambda_i u(\tau_i,x)\Big), \end{cases} \tag{4.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \leq \nu \leq 1, 0 < \mu < 1$, $t \in J = [0,b]$, $\lambda_i \neq 0, i = 1,2,\ldots,m$, integer $\mathbb{N} \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $f : J \times E \times E \to E$ is continuous and

$$p(D_x) = \sum_{|\alpha|\leq 2m} a_\alpha D_x^\alpha, \quad q(D_x) = \sum_{|\alpha|\leq 2m} b_\alpha D_x^\alpha, \quad a_\alpha, b_\alpha \in \mathbb{R},$$

are partial differential operators, here $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ is an n-dimensional multi-index, α denote their length, and

$$D_x^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\Big(\frac{\partial}{\partial x_2}\Big)^{\alpha_2}\cdots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n},$$

coefficient function $a_\alpha(x) \in C^{2m}(\overline{\Omega})$.

Let $E = L^p(\Omega)$ with $1 < p < \infty$, $P = \{u \in L^p(\Omega) : u(x) \geq 0, q.e.x \in \Omega\}$ and $A = p(D_x), B = q(D_x)$,

$$D(A) = \{f \in L^p(\Omega) : p(D_x)f \in L^p(\Omega)\},$$

$$D(B) = \{g \in L^p(\Omega) : q(D_x)g \in L^p(\Omega)\}.$$

Clearly, A and B are closed linear operators. The symbol of A, B will be denoted respectively by

$$p(\xi) = \sum_{|\alpha|\leq 2m} i^{|\alpha|}a_\alpha\xi^\alpha, q(\xi) = \sum_{|\alpha|\leq 2m} i^{|\alpha|}b_\alpha\xi^\alpha, \xi \in \mathbb{R}^n.$$

Then the above equation (4.1) can be reformulated as the abstract (1.1).

We deploy the following result in [43] (for the case $E_l = E, C_{r,l} = I$):

Theorem 4.1. *Assume that $q(\xi) \neq 0$ for each $\xi \in \mathbb{R}^n$ and*

$$\omega = \sup_{\xi\in\mathbb{R}^n} Re[p(\xi)q^{-1}(\xi)] < \infty.$$

Then the pair $(-A, B)$ generates propagation family $\{T(t)\}_{t\geq 0}$ mapping $D(B)$ into E such that

$$\|T(t)\| \leq Ce^{\omega t}, \quad t \geq 0,$$

where C is a positive constant.

Theorem 4.2. *If the following conditions*

(H1) Let $u_0(x) \geq 0, x \in \Omega$, and there exists a function $w = w(t,x) \in C_{1-\gamma}(J \times \Omega)$ such that

$$\begin{cases} D_{0+}^{\nu,\mu} q(D_x)w(t,x) + p(D_x)w(t,x) \geq q(D_x)f(t,x,w(t,x),Gw(t,x)), \\ I_{0+}^{(1-\nu)(1-\mu)} q(D_x)w(0,x) \geq q(D_x)\Big(u_0 + \sum_{i=1}^{m} \lambda_i w(\tau_i,x)\Big), \end{cases} \tag{4.2}$$

and the assumptions (F1)-(F4) are satisfied. Then the problem (4.1) has minimal and maximal mild solutions between 0 *and $w(x,t)$, which can be obtained by a monotone iterative procedure starting from* 0 *and $w(t)$, respectively.*

Proof. Assumption (H1) implies that $v_0 \equiv 0$ and $w_0 \equiv w(x,t)$ are lower and upper solutions of the problem (4.1), respectively. So our conclusion follows from Theorem 3.1.

5. Conclusions

In this paper, we focused on the existence of mild solutions for a class of evolution equations with Hilfer fractional derivative. By using monotone iterative technique, the fixed point theorem combined with noncompactness measure, we obtain some existence result of mild solutions for Hilfer fractional evolution equations with nonlocal conditions. Particularly, in this work, we do not assume that the solution operators generated by linear systems are compact. We study (1.1) without assuming B has bounded (or compact) inverse as well as without any assumption on the relation between $D(A)$ and $D(B)$.

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