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Solving Integral Equations via Admissible Contraction Mappings

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Abstract. In this article, we introduce a new concept of admissible contraction and prove fixed point theorems which generalize Banach contraction principle in a different way more than in the known results from the literature. The article includes an example which shows the validity of our results, and additionally we obtain a solution of integral equation by admissible contraction mapping in the setting of b-metric spaces.

1. Introduction

Ciric [10] introduced the quasi-contractivity and multivalued quasi-contractions and established fixed point results under these contractions. In 1989, Bakhtin [7] introduced the concept of b-metric space. Czerwik [12] first presented a generalization of the Banach fixed point theorem in b-metric spaces, which is a problem of the convergence of measurable functions concerning measure.

Using this idea, many researchers presented a generalization of the renowned Banach fixed point theorem in the b-metric space. Czerwik's [13], Audi, Bota and Karapinar [6], Sintunavaat, Plibtieng, and Katchang [34], Kir and Kiziltunc [22], Dubey, Shukla, and Dubey [14] extended the fixed point theorem in b-metric space. Latif et al. [23] explained Suzuki type theorems for nonlinear contraction conditions in the b-metric space configuration. Pant and Panicker [28] obtained some fixed point theorems for admissible mappings in b-metric space and also discussed an application to a nonlinear quadratic integral equation.

Many fixed point theorems, such as the well-known Geraghty and Ciric theorems on b-metric spaces by Mlaiki [27], were improved by his results. In recent years, many fixed point results for single-valued and multivalued operators in b-metric spaces have been extensively studied in [1, 4, 8, 15, 18, 19, 24, 25, 29, 32] and elsewhere. Alghamdi [2] was the first to talk about b-metric-like space as well as in a partially ordered b-metric-like space. Shukla [33] generalized both the concepts of b-metric and partial metric spaces by introducing the partial b-metric space and an analogy of the Banach contraction principle, as well as the Kannan type fixed point theorem in partial b-metric spaces, which he also proved. Chen, Dong, and Zhu [9] introduced the concept of quasi-b-metric-like spaces and some fixed point results are investigated in quasi-b-metric-like spaces. Many papers have dealt with fixed point for single and multivalued in b-metric-like spaces (see [20, 31]). In 2012, Samet et al. [30] initiated the concepts of α-admissible mappings and

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established many fixed point results for such mappings defined on complete metric spaces. Afterward, Alsulami et al. [3] and Karapinar et al. [21] modified the notion of admissible mapping with contractions and integral types of generalized metric spaces. The idea of α-admissible has been utilized by many researchers (see, [5, 11, 16, 17, 26, 35]).

In this article, using a mapping $\zeta : \mathbf{R}_0^{+\omega} \to \mathbf{R}_0^+$, we introduce a new type of contraction called $\alpha - \zeta$-contraction and prove a new fixed point theorem concerning $\alpha - \zeta$-contraction. The article includes the examples of $\alpha - \zeta$-contractions and give an integral equation application support by the nature of $\alpha - \zeta$-contractions.

2. Preliminaries

In this paper, we use the following notations. The sets of natural numbers, non-negative reals, and real numbers are denoted by $\mathbb{N}$, $\mathbf{R}_0^+$ and $\mathbf{R}$, respectively. Czerwik [7] formally defined the notion of a b-metric space as follows:

Definition 2.1. ([7]) *Let $\mathfrak{P} \neq \emptyset$. We say that a mapping $\eth : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ is a $\mathfrak{b}$-metric if there exists a positive number η such that $\forall \vartheta, \varsigma, \varrho \in \mathfrak{P}$,*

($\eth_1$) $\eth(\vartheta, \varsigma) = 0 \Longleftrightarrow \vartheta = \varsigma$;
($\eth_2$) $\eth(\vartheta, \varsigma) = \eth(\varsigma, \vartheta)$;
($\eth_3$) $\eth(\vartheta, \varrho) \leq \eta(\eth(\vartheta, \varsigma) + \eth(\varsigma, \varrho))$.

Then triplet $(\mathfrak{P}, \eth, \eta)$ is called a $\mathfrak{b}$-MS(shortly, $\mathfrak{b}$-MS).

The following is the main result in Aleksic [1].

Theorem 2.2. ([1]) *Let $(\mathfrak{P}, \eth)$ be a complete $\mathfrak{b}$-MS with a constant $\eta \geq 1$. If $\mathbb{G} : \mathfrak{P} \to \mathfrak{P}$ satisfies the inequality:*

$$\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq \tau_1 \eth(\vartheta, \varsigma) + \tau_2 \eth(\vartheta, \mathbb{G}\vartheta) + \tau_3 \eth(\varsigma, \mathbb{G}\varsigma) + \tau_4 \eth(\vartheta, \mathbb{G}\varsigma) + \eth(\mathbb{G}\vartheta, \varsigma),$$

where $\tau_\kappa \geq 0$, $\forall \kappa = 1, 2, 3, 4$ and $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$ for $\eta \in [1, 2]$ and $\frac{2}{\eta} < \tau_1 + \tau_2 + \tau_3 + 2\tau_4 < 1$, $\forall \eta \in [3, +\infty)$, then $\mathbb{G}$ has a unique fixed point.

Kirk [22] initiated the following concepts as follows.

Definition 2.3. ([22]) *Let $\{\vartheta_\upsilon\}$ be a sequence in $\mathfrak{b}$-MS $(\mathfrak{P}, \eth, \eta \geq 1)$.*
(i) If for any positive number ξ, there exists $\upsilon_0 \in \mathbb{N}$ such that $\eth(\vartheta_\upsilon, \vartheta_\omega) < \xi$, $\forall \upsilon, \omega \geq \upsilon_0$. Then the sequence $\{\vartheta_\upsilon\}$ is called Cauchy sequence.
(ii) If there exists $\hbar \in \mathfrak{P}$ such that any positive number ξ, there exists $\upsilon_0 \in \mathbb{N}$ such that $\eth(\vartheta_\upsilon, \hbar) < \xi$, $\forall \upsilon \geq \upsilon_0$. Then, we say that the sequence $\{\vartheta_\upsilon\}$ converges to $\hbar$.

Definition 2.4. ([22]) *We say that a $\mathfrak{b}$-MS $(\mathfrak{P}, \eth, \eta \geq 1)$ is complete if every Cauchy sequence is convergent.*

To prove our main results, we will use the following lemma in Latif [23], since $\mathfrak{b}$-metric is not continuous.

Lemma 2.5. ([23]) *Suppose that any two sequences $\{\vartheta_\upsilon\}$ and $\{\varsigma_\upsilon\}$ in $(\mathfrak{P}, \eth, \eta \geq 1)$ converge to ϑ and $\varsigma \in \mathfrak{P}$. Then*

$$\eta^2 \eth(\vartheta, \varsigma) \geq \lim_{\upsilon \to +\infty} \sup \eth(\vartheta_\upsilon, \varsigma_\upsilon) \geq \lim_{\upsilon \to +\infty} \inf \eth(\vartheta_\upsilon, \varsigma_\upsilon) \geq \frac{1}{\eta^2} \eth(\vartheta, \varsigma).$$

Particularly, if $\vartheta = \varsigma$, then $\lim_{\upsilon \to +\infty} \eth(\vartheta_\upsilon, \varsigma_\upsilon) = 0$. Moreover, for any $\varrho \in \mathfrak{P}$, we obtain

$$\eta \eth(\vartheta, \varrho) \geq \lim_{\upsilon \to +\infty} \sup \eth(\vartheta_\upsilon, \varrho) \geq \lim_{\upsilon \to +\infty} \inf \eth(\vartheta_\upsilon, \varrho) \geq \frac{1}{\eta} \eth(\vartheta, \varrho).$$

In [25], Miculescu proved the following interesting results.

Lemma 2.6. ([25]) *For each sequence $\{\vartheta_\upsilon\}$ of* b*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *is Cauchy if there exists* $\tau \in [0, 1)$ *such that* $\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g}) \leq \tau \eth(\vartheta_{\upsilon-g}, \vartheta_\upsilon)$, $\forall \upsilon \in \mathbb{N}$.

In [20], Jain introduced the following notion of new contractive mapping.

Definition 2.7. ([20]) *For any* $\omega \in \mathbb{N}$, $\mathbb{E}_\omega$ *denote the family of all functions* $\zeta : \mathbf{R}_0^{+\omega} \to \mathbf{R}_0^+$ *such that*
(i) $\zeta(\varpi_1, \varpi_2, \varpi_3, ..., \varpi_\omega) < \max\{\varpi_1, \varpi_2, \varpi_3, ..., \varpi_\omega\}$ *if* $(\varpi_1, \varpi_2, \varpi_3, ..., \varpi_\omega) \neq (0, 0, 0, ..., 0)$;
(ii) if $\{\varpi_\kappa{}^\upsilon\}_{\upsilon\in\mathbb{N}}$, $1 \leq \kappa \leq \omega$ *are* ω *sequences in* $\mathbf{R}_0^+$ *such that*

$$\lim_{\upsilon\to+\infty} \sup \varpi_\kappa^{(\upsilon)} = \varpi_\kappa < +\infty, \forall \kappa = 1 \textit{ to } \omega,$$

then

$$\lim_{\upsilon\to+\infty} \inf \zeta(\varpi_1^\upsilon, \varpi_2^\upsilon, \varpi_3^\upsilon, ..., \varpi_\omega^\upsilon) \leq \zeta(\varpi_1, \varpi_2, \varpi_3, ..., \varpi_\omega).$$

The following α-admissible mapping was first initiated by Samet et al. [30].

Definition 2.8. *Let* $\mathfrak{P} \neq \emptyset$ *and a mapping* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$. *Then* $\mathbb{G}$ *is said to be* α*-admissible if* $(\vartheta, \varsigma) \in \mathfrak{P} \times \mathfrak{P}$,

$$\alpha(\vartheta, \varsigma) \geq 1 \textit{ implies } \alpha(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \geq 1. \tag{1}$$

In this paper, we present the notion of admissible ζ- contraction mapping of types, which includes the ζ-contraction (resp. ζ-contraction of types) of Jain et al. [20]. Utilizing this class of mapping, we establish approximate fixed point and fixed point theorems in the setting of b-metric and b-metric-like spaces.

3. Main Results

We introduce α-admissible ζ-contraction map of type-I motivated by Jain et al. [20] as follows.

Definition 3.1. *Let* $\mathbb{G}$ *be a self-map on* b*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *and a mapping* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$. *We say that* $\mathbb{G}$ *is* ζ*-contractive map of type-I if there exists* $\zeta \in \mathbb{E}_4$ *and* $\forall \vartheta, \varsigma \in \mathfrak{P}$,

$$\alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq \frac{1}{\eta}\zeta(\vartheta, \varsigma), \tag{2}$$

where

$$\zeta(\vartheta, \varsigma) = \max\left(\eth(\vartheta, \varsigma), \eth(\vartheta, \mathbb{G}\vartheta), \eth(\varsigma, \mathbb{G}\varsigma), \frac{\eth(\vartheta, \mathbb{G}\varsigma) + \eth(\mathbb{G}\vartheta, \varsigma)}{2\eta}\right).$$

In the following main theorem, Jain et al. [20] proved fixed point theorems in ζ-contraction in b-metric space, we extend this our initiated admissible ζ-contractive mapping of type - I in the setting of b-metric space.

Theorem 3.2. *Let* $\mathbb{G}$ *be a self-map on complete* b*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *and let* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *be a function. Assume that the following conditions are true:*

(i) $\mathbb{G}$ *is* α*-admissible.*
(ii) $\exists\, \vartheta_1 \in \mathfrak{P}$ *such that* $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ *and* $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$.
(iii)

$$\alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq \frac{1}{\eta}\zeta(\vartheta, \varsigma),$$

where $\zeta(\vartheta, \varsigma) = \max\left(\eth(\vartheta, \varsigma), \eth(\vartheta, \mathbb{G}\vartheta), \eth(\varsigma, \mathbb{G}\varsigma), \frac{\eth(\vartheta, \mathbb{G}\varsigma) + \eth(\mathbb{G}\vartheta, \varsigma)}{2\eta}\right)$, $\forall \vartheta, \varsigma \in \mathfrak{P}$.

Then, $\mathbb{G}$ *has a unique fixed point.*

Proof. Let $\vartheta_1 \in \mathfrak{P}$ be such that $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ and $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$. Since Banach abstracted the fixed point theorem from the result of Picard, we define the Picard's iterative sequence $\{\vartheta_\upsilon\}$ in $\mathfrak{P}$ by the rule $\vartheta_\upsilon = \mathbb{G}\vartheta_{\upsilon-1} = \mathbb{G}^\upsilon\vartheta_1$, $\forall\ \upsilon \geq 1$. Obviously, if there exists $\upsilon_0 \geq 1$ for which $\mathbb{G}^{\upsilon_0}\vartheta_1 = \mathbb{G}^{\upsilon_0+1}\vartheta_1$ then $\mathbb{G}^{\upsilon_0}\vartheta_1$ has a fixed point of $\mathbb{G}$. Thus, we suppose that $\mathbb{G}^\upsilon\vartheta_1 \neq \mathbb{G}^{\upsilon+1}\vartheta_1$ for every $\upsilon \geq 1$.

Since $\mathbb{G}$ is α-admissible, the condition (ii) implies

$$\alpha(\vartheta_1, \vartheta_2) = \alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1 \implies \alpha(\mathbb{G}\vartheta_1, \mathbb{G}\vartheta_2) = \alpha(\vartheta_2, \vartheta_3) \geq 1,$$

continuing in this way,

$$\alpha(\vartheta_\upsilon, \vartheta_{\upsilon+1}) \geq 1,\ \forall \upsilon \in \mathbb{N}.$$

In a similar way, starting with

$$\alpha(\vartheta_1, \vartheta_3) = \alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1 \implies \alpha(\mathbb{G}\vartheta_1, \mathbb{G}\vartheta_3) = \alpha(\vartheta_2, \vartheta_4) \geq 1,$$

we deduce

$$\alpha(\vartheta_\upsilon, \vartheta_{\upsilon+2}) \geq 1, \forall \upsilon \in \mathbb{N}.$$

Assume that $\vartheta_\upsilon \neq \vartheta_{\upsilon+1}$ $\forall \upsilon \in \mathbb{N}$. Now, we prove $\{\vartheta_\upsilon\}$ is a Cauchy sequence. Let $\upsilon \in \mathbb{N}$. Consider

$$\begin{aligned}
\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g}) &= \eth(\mathbb{G}^\upsilon\vartheta_1, \mathbb{G}^{\upsilon+1}\vartheta_1) \\
&\leq \alpha(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1)\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1) \\
&\leq \frac{1}{\eta}\max\Big(\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1), \eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1), \eth(\mathbb{G}^\upsilon\vartheta_1, \mathbb{G}^{\upsilon+1}\vartheta_1), \\
&\quad \frac{\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^{\upsilon+1}\vartheta_1) + \eth(\mathbb{G}^\upsilon\vartheta_1, \mathbb{G}^\upsilon\vartheta_1)}{2\eta}\Big) \\
&= \frac{1}{\eta}\max\Big(\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1), \frac{\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^{\upsilon+1}\vartheta_1)}{2\eta}\Big) \\
&\leq \frac{1}{\eta}\max\Big(\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1), \frac{\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1) + \eth(\mathbb{G}^\upsilon\vartheta_1, \mathbb{G}^{\upsilon+1}\vartheta_1)}{2}\Big) \\
&\leq \frac{1}{\eta}\max\Big(\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon) + \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1})}{2}\Big),
\end{aligned} \tag{3}$$

by (3) implies that

$$\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g}) < \frac{1}{\eta}\eth(\vartheta_{\upsilon-g}, \vartheta_\upsilon),\ \forall \upsilon \geq 1. \tag{4}$$

Case 1: If $\eta > 1$, then, the sequence $\{\vartheta_\upsilon\}$ is Cauchy, by Lemma 2.6 in view of equation (4).
Case 2: If $\eta = 1$, then, by equation (4), we get monotonically decreasing and bounded below sequence $\{\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g})\}$. Now, we obtain, $\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g}) \to \flat$ for some $\flat \geq 0$. Suppose that $\flat > 0$ now, taking $\lim_{\upsilon\to+\infty}$ in (3), we have $\flat \leq \zeta(\flat, \flat, \flat, \flat')$, where

$$\flat' = \lim_{\upsilon\to+\infty}\sup\frac{\eth(\vartheta_{\upsilon-g}, \vartheta_{\upsilon+g})}{2} \leq \lim_{\upsilon\to+\infty}\sup\frac{\eth(\vartheta_{\upsilon-g}, \vartheta_\upsilon) + \eth(\vartheta_\upsilon, \vartheta_{\upsilon+g})}{2}.$$

Now, $\flat \leq \zeta(\flat, \flat, \flat, \flat') < \max(\flat, \flat, \flat, \flat') = \flat$, which is a contradiction, therefore,

$$\lim_{\upsilon\to+\infty}\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g}) = 0. \tag{5}$$

On contrary, we assume that the sequence $\{\vartheta_{\upsilon}\}$ is not Cauchy, then $\exists\ \xi > 0$ and sequences $\{\omega_{\mathfrak{n}}\}, \{\upsilon_{\mathfrak{n}}\}; \omega_{\mathfrak{n}} > \upsilon_{\mathfrak{n}} \geq \mathfrak{n}$ such that

$$\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}}) \geq \xi. \tag{6}$$

Now, take $\omega_{\mathfrak{n}} > \upsilon_{\mathfrak{n}}$ such that equation (6) holds. Then,

$$\begin{aligned}
\xi &\leq \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}}) \\
&\leq \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\omega_{\mathfrak{n}-g}}) + \eth(\vartheta_{\omega_{\mathfrak{n}-g}}, \vartheta_{\upsilon_{\mathfrak{n}}}) \\
&< \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}-g}}) + \xi \\
&< \eth(\vartheta_{\mathfrak{n}}, \vartheta_{\mathfrak{n}-g}) + \xi,
\end{aligned}$$

thus, taking $\lim \mathfrak{n} \to +\infty$ and by (4), we get

$$\lim_{\mathfrak{n}\to+\infty} \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon}) = \xi. \tag{7}$$

Now, consider

$$\begin{aligned}
\eth(\vartheta_{\omega_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}+1}) &\leq \alpha(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}})\eth(\mathbb{G}\vartheta_{\omega_{\mathfrak{n}}}, \mathbb{G}\vartheta_{\upsilon_{\mathfrak{n}}}) \\
&\leq \max\left(\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}}), \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\omega_{\mathfrak{n}}+1}), \eth(\vartheta_{\upsilon_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}+1}), \frac{\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}+1}) + \eth(\vartheta_{\omega_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}})}{2}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}}) &\leq \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\omega_{\mathfrak{n}}+1}) + \eth(\vartheta_{\omega_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}+1}) + \eth(\vartheta_{\upsilon_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}}) \\
&\leq \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\omega_{\mathfrak{n}}+1}) + \eth(\vartheta_{\upsilon_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}}) \\
&\quad + \max\left(\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}}), \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\omega_{\mathfrak{n}}+1}), \eth(\vartheta_{\upsilon_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}+1}), \frac{\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}+1}) + \eth(\vartheta_{\omega_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}})}{2}\right).
\end{aligned}$$

From the above, setting $\liminf_{\mathfrak{n}\to+\infty}$ and using equations (5) and (7). Thus, we get $\xi \leq 0 + 0 + \zeta(\xi, 0, 0, \xi')$, where

$$\begin{aligned}
\xi' &= \lim_{\mathfrak{n}\to+\infty} \sup \frac{\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}+1}) + \eth(\vartheta_{\omega_{\mathfrak{n}}+1}, \vartheta_{\upsilon_{\mathfrak{n}}})}{2} \\
&\leq \lim_{\mathfrak{n}\to+\infty} \sup \frac{\eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}}) + \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}+1}) + \eth(\vartheta_{\omega_{\mathfrak{n}}+1}, \vartheta_{\omega_{\mathfrak{n}}}) + \eth(\vartheta_{\omega_{\mathfrak{n}}}, \vartheta_{\upsilon_{\mathfrak{n}}})}{2} \\
&= \frac{\xi + 0 + 0 + \xi}{2} \\
&= \xi.
\end{aligned}$$

Thus, $\xi \leq \zeta(\xi, 0, 0, \xi') < \max\{\xi, 0, 0, \xi'\} = \xi$, a contradiction. Thus, the Cauchy sequence $\{\vartheta_{\upsilon}\}$ in $\mathfrak{b}$-MS $(\mathfrak{P}, \eth, \eta \geq 1)$ is complete. Therefore, $\exists\ \vartheta \in \mathfrak{P}$ such that $\vartheta_{\upsilon} \to \vartheta$.

Consider

$$\begin{aligned}
\eth(\mathbb{G}\vartheta_{\upsilon}, \mathbb{G}\vartheta) &\leq \alpha(\vartheta_{\upsilon}, \vartheta)\eth(\mathbb{G}\vartheta_{\upsilon}, \mathbb{G}\vartheta) \\
&\leq \frac{1}{\eta} \max\left(\eth(\vartheta_{\upsilon}, \vartheta), \eth(\vartheta_{\upsilon}, \mathbb{G}\vartheta_{\upsilon}), \eth(\vartheta, \mathbb{G}\vartheta), \frac{\eth(\vartheta_{\upsilon}, \mathbb{G}\vartheta) + \eth(\vartheta, \mathbb{G}\vartheta_{\upsilon})}{2\eta}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\eth(\vartheta_{\upsilon+1}, \mathbb{G}\vartheta) &= \eth(\mathbb{G}\vartheta_{\upsilon}, \mathbb{G}\vartheta) \\
&\leq \alpha(\vartheta_{\upsilon}, \vartheta)\eth(\mathbb{G}\vartheta_{\upsilon}, \mathbb{G}\vartheta) \\
&\leq \frac{1}{\eta} \max\left(\eth(\vartheta_{\upsilon}, \vartheta), \eth(\vartheta_{\upsilon}, \mathbb{G}\vartheta_{\upsilon+1}), \eth(\vartheta, \mathbb{G}\vartheta), \frac{\eth(\vartheta_{\upsilon}, \mathbb{G}\vartheta) + \eth(\vartheta, \mathbb{G}\vartheta_{\upsilon})}{2\eta}\right).
\end{aligned}$$

From the above inequality taking lim inf $\upsilon \to +\infty$ and by Lemma 2.5, we get

$$\frac{1}{\eta}\eth(\vartheta, \mathbb{G}\vartheta) \leq \frac{1}{\eta}\max(0, 0, \eth(\vartheta, \mathbb{G}\vartheta), \hbar),$$

i.e.,

$$\eth(\vartheta, \mathbb{G}\vartheta) \leq \max(0, 0, \eth(\vartheta, \mathbb{G}\vartheta), \hbar),$$

where

$$\hbar = \lim_{\upsilon\to+\infty}\sup\frac{\eth(\vartheta_\upsilon, \mathbb{G}\vartheta) + \eth(\vartheta, \mathbb{G}\vartheta_\upsilon)}{2\eta} \leq \lim_{\upsilon\to+\infty}\sup\frac{s\eth(\vartheta, \mathbb{G}\vartheta) + 0}{2\eta} = \frac{\eth(\vartheta, \mathbb{G}\vartheta)}{2}.$$

Thus

$$\eth(\vartheta, \mathbb{G}\vartheta) \leq \zeta(0, 0, \eth(\vartheta, \mathbb{G}\vartheta), \hbar) < \max\{0, 0, \eth(\vartheta, \mathbb{G}\vartheta), \hbar\} = \eth(\vartheta, \mathbb{G}\vartheta),$$

which is a contradiction. Hence $\mathbb{G}\vartheta = \vartheta$.

Suppose that ϑ, ς are two fixed points of $\mathbb{G}$ such that $\mathbb{G}\vartheta = \vartheta \neq \varsigma = \mathbb{G}\varsigma$. Then, for all $\vartheta, \varsigma \in \mathfrak{P}$ such that $\alpha(\vartheta, \varsigma) \geq 1$. If $\eth(\vartheta, \varsigma) > 0$ then, by the contractive condition (iii) with the fixed points ϑ and ς yields

$$\begin{aligned}
\eth(\vartheta, \varsigma) = \alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) &\leq \frac{1}{\eta}\max\Big(\eth(\vartheta, \varsigma), \eth(\vartheta, \mathbb{G}\vartheta), \eth(\varsigma, \mathbb{G}\varsigma), \frac{\eth(\vartheta, \mathbb{G}\varsigma) + \eth(\varsigma, \mathbb{G}\vartheta)}{2\eta}\Big) \\
&\leq \frac{1}{\eta}\max\Big(\eth(\vartheta, \varsigma), 0, 0, \frac{\eth(\vartheta, \varsigma)}{\eta}\Big) \\
&< \frac{1}{\eta}\max\Big\{(\eth(\vartheta, \varsigma), 0, 0, \frac{\eth(\vartheta, \varsigma)}{\eta}\Big\} \\
&= \frac{\eth(\vartheta, \varsigma)}{\eta},
\end{aligned}$$

which is a contradiction. Therefore, $\vartheta = \varsigma$. □

Now, the following corollary is an extension of Theorem 3.2.

Corollary 3.3. *Let $\mathbb{G}$ be a self-map on complete* $\mathfrak{b}$*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *and let* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *be a function. Suppose that there exists* $\mathrm{q} \in [0, \frac{1}{\eta})$ *such that the following assumptions are true:*

(*i*) $\mathbb{G}$ *is* α*-admissible;*
(*ii*) $\exists\ \vartheta_1 \in \mathfrak{P}$ *such that* $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ *and* $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$*;*
(*iii*)

$$\alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq \mathrm{q}\max\Big\{\eth(\vartheta, \varsigma), \eth(\vartheta, \mathbb{G}\vartheta), \eth(\varsigma, \mathbb{G}\varsigma), \frac{\eth(\vartheta, \mathbb{G}\varsigma) + \eth(\mathbb{G}\vartheta, \varsigma)}{2\eta}\Big\}, \quad \forall\ \vartheta, \varsigma \in \mathfrak{P} \tag{8}$$

Then, $\mathbb{G}$ *has a unique fixed point.*

Proof. Let $\zeta \in \mathbb{E}_4$ be defined by $\zeta(\varpi_1, \varpi_2, \varpi_3, \varpi_4) = \varsigma\eta\max\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$. Then $\mathbb{G}$ has a unique fixed point by Theorem 3.2. □

We see that all conditions are satisfied in Theorem 3.2, but it is not applicable in Corollary 3.3.

Example 3.4. *Let* $\mathfrak{P} = \Big\{\frac{1}{\sqrt{\upsilon}} : \upsilon \in \mathbb{N} \cup \{0\}\Big\}$*. Define* $\eth : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *by* $\eth(\vartheta, \varsigma) = |\vartheta - \varsigma|^2$, $\forall \vartheta, \varsigma \in \mathfrak{P}$*. Then* $\eth$ *is a* $\mathfrak{b}$*-metric on* $\mathfrak{P}$ *with* $\eta = 2$*. A self-map* $\mathbb{G}$ *on* $\mathfrak{P}$ *defined by*

$$\mathbb{G}(\frac{1}{\sqrt{\upsilon}}) = \frac{1}{\sqrt{2(\upsilon + 1)}}, \ \forall \upsilon \in \mathbb{N} \ \textit{and} \ \mathbb{G}(0) = 0.$$

Define

$$\zeta(\varpi_1,\varpi_2,\varpi_3,\varpi_4) = \begin{cases} \frac{\max\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}}{1+\varpi_1}, & \text{if } \varpi_1 > 0, \\ \frac{1}{2}\max\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}, & \text{otherwise.} \end{cases}$$

and define $\alpha : \mathfrak{P}\times\mathfrak{P} \to \mathbf{R}_0^+$ *by*

$$\alpha(\vartheta,\varsigma) = \begin{cases} 1, & \text{if } \vartheta \le \varsigma \text{ or } \varsigma \le \vartheta, \\ 0, & \text{if otherwise.} \end{cases}$$

Now, for all $\vartheta,\varsigma \in \mathfrak{P}$*, condition (iii) of Theorem 3.2 is satisfied, and all conditions of Theorem 3.2 are satisfied. However, if* (8) *is satisfied, then, we have*

$$\alpha(\vartheta,\varsigma)\eth(\mathbb{G}\vartheta,\mathbb{G}\varsigma) \le \mathrm{q}\mathbb{N}(\vartheta,\varsigma),\ \forall\vartheta,\varsigma\in\mathfrak{P},$$

where $\mathbb{N}(\vartheta,\varsigma) = \max\{\eth(\vartheta,\varsigma),\eth(\vartheta,\mathbb{G}\vartheta),\eth(\varsigma,\mathbb{G}\varsigma),\frac{\eth(\vartheta,\mathbb{G}\varsigma)+\eth(\mathbb{G}\vartheta,\varsigma)}{2\eta}\}$*. So, in particular, we have*

$$\alpha\left(\frac{1}{\sqrt{\upsilon}},\frac{1}{\sqrt{\omega}}\right)\eth\left(\frac{1}{\sqrt{2(\upsilon+1)}},\frac{1}{\sqrt{2(\omega+1)}}\right) \le \mathrm{q}\mathbb{N}\left(\frac{1}{\sqrt{\upsilon}},\frac{1}{\sqrt{\omega}}\right),\ \forall\omega,\upsilon\in\mathbb{N},\ \omega\ne\upsilon,$$

i.e.,

$$\frac{\left|\frac{1}{\sqrt{2(\upsilon+1)}},\frac{1}{\sqrt{2(\omega+1)}}\right|^2}{\mathbb{N}\left(\frac{1}{\sqrt{\upsilon}},\frac{1}{\sqrt{\omega}}\right)} \le 2\mathrm{q},\ \forall\omega,\upsilon\in\mathbb{N},\ \omega\ne\upsilon.$$

In the above inequality, take $\lim \upsilon,\omega\to+\infty$*, we have* $2\mathrm{q}\ge 1$*, a contradiction. Thus, this example is not applied for Corollary 3.3.*

3.1. Second Main Result

We introduce the another concept of α-admissible ζ-contraction mapping of type-II motivated by Jain et al. [20] as follows.

Definition 3.5. *Let* $\mathbb{G}$ *be a self-map on* $\mathfrak{b}$*-MS* $(\mathfrak{P},\eth,\eta\ge 1)$ *and a mapping* $\alpha : \mathfrak{P}\times\mathfrak{P}\to\mathbf{R}_0^+$*. We say that* $\mathbb{G}$ *is* ζ*-contractive map of type-II if there exists* $\zeta\in\mathbb{E}_5$*,*

$$\alpha(\vartheta,\varsigma)\eth(\mathbb{G}\vartheta,\mathbb{G}\varsigma) \le \frac{1}{\eta}\zeta(\vartheta,\varsigma),\ \forall\vartheta,\varsigma\in\mathfrak{P}, \tag{9}$$

where $\zeta(\vartheta,\varsigma) = \max\left(\eth(\vartheta,\varsigma),\eth(\vartheta,\mathbb{G}\vartheta),\eth(\varsigma,\mathbb{G}\varsigma),\frac{\eth(\vartheta,\mathbb{G}\varsigma)}{2\eta},\eth(\mathbb{G}\vartheta,\varsigma)\right)$*.*

In a similar way, the proof of our succeeding results proceeds as the proof of Theorem 3.2.

Theorem 3.6. *Let* $\mathbb{G}$ *be a self-map on complete* $\mathfrak{b}$*-MS* $(\mathfrak{P},\eth,\eta\ge 1)$ *and* $\alpha : \mathfrak{P}\times\mathfrak{P}\to\mathbf{R}_0^+$ *be a function. Assume that the following conditions are true:*

(*i*) $\mathbb{G}$ *is* α*-admissible;*
(*ii*) $\exists\ \vartheta_1\in\mathfrak{P}$ *such that* $\alpha(\vartheta_1,\mathbb{G}\vartheta_1)\ge 1$ *and* $\alpha(\vartheta_1,\mathbb{G}^2\vartheta_1)\ge 1$*;*
(*iii*)

$$\alpha(\vartheta,\varsigma)\eth(\mathbb{G}\vartheta,\mathbb{G}\varsigma) \le \frac{1}{\eta}\zeta(\vartheta,\varsigma),\ \forall\vartheta,\varsigma\in\mathfrak{P},$$

where $\zeta(\vartheta,\varsigma) = \max\left(\eth(\vartheta,\varsigma),\eth(\vartheta,\mathbb{G}\vartheta),\eth(\varsigma,\mathbb{G}\varsigma),\frac{\eth(\vartheta,\mathbb{G}\varsigma)}{2\eta},\eth(\mathbb{G}\vartheta,\varsigma)\right)$*.*

Then, $\mathbb{G}$ *has a unique fixed point.*

Corollary 3.7. *Let* $\mathbb{G}$ *be a self-map on complete* $\mathfrak{b}$*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *and* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *be a function. Assume that there exists* $q \in [0, \frac{1}{\eta})$ *such that the following results are true:*

(*i*) $\mathbb{G}$ *is* α*-admissible;*
(*ii*) $\exists\, \vartheta_1 \in \mathfrak{P}$ *such that* $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ *and* $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$*;*
(*iii*)

$$\alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq q \max\Big(\eth(\vartheta, \varsigma), \eth(\vartheta, \mathbb{G}\vartheta), \eth(\varsigma, \mathbb{G}\varsigma), \frac{\eth(\vartheta, \mathbb{G}\varsigma)}{2\eta}, \eth(\mathbb{G}\vartheta, \varsigma)\Big), \forall \vartheta, \varsigma \in \mathfrak{P}.$$

Then, $\mathbb{G}$ *has a unique fixed point.*

Proof. Let ζ in $\mathbb{E}_5$ defined by $\zeta(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \varsigma\eta \max\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$. Then, by Theorem 3.6, $\mathbb{G}$ has a unique fixed point. □

Corollary 3.8. *Let* $\mathbb{G}$ *be a self-map on complete* $\mathfrak{b}$*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *and* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *be a function. Assume the following conditions are true:*

(*i*) $\mathbb{G}$ *is* α*-admissible;*
(*ii*) $\exists\, \vartheta_1 \in \mathfrak{P}$ *such that* $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ *and* $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$*;*
(*iii*) $\forall\, \vartheta, \varsigma \in \mathfrak{P}$,

$$\alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq \tau_1\eth(\vartheta, \varsigma) + \tau_2\eth(\vartheta, \mathbb{G}\vartheta) + \tau_3\eth(\varsigma, \mathbb{G}\varsigma) + \tau_4\eth(\vartheta, \mathbb{G}\varsigma) + \tau_5\eth(\mathbb{G}\vartheta, \varsigma), \tag{10}$$

where $\tau_1 + \tau_2 + \tau_3 + \delta\eta\tau_4 + \tau_5 < \frac{1}{\eta}$ *and* $\tau_\kappa \geq 0$, $\forall \kappa = 1$ *to 5.*

Then, $\mathbb{G}$ *has a unique fixed point.*

Proof. Let ζ in $\mathbb{E}_5$ defined by $\zeta(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5) = \eta(\tau_1\eth(\vartheta, \varsigma) + \tau_2\eth(\vartheta, \mathbb{G}\vartheta) + \tau_3\eth(\varsigma, \mathbb{G}\varsigma) + \tau_4\eth(\vartheta, \mathbb{G}\varsigma) + \tau_5\eth(\mathbb{G}\vartheta, \varsigma))$. Then, by Theorem 3.6, $\mathbb{G}$ has a unique fixed point. □

We prove some fixed point results for α-admissible ζ-contractive mappings in $\mathfrak{b}$-metric-like spaces, inspired by the work in [18,19].

4. Fixed Point Results in $\mathfrak{b}$-MLSs

In 2014, Shukla [33] initiated the partial $\mathfrak{b}$-metric.

Definition 4.1. *[33] Let* $\mathfrak{P} \neq \emptyset$*. Then, we say that a mapping* $\eth : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *is partial* $\mathfrak{b}$*-metric if there exists a positive number* η *such that* $\forall\, \vartheta, \varsigma, \varrho \in \mathfrak{P}$,

(pb_1) $\eth(\vartheta, \varsigma) = 0 \Longleftrightarrow \eth(\vartheta, \vartheta) = \eth(\vartheta, \varsigma) = \eth(\varsigma, \varsigma)$*;*
(pb_2) $\eth(\vartheta, \vartheta) \leq \eth(\vartheta, \varsigma)$*;*
(pb_3) $\eth(\vartheta, \varsigma) = \eth(\varsigma, \vartheta)$*;*
(pb_4) $\eth(\vartheta, \varrho) \leq \eta(\eth(\vartheta, \varsigma) + \eth(\varsigma, \varrho)) - \eth(\varsigma, \varsigma)$.

Then, the triplet $(\mathfrak{P}, \eth, \eta)$ *is said to be a partial* $\mathfrak{b}$*-MS.*

In 2013, Alghamdi [2] initiated the concept of $\mathfrak{b}$-metric-like space.

Definition 4.2. *[2] Let* $\mathfrak{P} \neq \emptyset$*. Then, we say that a mapping* $\eth : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *is* $\mathfrak{b}$*-metric-like if there exists a positive number* η *such that* $\forall\, \vartheta, \varsigma, \varrho \in \mathfrak{P}$,

(bml_1) $\eth(\vartheta, \varsigma) = 0 \Longleftrightarrow \vartheta = \varsigma$*;*
(bml_2) $\eth(\vartheta, \varsigma) = \eth(\varsigma, \vartheta)$*;*

(bml_3) $\eth(\vartheta, \varrho) \leq \eta(\eth(\vartheta, \varsigma) + \eth(\varsigma, \varrho))$.

Then, the triplet $(\mathfrak{P}, \eth, \eta)$ *is called a* $\mathfrak{b}$*-metric-like space (shortly,* $\mathfrak{b}$*-MLS).*

Definition 4.3. *[9] Let* $\{\vartheta_\upsilon\}$ *be a sequence in* $\mathfrak{b}$*-MLS* $(\mathfrak{P}, \eth, \eta \geq 1)$*. We say that a point* $\vartheta \in \mathfrak{P}$ *is the limit point of* $\{\vartheta_\upsilon\}$ *if* $\lim_{\upsilon\to+\infty} \eth(\vartheta, \vartheta_\upsilon) = \eth(\vartheta, \vartheta)$*, and the sequence* $\{\vartheta_\upsilon\}$ *is said to be convergent to* ϑ *and it is denoted* $\vartheta_\upsilon \to \vartheta$ *as* $\upsilon \to +\infty$.

Definition 4.4. *[9]*

(*i*) *A sequence* $\{\vartheta_\upsilon\}$ *in a* $\mathfrak{b}$*-MLS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *is said to be Cauchy sequence if* $\lim_{\upsilon,\omega\to+\infty} \eth(\vartheta_\upsilon, \vartheta_\omega)$ *exists and is finite.*

(*ii*) *A* $\mathfrak{b}$*-MLS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *is called complete if for each Cauchy sequence* $\{\vartheta_\upsilon\}$ *in* $\mathfrak{P}$ *converges to* $\vartheta \in \mathfrak{P}$*. i.e.,*

$$\lim_{\upsilon,\omega\to+\infty} \eth(\vartheta_\upsilon, \vartheta_\omega) = \eth(\vartheta, \vartheta) = \lim_{\upsilon\to+\infty} \eth(\vartheta_\upsilon, , \vartheta).$$

The following proposition used by Alghamdi [2] for proving fixed point result.

Proposition 4.5. *[2] A sequence* $\{\vartheta_\upsilon\}$ *in* $\mathfrak{b}$*-MLS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *such that* $\lim_{\upsilon\to+\infty} \eth(\vartheta_\upsilon, \vartheta) = 0$*, for some* $\vartheta \in \mathfrak{P}$*. Then,*

(*i*) ϑ *is unique.*

(*ii*) $\frac{1}{\eta}\eth(\vartheta, \varsigma) \leq \lim_{\upsilon\to+\infty} \eth(\vartheta_\upsilon, \varsigma) \leq \eta\eth(\vartheta, \varsigma)$ *for all* $\varsigma \in \mathfrak{P}$.

In 2019, Sen [31] introduced the following lemma.

Lemma 4.6. *[31] A sequence* $\{\vartheta_\upsilon\}$ *in* $\mathfrak{b}$*-MLS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *such that for some* $\tau \in [0, 1)$*,*

$$\eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) \leq \tau\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon),\ \forall\ \upsilon \in \mathbb{N}.$$

Then, the sequence $\{\vartheta_\upsilon\}$ *is Cauchy with* $\lim_{\upsilon,\omega\to+\infty} \eth(\vartheta_\upsilon, \vartheta_\omega) = 0$.

Now, we extend Theorem 3.2 in the framework of admissible ζ-contraction in b-metric-like space and provide a supporting example at the end of the proof.

Theorem 4.7. *Let* $\mathbb{G}$ *be a self-map on complete* $\mathfrak{b}$*-MS* $(\mathfrak{P}, \eth, \eta \geq 1)$ *and* $\alpha : \mathfrak{P} \times \mathfrak{P} \to \mathbf{R}_0^+$ *be a mapping. Assume that there exists* $\zeta \in \mathbb{E}_4$ *such that the following assumptions are true:*

(*i*) $\mathbb{G}$ *is* α*-admissible;*

(*ii*) $\exists\ \vartheta_1 \in \mathfrak{P}$ *such that* $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ *and* $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$*;*

(*iii*)

$$\alpha(\vartheta, \varsigma)\eth(\mathbb{G}\vartheta, \mathbb{G}\varsigma) \leq \frac{1}{\eta}\zeta(\vartheta, \varsigma),\ \forall \vartheta, \varsigma \in \mathfrak{P},$$

where

$$\zeta(\vartheta, \varsigma) = \max\left(\eth(\vartheta, \varsigma), \eth(\vartheta, \mathbb{G}\vartheta), \eth(\varsigma, \mathbb{G}\varsigma), \frac{\eth(\vartheta, \mathbb{G}\varsigma) + \eth(\mathbb{G}\vartheta, \varsigma) - \eth(\varsigma, \varsigma)}{2\eta}\right)$$

Then, $\mathbb{G}$ *has a unique fixed point.*

Proof. Let $\vartheta_1 \in \mathfrak{P}$ be such that $\alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1$ and $\alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1$. We define the iterative sequence $\{\vartheta_\upsilon\}$ in $\mathfrak{P}$ by the rule $\vartheta_\upsilon = \mathbb{G}\vartheta_{\upsilon-1} = \mathbb{G}^\upsilon\vartheta_1$, $\forall \upsilon \geq 1$. Obviously, if there exists $\upsilon_0 \geq 1$ for which $\mathbb{G}^{\upsilon_0}\vartheta_1 = \mathbb{G}^{\upsilon_0+1}\vartheta_1$, then $\mathbb{G}^{\upsilon_0}\vartheta_1$ has a fixed point of $\mathbb{G}$. Thus, suppose $\mathbb{G}^\upsilon\vartheta_1 \neq \mathbb{G}^{\upsilon+1}\vartheta_1$ for every $\upsilon \geq 1$.

Since $\mathbb{G}$ is α-admissible, the condition (ii) implies

$$\alpha(\vartheta_1, \vartheta_2) = \alpha(\vartheta_1, \mathbb{G}\vartheta_1) \geq 1 \implies \alpha(\mathbb{G}\vartheta_1, \mathbb{G}\vartheta_2) = \alpha(\vartheta_2, \vartheta_3) \geq 1,$$

continuing in this way,

$$\alpha(\vartheta_\upsilon, \vartheta_{\upsilon+1}) \geq 1, \ \forall \upsilon \in \mathbb{N}.$$

In a similar way, starting with

$$\alpha(\vartheta_1, \vartheta_3) = \alpha(\vartheta_1, \mathbb{G}^2\vartheta_1) \geq 1 \implies \alpha(\mathbb{G}\vartheta_1, \mathbb{G}\vartheta_3) = \alpha(\vartheta_2, \vartheta_4) \geq 1,$$

we deduce

$$\alpha(\vartheta_\upsilon, \vartheta_{\upsilon+2}) \geq 1, \ \forall \upsilon \in \mathbb{N}.$$

Assume that $\vartheta_\upsilon \neq \vartheta_{\upsilon+1}$ $\forall \upsilon \in \mathbb{N}$. Now, we prove the sequence $\{\vartheta_\upsilon\}$ is Cauchy. Let $\upsilon \in \mathbb{N}$. Now,

$$\eth(\vartheta_{\upsilon-1}, \mathbb{G}\vartheta_\upsilon) + \eth(\mathbb{G}\vartheta_{\upsilon-1}, \vartheta_\upsilon) = \eth(\vartheta_{\upsilon-1}, \vartheta_{\upsilon+1}) + \eth(\vartheta_\upsilon, \vartheta_\upsilon) \geq \eth(\vartheta_\upsilon, \vartheta_\upsilon);$$

therefore, using (12), we have

$$\begin{aligned}
\eth(\vartheta_\upsilon, \vartheta_{\upsilon+g}) &= \eth(\mathbb{G}^\upsilon\vartheta_1, \mathbb{G}^{\upsilon+1}\vartheta_1) \\
&\leq \alpha(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1)\eth(\mathbb{G}^{\upsilon-1}\vartheta_1, \mathbb{G}^\upsilon\vartheta_1) \\
&\leq \frac{1}{\eta}\max\left\{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}), \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_{\upsilon+1}) + \eth(\vartheta_\upsilon, \vartheta_\upsilon) - \eth(\vartheta_\upsilon, \vartheta_\upsilon)}{2\eta}\right\} \\
&< \frac{1}{\eta}\max\left\{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}), \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_{\upsilon+1})}{2\eta})\right\} \\
&= \frac{1}{\eta}\max\left\{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_{\upsilon+1})}{2\eta}\right\} \\
&\leq \frac{1}{\eta}\max\left\{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon) + \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1})}{2}\right\},
\end{aligned} \tag{11}$$

which implies that

$$\eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) < \frac{1}{\eta}\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon), \ \forall \upsilon \geq 1. \tag{12}$$

Case 1: If $\eta > 1$, then the sequence $\{\vartheta_\upsilon\}$ is Cauchy, by Lemma 4.6 in view of equation (12).
Case 2: If $\eta = 1$, then by equation (12) we get monotonically decreasing and bounded below the sequence $\{\eth(\vartheta_\upsilon, \vartheta_{\upsilon+1})\}$. Here, we obtain $\eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) \to k$ for some $\flat \geq 0$. Suppose that $\flat > 0$; now, taking $\liminf \upsilon \to +\infty$ in (11), we have $\flat \leq \zeta(\flat, \flat, \flat, \flat')$ where

$$\flat' = \lim_{n\to+\infty}\sup \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_{\upsilon+1})}{2} \leq \lim_{\upsilon\to+\infty} \frac{\eth(\vartheta_{\upsilon-1}, \vartheta_\upsilon) + \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1})}{2} = \flat.$$

Now,

$$\flat \leq \zeta(\flat, \flat, \flat, \flat') < \max\{\flat, \flat, \flat, \flat'\} = \flat,$$

which is a contradiction; so

$$\lim_{\upsilon\to+\infty} \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) = 0. \tag{13}$$

Furthermore,

$$\eth(\vartheta_\upsilon, \vartheta_\upsilon) \leq \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) + \eth(\vartheta_{\upsilon+1}, \vartheta_\upsilon),$$

taking $\limsup \upsilon \to +\infty$, and using (13), we find

$$\lim_{\upsilon\to+\infty} \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) = 0. \tag{14}$$

Suppose that

$$\lim_{\upsilon\to+\infty} \eth(\vartheta_\upsilon, \vartheta_{\upsilon+1}) \neq 0.$$

On contrary, we assume that the sequence $\{\vartheta_\upsilon\}$ is not Cauchy, then $\exists\ \xi > 0$ and sequences $\{\omega_n\}, \{\upsilon_n\}; \omega_n > \upsilon_n \geq n$ such that

$$\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) \geq \xi. \tag{15}$$

Now, take $\omega_n > \upsilon_n$ such that equation (15) holds. Then,

$$\begin{aligned}\xi &\leq \eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n})\\ &\leq \eth(\vartheta_{\omega_n}, \vartheta_{\omega_n-1}) + \eth(\vartheta_{\omega_n-1}, \vartheta_{\upsilon_n})\\ &< \eth(\vartheta_{\omega_n-1}, \vartheta_{\omega_n}) + \xi\\ &< \eth(\vartheta_r, \vartheta_{n-1}) + \xi.\end{aligned}$$

Thus, taking $\lim n \to +\infty$ and by (13), we get

$$\lim_{n\to+\infty} \eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) = \xi. \tag{16}$$

Now, assume that there exist infinitely large n such that

$$\eth(\vartheta_{\omega_n}, \mathbb{G}\vartheta_{\upsilon_n}) + \eth(\mathbb{G}\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) < \eth(\vartheta_{\upsilon_n}, \vartheta_{\upsilon_n}).$$

Setting $\limsup_{n\to+\infty}$, and by (14), we get

$$\lim_{n\to+\infty} \eth(\vartheta_{\omega_n}, \mathbb{G}\vartheta_{\upsilon_n}) + \eth(\mathbb{G}\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) = 0,$$

which means that

$$\lim_{n\to+\infty} \eth(\vartheta_{\omega_n}, \mathbb{G}\vartheta_{\upsilon_n+1}) = \lim_{n\to+\infty} \eth(\mathbb{G}\vartheta_{\omega_n+1}, \vartheta_{\upsilon_n}) = 0.$$

Now,

$$\xi = \lim_{n\to+\infty} \eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) \leq \lim_{n\to+\infty} \sup(\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n+1}) + \eth(\vartheta_{\upsilon_n+1}, \vartheta_{\upsilon_n})) = 0,$$

a contradiction. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\forall\ n \geq n_0, \eth(\vartheta_{\omega_n}, \mathbb{G}\vartheta_{\upsilon_n}) + \eth(\mathbb{G}\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) \geq \eth(\vartheta_{\upsilon_n}, \vartheta_{\upsilon_n}).$$

Thus, for all $n \geq n_0$, using (12),

$$\begin{aligned}\eth(\vartheta_{\omega_n+1}, \vartheta_{\upsilon_n+1}) &\leq \alpha(\vartheta_{\omega_n}, \vartheta_{\upsilon_n})\eth(\mathbb{G}\vartheta_{\omega_n}, \mathbb{G}\vartheta_{\upsilon_n})\\ &\leq \max\Big(\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n}), \eth(\vartheta_{\omega_n}, \vartheta_{\omega_n+1}), \eth(\vartheta_{\upsilon_n}, \vartheta_{\upsilon_n+1}),\\ &\quad \frac{\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n+1}) + \eth(\vartheta_{\omega_n+1}, \vartheta_{\upsilon_n}) - \eth(\vartheta_{\upsilon_n}, \vartheta_{\upsilon_n})}{2}\Big).\end{aligned}$$

Now,

$$\begin{aligned}\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n}) &\leq \eth(\vartheta_{\omega_n}, \vartheta_{\omega_n+1}) + \eth(\vartheta_{\omega_n+1}, \vartheta_{\upsilon_n+1}) + \eth(\vartheta_{\upsilon_n+1}, \vartheta_{\upsilon_n})\\ &\leq \eth(\vartheta_{\omega_n}, \vartheta_{\omega_n+1}) + \eth(\vartheta_{\upsilon_n+1}, \vartheta_{\upsilon_n}) + \max\Big(\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n}), \eth(\vartheta_{\omega_n}, \vartheta_{\omega_n+1}), \eth(\vartheta_{\upsilon_n}, \vartheta_{\upsilon_n+1}),\\ &\quad \frac{\eth(\vartheta_{\omega_n}, \vartheta_{\upsilon_n+1}) + \eth(\vartheta_{\omega_n+1}, \vartheta_{\upsilon_n}) - \eth(\vartheta_{\upsilon_n}, \vartheta_{\upsilon_n})}{2}\Big).\end{aligned}$$

From the above, setting $\liminf_{\mathfrak{n}\to+\infty}$ and by equations (13) and (16). Thus, we get $\xi \leq 0+0+\zeta(\xi,0,0,\xi')$, where

$$\begin{aligned}\xi' &= \lim_{\mathfrak{n}\to+\infty}\sup \frac{\mho(\vartheta_{\omega_{\mathfrak{n}}},\vartheta_{\upsilon_{\mathfrak{n}}+1})+\mho(\vartheta_{\omega_{\mathfrak{n}}+1},\vartheta_{\upsilon_{\mathfrak{n}}})-\mho(\vartheta_{\upsilon_{\mathfrak{n}}},\vartheta_{\upsilon_{\mathfrak{n}}})}{2}\\ &\leq \lim_{\mathfrak{n}\to+\infty}\sup \frac{\mho(\vartheta_{\omega_{\mathfrak{n}}},\vartheta_{\upsilon_{\mathfrak{n}}})+\mho(\vartheta_{\omega_{\mathfrak{n}}},\vartheta_{\upsilon_{\mathfrak{n}}+1})+\mho(\vartheta_{\omega_{\mathfrak{n}}+1},\vartheta_{\omega_{\mathfrak{n}}})+\mho(\vartheta_{\omega_{\mathfrak{n}}},\vartheta_{\upsilon_{\mathfrak{n}}})-0}{2}\\ &= \frac{\xi+0+0+\xi}{2}\\ &= \xi.\end{aligned}$$

Thus, $\xi \leq \zeta(\xi,0,0,\xi') < \max\{\xi,0,0,\xi'\} = \xi$, a contradiction. Thus, $\{\vartheta_\upsilon\}$ is a Cauchy sequence. Since $(\mathfrak{P},\mho,\eta\geq 1)$ is complete $\mathfrak{b}$-MLS, there exists $\vartheta\in\mathfrak{P}$ such that $\vartheta_\upsilon\to\vartheta$

$$\mho(\vartheta,\vartheta) = \lim_{\upsilon\to+\infty}\mho(\vartheta_\upsilon,\vartheta) = \lim_{\upsilon,\omega\to+\infty}\mho(\vartheta_\upsilon,\vartheta_\omega) = 0.$$

Moreover, by Proposition 4.5, ϑ is unique. Assume that $\mathbb{G}\vartheta\neq\vartheta$. Consider

$$\begin{aligned}\mho(\mathbb{G}\vartheta_\upsilon,\mathbb{G}\vartheta) &\leq \alpha(\vartheta_\upsilon,\vartheta)\mho(\mathbb{G}\vartheta_\upsilon,\mathbb{G}\vartheta)\\ &\leq \frac{1}{\eta}\max\Big(\mho(\vartheta_\upsilon,\vartheta),\mho(\vartheta_\upsilon,\mathbb{G}\vartheta_\upsilon),\mho(\vartheta,\mathbb{G}\vartheta),\frac{\mho(\vartheta_\upsilon,\mathbb{G}\vartheta)+\mho(\vartheta,\mathbb{G}\vartheta_\upsilon)-\mho(\vartheta,\vartheta)}{2\eta}\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\mho(\vartheta_{\upsilon+1},\mathbb{G}\vartheta) &= \mho(\mathbb{G}\vartheta_\upsilon,\mathbb{G}\vartheta)\\ &\leq \alpha(\vartheta_\upsilon,\vartheta)\mho(\mathbb{G}\vartheta_\upsilon,\mathbb{G}\vartheta)\\ &\leq \frac{1}{\eta}\max\Big(\mho(\vartheta_\upsilon,\vartheta),\mho(\vartheta_\upsilon,\mathbb{G}\vartheta_{\upsilon+1}),\mho(\vartheta,\mathbb{G}\vartheta),\frac{\mho(\vartheta_\upsilon,\mathbb{G}\vartheta)+\mho(\vartheta,\vartheta_{\upsilon+1})}{2\eta}\Big).\end{aligned}$$

From the above inequality taking $\liminf \upsilon\to+\infty$ and by Proposition 4.5, we get

$$\frac{1}{\eta}\mho(\vartheta,\mathbb{G}\vartheta) \leq \frac{1}{\eta}\zeta(0,0,\mho(\vartheta,\mathbb{G}\vartheta),\hbar),$$

i.e.,

$$\mho(\vartheta,\mathbb{G}\vartheta) \leq \zeta(0,0,\mho(\vartheta,\mathbb{G}\vartheta),\hbar),$$

where

$$\hbar = \lim_{\upsilon\to+\infty}\sup\frac{\mho(\vartheta_\upsilon,\mathbb{G}\vartheta)+\mho(\vartheta,\vartheta_{\upsilon+1})}{2\eta} \leq \lim_{\upsilon\to+\infty}\sup\frac{\eta\mho(\vartheta,\mathbb{G}\vartheta)+0}{2\eta} = \frac{\mho(\vartheta,\mathbb{G}\vartheta)}{2}.$$

Thus

$$\mho(\vartheta,\mathbb{G}\vartheta) \leq \zeta(0,0,\mho(\vartheta,\mathbb{G}\vartheta),\hbar) < \max\{0,0,\mho(\vartheta,\mathbb{G}\vartheta),\hbar\} = \mho(\vartheta,\mathbb{G}\vartheta),$$

which is a contradiction. Therefore, $\mathbb{G}\vartheta=\vartheta$.

Suppose that ϑ,ς are two fixed points of $\mathbb{G}$ such that $\mathbb{G}\vartheta=\vartheta\neq\varsigma=\mathbb{G}\varsigma$. Then, for all $\vartheta,\varsigma\in\mathfrak{P}$ such that

$\alpha(\vartheta,\varsigma) \geq 1$. If $\eth(\vartheta,\varsigma) > 0$ then, by the contractive condition (iii) with the fixed points ϑ and ς yields

$$\begin{aligned}\eth(\vartheta,\varsigma) = \alpha(\vartheta,\varsigma)\eth(\mathbb{G}\vartheta,\mathbb{G}\varsigma) &\leq \frac{1}{\eta}\max\Big(\eth(\vartheta,\varsigma),\eth(\vartheta,\mathbb{G}\vartheta),\eth(\varsigma,\mathbb{G}\varsigma),\frac{\eth(\vartheta,\mathbb{G}\varsigma)+\eth(\varsigma,\mathbb{G}\vartheta)-\eth(\vartheta,\vartheta)}{2\eta}\Big)\\ &= \frac{1}{\eta}\max\Big(\eth(\vartheta,\varsigma),\eth(\vartheta,\mathbb{G}\vartheta),\eth(\varsigma,\mathbb{G}\varsigma),\frac{\eth(\vartheta,\mathbb{G}\varsigma)+\eth(\varsigma,\mathbb{G}\vartheta)}{2\eta}\Big)\\ &\leq \frac{1}{\eta}\max\Big(\eth(\vartheta,\varsigma),0,0,\frac{\eth(\vartheta,\varsigma)}{\eta}\Big)\\ &< \frac{1}{\eta}\max\Big\{(\eth(\vartheta,\varsigma),0,0,\frac{\eth(\vartheta,\varsigma)}{\eta}\Big\}\\ &= \frac{\eth(\vartheta,\varsigma)}{\eta},\end{aligned}$$

which is a contradiction. Therefore, $\vartheta = \varsigma$. □

Example 4.8. *Let $\mathfrak{P} = \mathbf{R}_0^+$. Define $\eth : \mathfrak{P}\times\mathfrak{P} \to \mathbf{R}_0^+$ by $\eth(\vartheta,\varsigma) = (\vartheta+\varsigma)^2$, $\forall \vartheta,\varsigma \in \mathfrak{P}$. Then, $\eth$ is b-ML on $\mathfrak{P}$ with $\eta = 2$, but $\eth$ is not b-metric on $\mathfrak{P}$. A mapping $\mathbb{G} : \mathfrak{P} \to \mathfrak{P}$ defined by $\mathbb{G} = \frac{\vartheta}{2}$. In addition, define $\mathfrak{I}(\varpi_1,\varpi_2,\varpi_3,\varpi_4) = \frac{\vartheta}{2}\max\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$ and define $\alpha : \mathfrak{P}\times\mathfrak{P} \to \mathbf{R}_0^+$ by*

$$\alpha(\vartheta,\varsigma) = \begin{cases} 1, & \text{if } \vartheta \leq \varsigma \text{ or } \varsigma \leq \vartheta,\\ 0, & \text{if otherwise.}\end{cases}$$

Now, $\forall\ \vartheta,\varsigma \in \mathfrak{P}$ with $\eth(\vartheta,\mathbb{G}\varsigma) + \eth(\mathbb{G}\vartheta,\varsigma) \geq \eth(\varsigma,\varsigma)$, condition (iii) of Theorem 4.7 is fulfilled and hence, 0 is the unique fixed point of $\mathbb{G}$.

5. Application

In this section, we arise an integral equation application of our main results. Consider the following integral equation:

$$\mathfrak{u}(\mathfrak{n}) = \mathfrak{v}(\mathfrak{n}) + \rho\int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\mathfrak{f}(\varrho,\mathfrak{u}(\varrho))\eth\varrho,\ \mathfrak{n} \in \mathbb{I} = [\mathfrak{a},\mathfrak{b}], \tag{17}$$

where ρ is a constant such that $\rho \geq 0$ and $\mathfrak{v} : [\mathfrak{a},\mathfrak{b}] \to \mathbf{R}$, $\mathbb{H} : [\mathfrak{a},\mathfrak{b}]\times[\mathfrak{a},\mathfrak{b}] \to \mathbf{R}$ and $\mathfrak{f} : [\mathfrak{a},\mathfrak{b}]\times\mathbf{R} \to \mathbf{R}$ are given continuous functions.

The set of all real valued continuous functions $\mathfrak{P}$ defined on $[\mathfrak{a},\mathfrak{b}]$. Define the $\mathfrak{b}$-metric by the following:

$$\eth(\mathfrak{u},\mathfrak{v}) = \frac{1}{\eta}\sup_{\mathfrak{n}\in\mathbb{I}}|\mathfrak{u}(\mathfrak{n}) - \mathfrak{v}(\mathfrak{n})|,\ \forall \mathfrak{a},\mathfrak{b} \in \mathfrak{P}. \tag{18}$$

Consider $\eta > 1$. Then, $(\mathfrak{P},\eth)$ is a complete $\mathfrak{b}$-MS. Now, a self-map $\mathbb{G}$ defined on $\mathfrak{P}$ by

$$\mathbb{G}\mathfrak{u}(\mathfrak{n}) = \mathfrak{v}(\mathfrak{n}) + \rho\int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\mathfrak{f}(\varrho,\mathfrak{u}(\varrho))\eth\varrho,\ \mathfrak{n} \in [\mathfrak{a},\mathfrak{b}]. \tag{19}$$

Assume that the following to prove the existence of a solution of Equation (17):

(*a*) $\rho \leq \frac{1}{\eta}$

(*b*) $\sup_{\mathfrak{n}\in[\mathfrak{a},\mathfrak{b}]}\int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\eth\varrho \leq \frac{1}{\mathfrak{b}-\mathfrak{a}}$

(*c*) $\forall\ \mathfrak{u},\mathfrak{v} \in \mathbf{R}, |\mathfrak{f}(\varrho,\mathfrak{u}) - \mathfrak{f}(\varrho,\mathfrak{v})| \leq |\mathfrak{u} - \mathfrak{v}|$

(*d*) There exists a mapping $\zeta : \mathfrak{P}\times\mathfrak{P} \to \mathbf{R}_0^+$ such that $\forall\ \mathfrak{n} \in [\mathfrak{a},\mathfrak{b}]$ and $\forall\ \mathfrak{a},\mathfrak{b} \in \mathfrak{P}$ with $\zeta(\mathfrak{a},\mathfrak{b}) \geq 0$.

A solution to Equation (17) is equal to the existence of a fixed point of $\mathbb{G}$. We will now present the following results.

Theorem 5.1. *Equation* (17) *has a unique solution in* $\mathfrak{P}$*, under the above assumptions (a) - (d).*

Proof.

$$\begin{aligned}
\eth(\mathbb{G}\mathfrak{u}_1, \mathbb{G}\mathfrak{u}_2) &= \frac{1}{\eta} \sup_{\mathfrak{n}\in\mathbb{I}} |\mathbb{G}\mathfrak{u}_1(\mathfrak{n}) - \mathbb{G}\mathfrak{u}_2(\mathfrak{n})| \\
&= \frac{1}{\eta} \sup_{\mathfrak{n}\in\mathbb{I}} \left|\Big(\mathfrak{v}(\mathfrak{n}) + \rho \int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\mathfrak{f}(\varrho, \mathfrak{u}_1(\varrho))\eth\varrho\Big) - \Big(\mathfrak{v}(\mathfrak{n}) + \rho \int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\mathfrak{f}(\varrho, \mathfrak{u}_2(\varrho))\eth\varrho\Big)\right| \\
&= \frac{1}{\eta} \sup_{\mathfrak{n}\in\mathbb{I}} \left|\rho \int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)[\mathfrak{f}(\varrho, \mathfrak{u}_1(\varrho)) - \mathfrak{f}(\varrho, \mathfrak{u}_2(\varrho))]\eth\varrho\right| \\
&\leq \frac{1}{\eta^2}\{\sup_{\mathfrak{n}\in\mathbb{I}} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\}\Big(\int_{\mathfrak{a}}^{\mathfrak{b}} \left|\mathfrak{f}(\varrho, \mathfrak{u}_1(\varrho)) - \mathfrak{f}(\varrho, \mathfrak{u}_2(\varrho))\right|\eth\varrho\Big) \\
&\leq \frac{1}{\eta^2}\{\sup_{\mathfrak{n}\in\mathbb{I}} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathbb{H}(\mathfrak{n},\varrho)\} \int_{\mathfrak{a}}^{\mathfrak{b}} |\mathfrak{u}_1 - \mathfrak{u}_2|\eth\varrho \\
&\leq \frac{1}{\eta^2}|\mathfrak{u}_1 - \mathfrak{u}_2|(\frac{1}{b-a}) \int_{\mathfrak{a}}^{\mathfrak{b}} \eth\varrho \\
&= \frac{1}{\eta}\eth(\mathfrak{u}_1, \mathfrak{u}_2). \qquad (20)
\end{aligned}$$

So, Equation (17) has a solution in $\mathfrak{P}$, which means that $\mathbb{G}$ has a fixed point. □

6. Conclusion

In this study, we introduce the notion of admissible ζ-contraction mapping of types, which includes the admissible ζ-contraction of Jain et al. [20] and the α-admissible mapping of Samet et al. [30]. Utilizing this class of mappings, we establish approximate fixed point and fixed point theorems in the setting of b-metric and b-metric-like spaces. Finally, we use some examples to prove the established theorems and our results can be used to solve an integral equation.

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