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n-Isoclinic Lie Crossed Modules

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Abstract. We define the notion of n-isoclinic Lie crossed modules and give the relation between the n-isoclinic Lie crossed modules and n-isoclinic Lie algebras.

Introduction

In [16], Hall introduced the notion of isoclinism which is an equivalence relation weaker than isomorphism. After this, a number of authors has been studied about isoclinism in [7, 11, 14, 15]. The Lie algebra version of isoclinism was defined in [13] and investigated other properties in [2, 5].

Crossed modules, defined in [12], play an important role in many areas such that group presentation, algebraic K-theory and homological algebra. Many of properties about crossed modules were given in [17, 18]. Also computational analogues of crossed modules have been given in [19, 20]. The notion of isoclinic crossed modules was given in [3] and Lie crossed modules analogues of isoclinic Lie algebras was defined in [4]. Also, relations between commutativity degree and isoclinism of crossed modules (on groups) have been obtained in [21].

For groups G and H, if there exist isomorphisms $\alpha : \frac{G}{Z_n(G)} \longrightarrow \frac{H}{Z_n(H)}$ and $\beta : [G,G]_{n+1} \longrightarrow [H,H]_{n+1}$ in such that β is compatible with α, then G and H are called n-isoclinic, $G \underset{n}{\sim} H$. Also the pair (α, β) is called n-isoclinism between G and H. n-isoclinism is an equivalence relation same as isoclinism, and produces a partition on the class of groups. In [7], all groups occurring in an n-isoclinism class of a given group was determined and each n-isoclinism class of groups contains at least a group called n-stem group in [15]. Also, in [6], authors give the crossed modules (of groups) analogues of the n-isoclinism and obtain some results about it.

In this work, we give the notions of n-isoclinic Lie crossed modules and obtain the relation between the n-isoclinic Lie crossed modules and the n-isoclinic Lie algebras in Proposition 20.

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1. Preliminaries

In this section we recall the basic properties of Lie crossed modules. See [8–10], for comprehensive research about the notions.

A Lie crossed module is a Lie algebra homomorphism

$$d : L_1 \longrightarrow L_0$$

with a Lie action of L_0 on L_1 written $(l_0, l_1) \mapsto [l_0, l_1]$, for $l_0 \in L_0, l_1 \in L_1$ satisfying the following conditions:

$$\begin{array}{rrcl} 1) & d([l_0, l_1]) & = & [l_0, d(l_1)], \\ 2) & [d(l_1), l_1'] & = & [l_1, l_1'], \end{array}$$

for all $l_0 \in L_0, l_1, l_1' \in L_1$.

We will denote such a Lie crossed module by $L : L_1 \xrightarrow{d} L_0$.

Examples 1.
(1) $L \xrightarrow{ad} Der(L)$ is a crossed module, where ad assigns to each element $l \in L$, the inner derivation of L, $ad(l) : x \longmapsto [l, x]$ for all $x \in L$.
(2) $N \xhookrightarrow{inc.} L$, where N is an ideal of a Lie algebra L and L acts on N via adjoint representation. Consequently, every Lie algebra L can be thought as a crossed module in the two obvious way: $0 \xhookrightarrow{inc.} L$ or $L \xrightarrow{id} L$.
(3) $A \xrightarrow{0} L$ is a crossed module, where A is a L-module and the boundary map is the zero map.

A Lie crossed module $L : L_1 \xrightarrow{d} L_0$ is called *aspherical* if $\ker(d) = 0$, i.e d is injective, and *simply connected* if $\operatorname{coker}(d) = 0$, i.e d is surjective.

A morphism between Lie crossed modules $L : L_1 \xrightarrow{d} L_0$ and $M : M_1 \xrightarrow{d'} M_0$ is a pair (α, β) of Lie algebra homomorphisms $\alpha : L_1 \longrightarrow M_1$, $\beta : L_0 \longrightarrow M_0$ such that $\beta d = d'\alpha$ and $\alpha([l_0, l_1]) = [\beta(l_0), \alpha(l_1)]$, for all $l_0 \in L_0$, $l_1 \in L_1$. Consequently, we have a category **XLie** whose objects are Lie crossed modules and morphisms are morphisms of Lie crossed modules .

A Lie crossed module $L' : L_1' \xrightarrow{d'} L_0'$ is a *subcrossed module* of a crossed module $L : L_1 \xrightarrow{d} L_0$ if L_1', L_0' are Lie subalgebras of L_1, L_0, respectively, $d' = d|_{L_1'}$ and the action of L_0' on L_1' is induced from the action of L_0 on L_1. Additionally, if L_1' and L_0' are ideals of L_1 and L_0, respectively, $[l_0, l_1'] \in L_1'$ and $[l_0', l_1] \in L_1'$, for all $l_0 \in L_0, l_1 \in L_1, l_0' \in L_0', l_1' \in L_1'$ then L' is called an *ideal* of L. Consequently, we have the *quotient crossed module* $L/L' : L_1/L_1' \xrightarrow{\bar{d}} L_0/L_0'$ with the induced boundary map and action.

Let $(\alpha, \beta) : (L : L_1 \xrightarrow{d} L_0) \longrightarrow (L' : L_1' \xrightarrow{d'} L_0')$ be a Lie algebra crossed module morphism. The *kernel* of (α, β) is the ideal $(\ker\alpha, \ker\beta, d|)$ of L, denoted by $\ker(\alpha, \beta)$ and the *image* $\operatorname{Im}(\alpha, \beta)$ is the subcrossed module $(\operatorname{Im}\alpha, \operatorname{Im}\beta, d'|)$ of L'.

We have the second isomorphism theorem for Lie crossed modules given in [8]:

Let $M : M_1 \xrightarrow{d} M_0$ and $N : N_1 \xrightarrow{d} N_0$ be a subcrossed module of $L : L_1 \xrightarrow{d} L_0$. Then the *intersection* of M and N defined by

$$M \cap N : M_1 \cap N_1 \xrightarrow{d} M_0 \cap N_0,$$

is an ideal of L. Also, we have the subcrossed module $M+N : M_1+N_1 \xrightarrow{d} M_0+N_0$. Consequently, we have

$$\frac{M}{M\cap N} \cong \frac{M+N}{N}.$$

Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. Then the *center* of L is the crossed module $Z(L) : L_1^{L_0} \xrightarrow{d|} (St_{L_0}(L_1) \cap Z(L_0))$ where

$$L_1^{L_0} = \{l_1 \in L_1 : [l_0, l_1] = 0, \text{ for all } l_0 \in L_0\}$$

and

$$St_{L_0}(L_1) = \{l_0 \in L_0 : [l_0, l_1] = 0, \text{ for all } l_1 \in L_1\}.$$

Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. The *commutator subcrossed module* $[L, L]$ of L is defined by

$$[L, L] : D_{L_0}(L_1) \xrightarrow{d|} [L_0, L_0]$$

where $D_{L_0}(L_1) = \{[l_0, l_1] : l_0 \in L_0, l_1 \in L_1\}$ and $[L_0, L_0]$ is the commutator subalgebra of L_0.

Proposition 2. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. Then we have the following;*
(i) If L is simply connected, then $L_1^{L_0} = Z(L_1)$ and $D_{L_0}(L_1) = [L_1, L_1]$.
(ii) If L is aspherical, then $Z(L_0) = St_{L_0}(L_1) \cap Z(L_0)$.

Proof. (i) Let $l_1 \in L_1^{L_0}$. Since L is simply connected, for every $l_0 \in L_0$ there exists $l_1' \in L_1$ such that $d(l_1') = l_0$. Then $[l_0, l_1] = [d(l_1'), l_1] = 0$ and $[l_1', l_1] = 0$. So $l_1 \in Z(L_1)$ i.e $L_1^{L_0} \subseteq Z(L_1)$. Conversely, let $l_1 \in Z(L_1)$. From the hypothesis, we have $[l_0, l_1] = [d(l_1'), l_1] = [l_1', l_1] = 0$ ($\because$ $l_1 \in Z(L_1)$). So $l_1 \in L_1^{L_0}$ i.e $Z(L_1) \subseteq L_1^{L_0}$. Let $[l_0, l_1] \in D_{L_0}(L_1)$. From the hypothesis, we can say that $[l_0, l_1] = [d(l_1'), l_1] = [l_1', l_1] \in [L_1, L_1]$. So we have $D_{L_0}(L_1) \subseteq [L_1, L_1]$. Let $[l_1, l_1'] \in [L_1, L_1]$. Then $[l_0, l_1] = [d(l_1'), l_1] \in D_{L_0}(L_1)$ i.e $[L_1, L_1] \subseteq D_{L_0}(L_1)$.

(ii) Let $l_0 \in Z(L_0)$. Then we have $d([l_0, l_1]) = [l_0, d(l_1)] = 0 = d(0)$. Since L is aspherical, $[l_0, l_1] = 0$ i.e $l_0 \in St_{L_0}(L_1)$. So, we have $Z(L_0) \subseteq St_{L_0}(L_1)$ i.e $Z(L_0) = St_{L_0}(L_1) \cap Z(L_0)$. □

Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. If there exists $n \in \mathbb{Z}^+$ such that $(L_1, L_0, d)^{(n)} = 0$, L is called *solvable Lie crossed module. Also, the least positive integer n* satisfying $(L_1, L_0, d)^{(n)} = 0$ is called *derived length* of the Lie crossed module L.

Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. If there exists $n \in \mathbb{N}$ such that $(L_1, L_0, d)^n = 0$, L is called *nilpotent Lie crossed module. Also, the least natural n* satisfying $(L_1, L_0, d)^n = 0$ is called *nilpotency class* of the Lie crossed module L.

2. Isoclinic Lie crossed modules

In this section, we give the notion of isoclinism among Lie crossed modules from [4].

Definition 3. *[13] Let L_1 and L_2 be two Lie algebras. L_1 and L_2 are isoclinic if there exist isomorphisms $\eta : L_1/Z(L_1) \longrightarrow L_2/Z(L_2)$ and $\xi : [L_1, L_1] \longrightarrow [L_2, L_2]$ between central quotients and derived subalgebras, respectively,* *such that, the following diagram*

$$\begin{array}{ccc} L_1/Z(L_1)\times L_1/Z(L_1) & \xrightarrow{c_{L_1}} & [L_1,L_1] \\ {\scriptstyle \eta\times\eta}\downarrow & & \downarrow{\scriptstyle \xi} \\ L_2/Z(L_2)\times L_2/Z(L_2) & \xrightarrow[c_{L_2}]{} & [L_2,L_2] \end{array}$$

is commutative where c_{L_1}, c_{L_2} are commutator maps of Lie crossed modules. The pair (η,ξ) is called an isoclinism from L_1 to L_2, and denoted by $(\eta,\xi) : L_1 \sim L_2$.

Remark 4. *As expected, isoclinism is an equivalence relation.*

Examples 5.
(1) All abelian Lie algebras are isoclinic to each other. The commutator maps are and the pairs (η,ξ) consist of trivial homomorphisms.
(2) Every Lie algebra is isoclinic to a stem Lie algebra, such that its center is contained in its derived subalgebra.

Now we are going to define the notion of isoclinic Lie crossed modules.

Notation In the sequel of the paper, for a given Lie crossed module $L : L_1 \xrightarrow{d} L_0$, we denote $L/Z(L)$ by $\overline{L_1} \xrightarrow{\overline{d}} \overline{L_0}$ where $\overline{L_1} = L_1/L_1^{L_0}$ and $\overline{L_0} = L_0/(St_{L_0}(L_1)\cap Z(L_0))$, for shortness.

Definition 6. *The Lie crossed modules $L : L_1 \xrightarrow{d_L} L_0$ and $M : M_1 \xrightarrow{d_M} M_0$ are isoclinic if there exist isomorphisms*

$$(\eta_1,\eta_0) : (\overline{L_1} \xrightarrow{\overline{d_L}} \overline{L_0}) \longrightarrow (\overline{M_1} \xrightarrow{\overline{d_M}} \overline{M_0})$$

and

$$(\xi_1,\xi_0) : (D_{L_0}(L_1) \xrightarrow{d_L|} [L_0,L_0]) \longrightarrow (D_{M_0}(M_1) \xrightarrow{d_M|} [M_0,M_0])$$

such that the diagrams

$$\begin{array}{ccc} \overline{L_1}\times\overline{L_0} & \xrightarrow{c_1} & D_{L_0}(L_1) \\ {\scriptstyle \eta_1\times\eta_0}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ \overline{M_1}\times\overline{M_0} & \xrightarrow[c_1']{} & D_{M_0}(M_1) \end{array} \tag{1}$$

and

$$\begin{array}{ccc} \overline{L_0}\times\overline{L_0} & \xrightarrow{c_0} & [L_0,L_0] \\ {\scriptstyle \eta_0\times\eta_0}\downarrow & & \downarrow{\scriptstyle \xi_0} \\ \overline{M_0}\times\overline{M_0} & \xrightarrow[c_0']{} & [M_0,M_0] \end{array} \tag{2}$$

are commutative where (c_1, c_0) and (c_1', c_0') are commutator maps, defined in Proposition 14 in [4], of the Lie crossed modules L and M , respectively.

The pair $((\eta_1,\eta_0),(\xi_1,\xi_0))$ will be called an *isoclinism* from L to M and this situation will be denoted by $((\eta_1,\eta_0),(\xi_1,\xi_0)) : L \sim M$.

Examples 7.
(1) All abelian Lie crossed modules (crossed modules coincide with their center) are isoclinic. All commutator maps are and the pairs $((\eta_1, \eta_0), (\xi_1, \xi_0))$ *consist of trivial homomorphisms.*

(2) Let (η, ξ) *be an isoclinism from L to M with commutator maps* c_L *and* c_M*. Then* $L \xrightarrow{id} L$ *is isoclinic to* $M \xrightarrow{id} M$*. Here,* $(\eta_1, \eta_0) = (\eta, \eta)$, $(\xi_1, \xi_0) = (\xi, \xi)$ *and* $c_1 = c_0 = c_L$, $c_1' = c_0' = c_M$*.*

(3) Let L be a Lie algebra and let N be an ideal of L with $N + Z(L) = L$*. Then* $N \xhookrightarrow{inc.} L$ *is isoclinic to* $L \xrightarrow{id} L$*. Here* (η_1, η_0) *and* (ξ_1, ξ_0) *are defined by* $(inc., inc.)$, (id, id)*, respectively.*

Remark 8. *If the Lie crossed modules L and M are simply connected or finite dimensional, then the commutativity of diagrams* (1) *with* (2) *in Definition 6 are equivalent to the commutativity of following diagram.*

$$\begin{array}{ccc} L/Z(L) \times L/Z(L) & \longrightarrow & [L, L] \\ \Big\downarrow {\scriptstyle (\eta_1,\eta_0)\times(\eta_1,\eta_0)} & & \Big\downarrow {\scriptstyle (\xi_1,\xi_0)} \\ M/Z(M) \times M/Z(M) & \longrightarrow & [M, M] \end{array}$$

Proposition 9. *Let* $L : L_1 \xrightarrow{d} L_0$ *be a Lie crossed module and* $M : M_1 \xrightarrow{d|} M_0$ *be its subcrossed module. If* $L = M + Z(L)$*, i.e* $L_1 = M_1 + L_1^{L_0}$ *and* $L_0 = M_0 + (St_{L_0}(L_1) \cap Z(L_0))$*, then L is isoclinic to M.*

Proof. First, we show that $M_1^{M_0} = M_1 \cap M_1^{M_0}$ and $St_{M_0}(M_1) \cap Z(M_0) = M_0 \cap (St_{L_0}(L_1) \cap Z(L_0))$.
Let $m_1 \in M_1^{M_0}$. For any $l_0 \in L_0$, since $L_0 = M_0 + (St_{L_0}(L_1) \cap Z(L_0))$ there exist $a_0 \in St_{L_0}(L_1) \cap Z(L_0)$ and $m_0' \in m_0$ such that $l_0 = m_0' + a_0$. We have $[l_0, m_1] = [(m_0' + a_0), m_1] = [m_0', m_1] + [a_0, m_1] = 0 + 0 = 0$ ($\because$ $m_1 \in M_1^{M_0}$ and $a_0 \in St_{L_0}(L_1)$), so $m_1 \in M_1 \cap L_1^{L_0}$. Conversely, for any $m_1 \in M_1 \cap L_1^{L_0}$, we have $m_1 \in M_1^{M_0}$. So, $M_1^{M_0} = M_1 \cap L_1^{L_0}$.
Let $m_0 \in St_{M_0}(M_1) \cap Z(M_0)$. For any $l_1 \in L_1$, there exist $k_1 \in M_1$ and $a_1 \in L_1^{L_0}$ such that $l_1 = k_1 + a_1$. Then

$$[m_0, l_1] = [m_0, (k_1 + a_1)] = [m_0, k_1] + [m_0, a_1] = 0 \ (\because m_0 \in St_{M_0}(M_1) \text{ and } a_1 \in L_1^{L_0}),$$

which means that $m_0 \in St_{L_0}(L_1)$. On the other hand, it is clear that $m_0 \in Z(L_0)$. Then, we obtain $m_0 \in M_0 \cap (St_{L_0}(L_1) \cap Z(L_0))$. By a direct calculation, we get $St_{M_0}(M_1) \cap Z(M_0) = M_0 \cap (St_{L_0}(L_1) \cap Z(L_0))$. By the second isomorphism theorem for Lie crossed modules, we have

$$\begin{aligned} \frac{M}{Z(M)} &= \frac{(M_1, M_0, d|)}{(M_1^{M_0}, St_{M_0}(M_1) \cap Z(M_0), d|)} \\ &= \frac{(M_1, M_0, d|)}{(M_1 \cap L_1^{L_0}, M_0 \cap (St_{L_0}(L_1) \cap Z(L_0)), d|)} \\ &= \frac{(M_1, M_0, d|)}{(L_1^{L_0}, St_{L_0}(L_1) \cap Z(L_0)), d|) \cap (M_1, M_0, d|)} \\ &\cong \frac{(M_1, M_0, d|) + (L_1^{L_0}, St_{L_0}(L_1) \cap Z(L_0)), d|)}{(L_1^{L_0}, St_{L_0}(L_1) \cap Z(L_0)), d|)} \\ &= \frac{M + Z(L)}{Z(L)} \\ &= \frac{L}{Z(L)}, \end{aligned}$$

as required.
Let $[l_0, l_1] \in D_{L_0}(L_1)$, then there exist $m_1 \in M_1$, $a_1 \in L_1^{L_0}$, $m_0 \in M_0$, $a_0 \in (St_{L_0}(L_1) \cap Z(L_0))$ such that $l_1 = m_1 + a_1$

and $l_0 = m_0 + a_0$. Since

$$\begin{aligned}[l_0, l_1] &= [(m_1 + a_1), (m_0 + a_0)] \\ &= [(m_1 + a_1), m_0] + [(m_1 + a_1), a_0] \\ &= [m_1, m_0] + [a_1, m_0] + [m_1, a_0] + [a_1, a_0] \\ &= [m_1, m_0] \ (\because a_1 \in L_1^{L_0}, a_0 \in St_{L_0}(L_1)),\end{aligned}$$

we have $[l_0, l_1] \in D_{M_0}(M_1)$. On the other hand, for any $[l_0, l_0'] \in [L_0, L_0]$ there exist $m_0, m_0' \in M_0$, $a_0, a_0' \in (St_{L_0}(L_1) \cap Z(L_0))$ such that $l_0 = m_0 + a_0$, $l_0' = m_0' + a_0'$, from which we get

$$\begin{aligned}[l_0, l_0'] &= [m_0 + a_0, m_0' + a_0'] \\ &= [m_0, m_0'] + [a_0, m_0'] + [m_0, a_0'] + [a_0, a_0'] \\ &= [m_0, m_0']. \ (\because a_0, a_0' \in Z(L_0))\end{aligned}$$

Finally, the Lie crossed modules L and M are isoclinic where the isomorphisms (η_1, η_0) and (ξ_1, ξ_0) are defined by $(inc., inc.)$, (id, id), respectively. □

Remark 10. *If $M : M_1 \xrightarrow{d|} M_0$ is a finite dimensional Lie crossed module, then the converse of Proposition 9 is true.*

Proposition 11. *Let $L : L_1 \xrightarrow{d_L} L_0$ and $M : M_1 \xrightarrow{d_M} M_0$ be isoclinic crossed modules.*
(i) If L and M are aspherical, then L_0 and M_0 are isoclinic Lie algebras.
(ii) If L and M are simply connected, then L_1 and M_1 are isoclinic Lie algebras.

Proof. Let $L : L_1 \xrightarrow{d_L} L_0$ and $M : M_1 \xrightarrow{d_M} M_0$ be isoclinic Lie crossed modules. Then we have the isomorphisms

$$\begin{aligned}(\eta_1, \eta_0) &: (\overline{L_1} \xrightarrow{\overline{d_L}} \overline{L_0}) \longrightarrow (\overline{M_1} \xrightarrow{\overline{d_M}} \overline{M_0}) \\ (\xi_1, \xi_0) &: (D_{L_0}(L_1) \xrightarrow{d_L|} [L_0, L_0]) \longrightarrow (D_{M_0}(M_1) \xrightarrow{d_M|} [M_0, M_0])\end{aligned}$$

which makes diagrams (1) and (2) commutative.

(i) Since L and M are aspherical, we have $Z(L_0) \subseteq St_{L_0}(L_1)$, $Z(M_0) \subseteq St_{M_0}(M_1)$. Consequently, η_0 is an isomorphism between $L_0/Z(L_0)$ and $M_0/Z(M_0)$. So the pair (η_0, ξ_0) is an isoclinism from L_0 to M_0.

(ii) Since L and M are simply connected, we have $L_1^{L_0} = Z(L_1)$, $M_1^{M_0} = Z(M_1)$, $D_{L_0}(L_1) = [L_1, L_1]$ and $D_{M_0}(M_1) = [M_1, M_1]$. So we have the isomorphisms $\eta_1 : L_1/Z(L_1) \longrightarrow M_1/Z(M_1)$, $\xi_1 : [L_1, L_1] \longrightarrow [M_1, M_1]$. The pair (η_1, ξ_1) is an isoclinism from L_1 to M_1, as required. □

Proposition 12. *Let $L : L_1 \xrightarrow{d_L} L_0$ and $M : M_1 \xrightarrow{d_M} M_0$ be isoclinic finite dimensional Lie crossed modules. Then L_1 and L_0 are isoclinic to M_1 and M_0, respectively.*

Proof. Let $L : L_1 \xrightarrow{d_L} L_0$ and $M : M_1 \xrightarrow{d_M} M_0$ be isoclinic Lie crossed modules. Then we have the crossed module isomorphisms

$$\begin{aligned}(\eta_1, \eta_0) &: (\overline{L_1} \xrightarrow{\overline{d_L}} \overline{L_0}) \longrightarrow (\overline{M_1} \xrightarrow{\overline{d_M}} \overline{M_0}) \\ (\xi_1, \xi_0) &: (D_{L_0}(L_1) \xrightarrow{d_L|} [L_0, L_0]) \longrightarrow (D_{M_0}(M_1) \xrightarrow{d_M|} [M_0, M_0])\end{aligned}$$

which makes diagrams (1) and (2) commutative. Since L_1 and M_1 are finite dimensional, the restriction $\xi_1| : [L_1, L_1] \longrightarrow [M_1, M_1]$ is also an isomorphism. Similarly, we have the isomorphisms $\eta_1' : L_1/Z(L_1) \longrightarrow M_1/Z(M_1)$, $\eta_1'(l_1 Z(L_1)) = m_1 Z(M_1)$, $\eta_0' : L_0/Z(L_0) \longrightarrow M_0/Z(M_0)$, $\eta_0'(L_0 Z(L_0)) = m_0 Z(M_0)$, and ξ_0 which make L_1 and L_0 isoclinic to M_1 and M_0, respectively. □

3. n-Isoclinic Lie Crossed Modules

In this section, our aim is that define the notion of n-isoclinic Lie crossed modules. Firstly, we recall the n-isoclinic Lie algebras, see [2, 5] for details.

Let L_1 and L_2 be Lie algebras and n be a non-negative integer. Then, L_1 and L_2 are said to be *n-isoclinic*, $L_1 \underset{n}{\sim} L_2$, if there exist isomorphisms $\eta : L_1/Z_n(L_1) \longrightarrow L_2/Z_n(L_2)$ and $\xi : [L_1, L_1]_{n+1} \longrightarrow [L_2, L_2]_{n+1}$ in such a way that ξ is compatible with η, that is, the $(n+1)$-fold commutator $[\cdots[[b_1, b_2], b_3], \cdots, b_{n+1}]$ equals $\xi([\cdots[[a_1, a_2], a_3], \cdots, a_{n+1}])$ for any $b_i \in \eta(a_i Z_n(L_1))$ and $a_i \in L_1$ for $i = 1, \ldots, n+1$. The pair (η, ξ) is called an n-isoclinism between L_1 and L_0. Also, L_1 and L_2 are called n-isoclinic Lie algebras.

Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. We use the following notations in this section:

- $[L, L]_n$ denotes the n-th term of the lower central series of L defined inductively by $[L, L]_1 = L$ and $[L, L]_{n+1} = [[L, L]_n, L]$, for $n \geq 1$.
- $Z_n(L)$ denotes the n-th term of the upper central series of L defined inductively by $Z_0(L) = 1$ and $Z_{n+1}(L)/Z_n(L)$ is the centre of $L/Z_n(L)$, for $n \geq 0$.
- $\zeta_n(L_1) = \{l_1 \in L_1 \mid [{}_nL_0, l_1] = 1\}$, where $[{}_1L_0, l_1] =< [l_0, l_1] \mid l_0 \in L_0 >$ and inductively $[{}_{n+1}L_0, l_1] = [L_0, [{}_nL_0, l_1]]$.
- $\kappa_n(L_0) = Z_n(L_0) \cap \{l_0 \in L_0 \mid [{}_iL_0, [[{}_{n-1-i}L_0, l_0], L_1]] = 1 \text{ for all } 0 < i < n-1\}$, where $[{}_0L_0, L_1'] = L_1'$ for each subalgebra L_1 of L, $[l_0, L_1] =< [l_0, l_1] \mid l_1 \in L_1 >$, $[{}_0L_0, l_0] = l_0$ and inductively $[{}_nL_0, l_0] = [L_0, [{}_{n-1}L_0, l_0]]$.
- $\Gamma_n(L_1, L_0) = [{}_{n-1}L_0, L_1]$ where $[{}_0L_0, L_1] = L_1$ and inductively, $[{}_nL_0, L_1] = [L_0, [{}_{n-1}L_0, L_1]]$.

Lemma 13. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. Then*
(i) for all $l_1, l_1' \in L_1$ and $l_0, l_0' \in L_0$, the following identities hold:

$$\begin{aligned} [l_0 l_0', l_1] &= [l_0, [l_0', l_1]], \\ [l_0, l_1 l_1'] &= [[l_0, l_1], l_1']. \end{aligned}$$

(ii) for any $L_1' \trianglelefteq L_1$ and $L_0', L_0'' \trianglelefteq L_0$,

$$\begin{aligned} (a)\ [L_0', [L_0'', L_1']] &\subseteq [L_0'', [L_0', L_1']] + [[L_0', L_0''], L_1'], \\ (b)\ [[L_0', L_0''], L_1'] &\subseteq [L_0', [L_0'', L_1']] + [L_0'', [L_0', L_1']]\ . \end{aligned}$$

Proof. (i) It is clear from the conditions of Lie action.

(ii) Firstly, we show that $[L_0', L_1'] \trianglelefteq L_1$. For all $l_0' \in L_0'$, $l_1' \in L_1'$ and $l_1 \in L_1$, as

$$\begin{aligned} [l_1, [l_0', l_1']] &= [d(l_1), [l_0', l_1']]\ (\because d \sim \text{Lie crossed module}) \\ &= [d(l_1) l_0', l_1']\ (\because (i)) \\ &\in [L_0', L_1']\ (\because L_0' \trianglelefteq L_0), \end{aligned}$$

we get $[L_0', L_1'] \trianglelefteq L_1$. Similarly, $[L_0', [L_0'', L_1']]$, $[L_0'', [L_0', L_1']]$ and $[[L_0', L_0''], L_1']$ are ideal in L_1.

(a) For all $l_0' \in L_0'$, $l_1' \in L_1'$ and $l_1' \in L_1'$, we have

$$\begin{aligned} [l_0', [l_0'', l_1']] &= [[l_0', l_0''], l_1'] + [l_0'', [l_0', l_1']] \\ &\in [[L_0', L_0''], L_1'] + [L_0'', [L_0', L_1']]. \end{aligned}$$

So, we get $[L'_0, [L''_0, L'_1]] \subseteq [L''_0, [L'_0, L'_1]] + [[L'_0, L''_0], L'_1]$.

Similarly, (b) can be checked. □

Proposition 14. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module and $n \geq 1$. Then,*

$$\begin{aligned} (i)\ Z_n(L) &= (\zeta_n(L_1), \kappa_n(L_0), d) \\ (ii)\ [L, L]_n &= (\Gamma_n(L_1, L_0), [L_0, L_0]_n, d). \end{aligned}$$

Proof. It is Lie crossed modules analogues of the Lemma 2.1 in [1]. □

Proposition 15. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module and i, j be positive integers with $j \geq i$. Then,*

$$\begin{aligned} (i)[[L_0, L_0]_i, \kappa_j(L_0)] &\leq \kappa_{j-i}(L_0), \\ (ii)\ [[L_0, L_0]_i, \zeta_j(L_1)] &\leq \zeta_{j-i}(L_1), \\ (iii)[\kappa_j(L_0), \Gamma_i(L_1, L_0)] &\leq \zeta_{j-i}(L_1). \end{aligned}$$

Proof. (i) Using induction on i, the case $i = 1$ is clear. The three Lie subalgebra lemma show that

$$[[L_0, L_0]_{i+1}, \kappa_j(L_0)] = [[[L_0, L_0]_i, L_0], \kappa_j(L_0)]$$

is contained in the sum

$$[[L_0, \kappa_j(L_0)], [L_0, L_0]_i] + [\kappa_j(L_0), [L_0, L_0]_i], L_0];$$

by induction, the latter is contained in $\kappa_{j-i-1}(L_0)$.

(ii) It is proved by using Lemma 13 (ii) and similar to part of (i).

(iii) We have $[_{j-1}L_0, [\kappa_j(L_0), L_1]] = 1$ from the definition of $\kappa_j(L_0)$ and implying that $[\kappa_j(L_0), \Gamma_1(L_1, L_0)] \leq \zeta_{j-1}(L_0)$. From Lemma 13 (ii), induction argument on $i \geq 1$ and parts (i),(ii), we have

$$\begin{aligned} [\kappa_j(L_0), \Gamma_1(L_1, L_0)] &= [\kappa_j(L_0), [L_0, \Gamma_i(L_1, L_0)]] \\ &\leq [L_0, [\kappa_j(L_0), \Gamma_i(L_1, L_0)]] + [[L_0, \kappa_j(L_0)], \Gamma_i(L_1, L_0)] \\ &\leq [L_0, \zeta_{j-1}(L_1)] + [\kappa_{j-i}(L_0), \Gamma_i(L_1, L_0)] \\ &\leq \zeta_{j-i-1}(L_1). \end{aligned}$$

□

Corollary 16. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module. Then, for all positive integers i, j with $j \geq i$, $[Z_j(L), [L, L]_i] \leq Z_{j-i}(L)$.*

Lemma 17. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module, $l \in \zeta_n(L_1)$ and $k \in \kappa_n(L_0)$. Then for all $a \in L_1$ and $b_1, b_2, \ldots, b_{n+1} \in L_0$, we have*
(i) $[b_n, \cdots, [b_2, [b_1, l + a]], \cdots] = [b_n, \cdots, [b_2, [b_1, a]], \cdots]$,
(ii) for all $1 \leq i \leq n$, $[b_n, \cdots, [b_i + k, \cdots, [b_1, a] \cdots] \cdots] = [b_n, \cdots, [b_i, \cdots, [b_1, a] \cdots] \ldots]$,
(iii) for all $1 \leq i \leq n+1$, $[\cdots[\cdots[b_1, b_2], \cdots, b_i + k] \cdots, b_{n+1}] = [\cdots[b_1, b_2], \cdots, b_{n+1}]$,
(iv) $[b_n, \cdots, b_{n-i+1}, [[b_{n-i}, \cdots, [b_2, b_1 + k] \cdots], a] \cdots] = [b_n, \cdots, b_{n-i+1}, [[b_{n-i}, \cdots, [b_2, b_1] \cdots], a] \cdots]$.

Proof. By using Lemma 13, induction argument on $i \geq 1$ and Proposition 15, one can easily check these arguments. □

As an immediate consequence of the above lemma, we deduce that for any crossed module $L : L_1 \xrightarrow{d} L_0$ and $n \geq 1$, there exist well-defined maps

$$\eta_L^{n+1} : \frac{L_1}{\zeta_n(L_1)} \times \underbrace{\frac{L_0}{\kappa_n(L_0)} \times \cdots \times \frac{L_0}{\kappa_n(L_0)}}_{n-copies} \longrightarrow \Gamma_{n+1}(L_1, L_0),$$

and

$$\theta_L^{n+1} : \underbrace{\frac{L_0}{\kappa_n(L_0)} \times \cdots \times \frac{L_0}{\kappa_n(L_0)}}_{n+1-copies} \longrightarrow [L_0, L_0]_{n+1}$$

given by

$$\begin{aligned} \eta_L^{n+1}(a + \zeta_n(L_1), b_1 + \kappa_n(L_0), \dots, b_n + \kappa_n(L_0)) &= [b_n, \dots, [b_2, [b_1, a]] \dots], \\ \theta_L^{n+1}(b_1 + \kappa_n(L_0), \dots, b_n + \kappa_n(L_0), b_{n+1} + \kappa_{n+1}(L_0)) &= [\dots, [[b_1, b_2], b_3], \dots, b_{n+1}]. \end{aligned}$$

Proposition 18. *Let $L : L_1 \xrightarrow{d} L_0$ be a Lie crossed module with a Lie crossed submodule $M : M_1 \longrightarrow M_0$. Then,*
(i) $Z_n(M + Z_n(L)) = Z_n(M) + Z_n(L)$,
(ii) $[M + Z_n(L), M + Z_n(L)]_{n+1} = [L, L]_{n+1}$,
(iii) $[M + Z_n(L), M + Z_n(L)]_{n+1} \cap Z_n(M + Z_n(L)) = [M \cap Z_n(M), M \cap Z_n(M)]_{n+1}$.

Proof. (i) By using Lemma 17 (i), (ii), we get $\zeta_n(M_1 + \zeta_n(L_1))$. So, we will show that $\kappa_n(M_0 + \kappa_n(L_0)) = \kappa_n(M_0) + \kappa_n(L_0)$.

As $[\kappa_n(M_0) + \kappa_n(L_0),_n M_0 + \kappa_n(L_0)] \subseteq [Z_n(M_0 + Z_n(L_0)),_n M_0 + Z_n(L_0)] = 1$, we can write $\kappa_n(M_0) + \kappa_n(L_0) \subseteq Z_n(M_0 + \kappa_n(L_0))$. On the other hand, $1 \leq i \leq n-1$, we have the following results:

(1) From the induction, we have

$$[_{n-i-1}M_0 + \kappa_n(L_0), \kappa_n(M_0) + \kappa_n(L_0)] \subseteq [_{n-i-1}M_0, \kappa_n(M_0)] + \kappa_{i+1}(L_0).$$

(2) From the Proposition 15 (ii), (iii), we have

$$\begin{aligned} [[_{n-i-1}M_0, \kappa_n(M_0)], \zeta_n(L_1)] &\subseteq [[L_0, L_0]_{n-i}, \zeta_n(L_1)] \\ &\subseteq \zeta_i(L_1) \end{aligned}$$

and

$$[\kappa_{i+1}(L_0), M_1 + \zeta_n(L_1)] \subseteq [\kappa_{i+1}(L_0), L_1] \subseteq \zeta_i(L_1).$$

(3) By the definition of $\kappa_n(M_0)$, we get

$$[_iM_0, [[_{n-i-1}M_0, \kappa_n(M_0)], M_1]] = 1$$

that is $[[_{n-i-1}M_0, \kappa_n(M_0)], M_1]$ is contained in $\zeta_i(M_1)$.

(4) $\zeta_i(M_1 + \zeta_i(L_1)) \subseteq \zeta_1(M_1 + \zeta_n(L_1))$.

By using these, we get $\kappa_n(M_0) + \kappa_n(L_0) \subseteq \kappa_n(M_0 + \kappa_n(L_0))$. The reverse is proved easily by using Lemma 17 (iii) and (iv).

(ii) Using Proposition 14 (ii) with Lemma 17, we have

$$\begin{aligned}[M+Z_n(L),M+Z_n(L)]_{n+1} &= (\Gamma_{n+1}(M_1+\zeta_n(L_1),M_0+\kappa_n(L_0)),[M_0+\kappa_n(L_0),M_0+\kappa_n(L_0)]_{n+1},d)\\ &= (\Gamma_{n+1}(M_1,M_0),[M_0,M_0]_{n+1},d)\\ &= [M,M]_{n+1}.\end{aligned}$$

(iii) It is clear from the (i).
(iv) It is clear from the (ii) and (iii). □

Definition 19. *The Lie crossed modules* $L : L_1 \xrightarrow{d_L} L_0$ *and* $L' : L'_1 \xrightarrow{d_{L'}} L'_0$ *are said to be n-isoclinic* ($n \geq 0$) *,* $L \underset{n}{\sim} L'$*, if there exists a pair of isomorphisms of Lie crossed modules*

$$\begin{aligned}\alpha &= (\alpha_1,\alpha_2) : \frac{L}{Z_n(L)} \longrightarrow \frac{L'}{Z_n(L')},\\ \beta &= (\beta_1,\beta_2) : [L,L]_{n+1} \longrightarrow [L',L']_{n+1},\end{aligned}$$

such that the following diagrams are commutative

$$\begin{array}{ccc}\frac{L_1}{\zeta_n(L_1)} \times \frac{L_0}{\kappa_n(L_0)} \times \cdots \times \frac{L_0}{\kappa_n(L_0)} & \xrightarrow{\eta_L^{n+1}} & \Gamma_{n+1}(L_1,L_0)\\ \alpha_1 \times \alpha_2^n \downarrow & & \downarrow \beta_1\\ \frac{L'_1}{\zeta_n(L'_1)} \times \frac{L'_0}{\kappa_n(L'_0)} \times \cdots \times \frac{L'_0}{\kappa_n(L'_0)} & \xrightarrow{\eta_{L'}^{n+1}} & \Gamma_{n+1}(L'_1,L'_0)\end{array}$$

and

$$\begin{array}{ccc}\frac{L_0}{\kappa_n(L_0)} \times \cdots \times \frac{L_0}{\kappa_n(L_0)} & \xrightarrow{\theta_L^{n+1}} & [L_0,L_0]_{n+1}\\ \alpha_2^{n+1} \downarrow & & \downarrow \beta_2\\ \frac{L'_0}{\kappa_n(L'_0)} \times \cdots \times \frac{L'_0}{\kappa_n(L'_0)} & \xrightarrow{\theta_{L'}^{n+1}} & [L'_0,L'_0]_{n+1}.\end{array}$$

In other words, for all $l_1 \in L_1$ *and* $b_1, b_2, \ldots, b_{n+1} \in L_0$*, we have*

$$\begin{aligned}\beta_1([b_n,\ldots,[b_2,[b_1,l_1]]\ldots]) &= [b'_n,\ldots,[b'_2,[b'_1,l'_1]]\ldots],\\ \beta_2([\ldots,[[b_1,b_2],b_3],\ldots,b_{n+1}]) &= [\ldots,[[b'_1,b'_2],b'_3],\ldots,b'_{n+1}],\end{aligned}$$

where $l'_1 \in \alpha_1(l_1+\zeta_n(L_1))$ *and* $b'_i \in \alpha_2(b_i+\kappa_n(L_0))$ *for* $i = 1,\ldots,n+1$*. The pair* (α,β) *is called an n-isoclinism between* L *and* L'*.*

As Lie algebras are considered as Lie crossed modules, we obtain the definition of n-isoclinic Lie algebras. Since n-isoclinism between Lie crossed modules is an equivalence relation, we can say that it divides the class of all Lie crossed modules into n-isoclinism equivalence classes.

In the following proposition, we get a relation between the n-isoclinic Lie crossed modules and the n-isoclinic Lie algebras. By using above definition, one can easily check that all results obtained in [6] correct for Lie crossed modules.

Proposition 20. *Let* $L : L_1 \xrightarrow{d_L} L_0$ *and* $L' : L'_1 \xrightarrow{d_{L'}} L'_0$ *be two n-isoclinic Lie crossed modules. Then* $L_1 \underset{n}{\sim} L'_1$ *and* $L_0 \underset{n}{\sim} L'_0$*.*

Proof. Let (α,β) be an n-isoclinism between L and L'. Since $[L_1,L_1]_{n+1}$ and $[L'_1,L'_1]_{n+1}$ are Lie subalgebras of $\Gamma_{n+1}(L_1,L_0)$ and $\Gamma_{n+1}(L'_1,L'_0)$, we show that β_1 maps any generator of $[L_1,L_1]_{n+1}$ to a generator of $[L'_1,L'_1]_{n+1}$.

Suppose $l_1, \dots l_{n+1}$ are arbitrary elements of L_1 and choose $l'_i \in \alpha_1(l_i\zeta_n(L_1))$ for $1 \le i \le n+1$. Then $\alpha_2(d_L(l_i)\kappa_n(L_0)) = d'_L(l'_i) + \kappa_n(L'_0)$ for all i. Now, if $n = 1$, then

$$\begin{aligned} \beta_1([l_1, l_2]) &= \beta_1([d_L(l_1), l_2]) \\ &= [d'_L(l'_1), l'_2] \\ &= [l'_1, l'_2] \end{aligned}$$

from the above definition. We assume that $n \ge 2$. Setting $x_i = [\dots[[l_1, l_2], l_3], \dots l_i]$ for $i = 2, \dots n$, an easy inductive argument establishes that

$$[\dots[[l_1, l_2], l_3], \dots, l_{n+1}] = \begin{cases} [[x_n, l_{n+1}], [l_n, [x_{n-2}, l_{n-1}], \dots [x_2, l_3], [l_2, l_1] \dots]]] & \text{when } n \text{ is even} \\ -[l_{n+1}, [[x_{n-1}, l_n], [l_{n-1}, \dots [x_2, l_3], [l_2, l_1] \dots]]] & \text{when } n \text{ is odd.} \end{cases}$$

Also setting $y_i = [\dots[[l'_1, l'_2], l'_3], \dots l'_i]$ for $i = 2, \dots n$ a similar result holds for $[\dots[[l'_1, l'_2], l'_3], \dots l'_n]$. It is easily verified that $\alpha_1(x_i + \zeta_n(L_1)) = y_i + \zeta_n(L'_1)$ and $\alpha_2(d_L([x_i, l_{i+1}]) + \kappa_n(L_0)) = d'_L([y_i, l'_{i+1}]) + \kappa_n(L'_0)$ for $2 \le i \le n$.

Consequently, we have

$$\beta_1([\dots[[l_1, l_2], l_3], \dots l_{n+1}]) = [\dots[[l'_1, l'_2], l'_3], \dots l'_{n+1}],$$

whenever n is even and analogously, the above equality holds when n is odd.

Now, one readily sees that the restriction of β_1 to $[L_1, L_1]_{n+1}$ is an isomorphism of $[L_1, L_1]_{n+1}$ onto $[L'_1, L'_1]_{n+1}$, and α_1 induces an isomorphism $\overline{\alpha_1} : L_1/Z_n(L_1) \longrightarrow L'_1/Z_n(L'_1)$ given by $\overline{\alpha_1}(l_1 + Z_n(L_1)) = l'_1 + Z_n(L'_1)$. Also, using the isomorphism β_2, the maps $\overline{\alpha_2} : L_0/Z_n(L_0) \longrightarrow L'_0/Z_n(L'_0)$ defined by $\overline{\alpha_2}(l_0 + Z_n(L_0)) = l'_0 + Z_n(L'_0)$, where $l_0 \in L_0$ and $l'_0 \in \alpha_2(l_0 + Z_n(L_0))$, is an isomorphism. So, the pair $(\overline{\alpha_1}, \beta_1|_{[L_1, L_1]_{n+!}})$ is an n-isoclinism between the Lie algebras L_1 and L'_1, and the pair $(\overline{\alpha_2}, \beta_2)$ is an n-isoclinism between the Lie algebras L_0 and L'_0. $\square$

Remark: When $n = 1$, Proposition 20 improves Proposition 23 in [4]. Also, it follows from the above proposition that for any two Lie algebra L and M, if $(L \xrightarrow{i} L) \underset{n}{\sim} (L \xrightarrow{i} M)$ or $(L \xrightarrow{id} L) \underset{n}{\sim} (M \xrightarrow{id} M)$, then

$L \underset{n}{\sim} M.$

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