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Variations of Star Selection Principles on Small Spaces

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Abstract. In this work, we introduce the notions of Star-$\sigma\mathcal{K}$ and absolutely Star-$\sigma\mathcal{K}$ spaces which allow us to unify results among several properties in the theory of star selection principles on small spaces. In particular, results on star selective versions of the Menger, Hurewicz and Rothberger properties and selective versions of property (a) regarding the size of the space. Connections to other well-known star properties are mentioned. Furthermore, the absolute and selective version of the neighbourhood star selection principle are introduced. As an application, it is obtained that the extent of a separable absolutely strongly star-Menger (absolutely strongly star-Hurewicz) space is at most the dominating number $\mathfrak{d}$ (the bounding number $\mathfrak{b}$).

1. Introduction

In this section, we recall some classic definitions and important results in the theory of star selection principles that are central to our work. In addition, we introduce useful notation and terminology that will help us to deal with variations of the classical star versions of Menger, Hurewicz and Rothberger properties and selective variations property (a). Main results, consequences and applications are in Section 2 and 3. In section 4 we introduce new variations of neighbourhood star selection principles.

1.1. *Notation and terminology*

Let X be a topological space. We denote by $[X]^{<\omega}$ the collection of all finite subsets of X. For a subset A of X and a collection $\mathcal{U}$ of subsets of X, the star of A with respect to $\mathcal{U}$, denoted by $St(A,\mathcal{U})$, is the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$; for $A = \{x\}$ with $x \in X$, we write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$. Throughout this paper, all spaces are assumed to be regular, unless a specific separation axiom is indicated. For notation and terminology, we refer to [13].

In the context of classical star covering properties, we follow the notation of [12]. Recall that a space X is said to be strongly starcompact (strongly star-Lindelöf), briefly *SSC* (*SSL*), if for every open cover $\mathcal{U}$ of X there exists a finite (countable) subset F of X such that $St(F,\mathcal{U}) = X$. A space X is starcompact (star-Lindelöf), briefly *SC* (*SL*), if for every open cover $\mathcal{U}$ of X there exists a finite (countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\bigcup\mathcal{V},\mathcal{U}) = X$. We refer the reader to the survey of Matveev [22] for a more detailed treatment of these star covering properties.

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1.2. Classical (star) selection principles

Given a topological space X, we denote by $\mathcal{O}$ the collection of all open covers of X and by Γ the collection of all γ-covers of X; an open cover $\mathcal{U}$ of X is a γ-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Henceforth, $\mathcal{A}$ and $\mathcal{B}$ will denote some collections of open covers of a space X and $\mathcal{K}$ a family of subsets of X. We recall the definition of three classical well-known star selection principles introduced in [15, Definition 1.1, Definition 1.2]:

$S_1^*(\mathcal{A},\mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ there exists a sequence $\{U_n : n \in \omega\}$ such that $U_n \in \mathcal{U}_n$, $n \in \omega$, and $\{St(U_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

$S_{fin}^*(\mathcal{A},\mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$, $n \in \omega$, and $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n : n \in \omega\} \in \mathcal{B}$.

$SS_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ there exists a sequence $\{K_n : n \in \omega\} \subseteq \mathcal{K}$ such that $\{St(K_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all finite (resp. one-point) subsets of X, it is denoted by $SS_{fin}^*(\mathcal{A},\mathcal{B})$ (resp. $SS_1^*(\mathcal{A},\mathcal{B})$) instead of $SS_{\mathcal{K}}^*(\mathcal{A},\mathcal{B})$. Following this terminology, the star versions for the cases Menger and Rothberger were defined in [15, Definition 1.4] and the star versions for the Hurewicz case were defined in [4]:

SM : $S_{fin}^*(\mathcal{O},\mathcal{O})$ defines the star-Menger property ([15]);

SSM : $SS_{fin}^*(\mathcal{O},\mathcal{O})$ defines the strongly star-Menger property ([15]);

SR : $S_1^*(\mathcal{O},\mathcal{O})$ defines the star-Rothberger property ([15]);

SSR : $SS_1^*(\mathcal{O},\mathcal{O})$ defines the strongly star-Rothberger property ([15]);

SH : $S_{fin}^*(\mathcal{O},\Gamma)$ defines the star-Hurewicz property ([4]);

SSH : $SS_{fin}^*(\mathcal{O},\Gamma)$ defines the strongly star-Hurewicz property ([4]).

For paracompact Hausdorff spaces the three Menger-type properties, SM, SSM and M are equivalent and the same situation holds for the three Rothberger-type properties and the three Hurewicz-type properties (see [15, Theorem 2.8] and [4, Proposition 4.1]). In fact, the previous equivalences also holds in paraLindelöf spaces (see [9, Theorem 2.10]).

Figure 1 shows the relationships among these properties (in the diagram C, H, M, R and L are used to denote compactness, Hurewicz, Menger, Rothberger and the Lindelöf property, respectively). We refer the reader to [17] to see the current state of knowledge about these relationships with others.

Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n. A subset B of ω^ω is bounded if there is $g \in \omega^\omega$ such that $f \leq^* g$ for each $f \in B$. A subset D of ω^ω is dominating if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of ω^ω is denoted by $\mathfrak{b}$, and the minimal cardinality of a dominating subset of ω^ω is denoted by $\mathfrak{d}$. The family of all meager subsets of $\mathbb{R}$ is denoted by $\mathcal{M}$ and the minimum of the cardinalities of subfamilies $\mathcal{U} \subset \mathcal{M}$ such that $\bigcup \mathcal{U} = \mathbb{R}$ is denoted by $cov(\mathcal{M})$. We mention that the invariant cardinal $cov(\mathcal{M})$ is the minimum cardinality of a family $C \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in C$ such that $f(n) \neq g(n)$ for all but finitely many n (see [1]). Observe that, by this result, if $C \subseteq \omega^\omega$ is of size less than $cov(\mathcal{M})$, then there exists a function $g \in \omega^\omega$ such that for every $f \in C$, $f(n) = g(n)$ for infinitely many n and, in this case, it is said that the function g guesses the family C.

The first important characterizations of some of these star selection principles for the case of Ψ-spaces were obtained by Bonanzinga and Matveev:

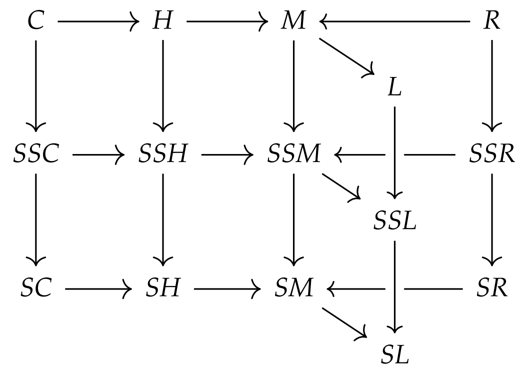


Figure 1: Classical star versions of selection principles. None of the arrows reverse

Proposition 1.1. ([8, Proposition 2, Proposition 3, Proposition 4]) *Given any almost disjoint family $\mathcal{A}$, the following assertions hold.*

1. $\Psi(\mathcal{A})$ *is strongly star-Menger if and only if* $|\mathcal{A}| < \mathfrak{d}$.
2. $\Psi(\mathcal{A})$ *is strongly star-Hurewicz if and only if* $|\mathcal{A}| < \mathfrak{b}$.
3. *If* $|\mathcal{A}| < cov(\mathcal{M})$, *then* $\Psi(\mathcal{A})$ *is strongly star-Rothberger.*

The following generalization of one direction of Proposition 1.1 (1) was given by Sakai in [27, Proposition 1.7]. It can be viewed as a selective version of the fact that Lindelöf spaces of size less than $\mathfrak{d}$ are Menger:

Proposition 1.2. ([27, Proposition 1.7]) *Every strongly star-Lindelöf space of cardinality less than* $\mathfrak{d}$ *is strongly star-Menger.*

The Hurewicz and Rothberger cases of Proposition 1.2 can be proved using similar ideas:

Proposition 1.3. ([9, Lemma 3.9]) *Every strongly star-Lindelöf space of cardinality less than* $\mathfrak{b}$ *is strongly star-Hurewicz.*

Proposition 1.4. *Every strongly star-Lindelöf space of cardinality less than* $cov(\mathcal{M})$ *is strongly star-Rothberger.*

Other generalizations of (1) and (2) of Proposition 1.1 that also characterizes star selection principles on the Niemytzki plane were given in [9]:

Theorem 1.5. ([9, Theorem 3.5]) *Let X be a topological space of the form $Y \cup Z$, where $Y \cap Z = \emptyset$, Z is a σ-compact subspace and Y is a closed discrete set. If X is strongly star-Lindelöf, then $|Y| < \mathfrak{d}$ if and only if X is strongly star-Menger.*

Theorem 1.6. ([9, Theorem 3.12]) *Let X be a topological space of the form $Y \cup Z$, where $Y \cap Z = \emptyset$, Z is a σ-compact subspace and Y is a closed discrete set. If X is strongly star-Lindelöf, then $|Y| < \mathfrak{b}$ if and only if X is strongly star-Hurewicz.*

1.3. Absolute and selective versions of star selection principles

In this section we recall the absolute and selective versions of the classical star selection principles. We start by mentioning the definition of the absolute and selective versions of the strongly star-Lindelöf property. In [3], Bonanzinga defined and studied the absolute version of the strongly star-Lindelöf property.

Definition 1.7. ([3])A space X is absolutely strongly star-Lindelöf (*aSSL*) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there is a countable set $C \subseteq D$ such that $St(C, \mathcal{U}) = X$.

On the other hand, the selective version of the strongly star-Lindelöf property was defined first by S. Bhowmik in [2] and later studied in [6, Definition 3] with a different name.

Definition 1.8. ([2])A space X is selectively strongly star-Lindelöf (*selSSL*) if for every open cover $\mathcal{U}$ of X and for every sequence $\{D_n : n \in \omega\}$ of dense sets of X, there is a sequence $\{F_n : n \in \omega\}$ of finite sets such that $F_n \subseteq D_n$, $n \in \omega$, and $\{St(F_n, \mathcal{U}) : n \in \omega\}$ is an open cover of X.

Caserta, Di Maio and Kočinac introduced in [10, Definition 2.1] the absolute versions of the classical star selection principles in a general form. Here, we will use a different notation that seems to be simpler and naturally relates to the star selection principle $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$: [1)]

Definition 1.9. Given a space X, the following selection hypothesis is defined:

absolutely $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ and each dense subset D of X, there exists a sequence $\{K_n : n \in \omega\} \subseteq \mathcal{K}$ such that each $K_n \subseteq D$, $n \in \omega$, and $\{St(K_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

For shortness, we write $aSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ instead of absolutely $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ and, as usual, when $\mathcal{K}$ is the collection of all finite (resp. one-point, closed discrete) subsets of X, we write $aSS^*_{fin}(\mathcal{A}, \mathcal{B})$ (resp. $aSS^*_1(\mathcal{A}, \mathcal{B})$, $aSS^*_{cd}(\mathcal{A}, \mathcal{B})$) instead of $aSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$. Following this terminology, the absolute versions of the classical star selection principles (defined in [10]) are given as follows:

aSSM : $aSS^*_{fin}(\mathcal{O}, \mathcal{O})$ defines the absolutely strongly star-Menger property;

aSSR : $aSS^*_1(\mathcal{O}, \mathcal{O})$ defines the absolutely strongly star-Rothberger property;

aSSH : $aSS^*_{fin}(\mathcal{O}, \Gamma)$ defines the absolutely strongly star-Hurewicz property.

More recently, Bonanzinga et al. defined and studied the selective version of the strongly star-Menger property in [7] and [11, Definition 1.1.4]. Furthermore, they asked the question whether absolutely strongly star-Menger and selectively strongly star-Menger are equivalent properties. This selective principle naturally gives birth to the selective version for the Hurewicz and Rothberger cases. We introduce the following general notation that includes these kind of selection principles.[2)]

Definition 1.10. Given a space X, the following selection hypothesis is defined:

selectively $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ and each sequence $\{D_n : n \in \omega\}$ of dense sets of X, there exists a sequence $\{K_n : n \in \omega\} \subseteq \mathcal{K}$ such that each $K_n \subseteq D_n$, $n \in \omega$, and $\{St(K_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

For shortness, we write $selSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ instead of selectively $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ and, again, when $\mathcal{K}$ is the collection of all finite (resp. one-point, closed discrete) subsets of X, we write $selSS^*_{fin}(\mathcal{A}, \mathcal{B})$ (resp. $selSS^*_1(\mathcal{A}, \mathcal{B})$, $selSS^*_{cd}(\mathcal{A}, \mathcal{B})$) instead of $selSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$. With this notation, the selective versions of some classical star selection principles are given as follow:[3)]

selSSM ([7], [11]): $selSS^*_{fin}(\mathcal{O}, \mathcal{O})$ defines the selectively strongly star-Menger property;

selSSR : $selSS^*_1(\mathcal{O}, \mathcal{O})$ defines the selectively strongly star-Rothberger property;

selSSH : $selSS^*_{fin}(\mathcal{O}, \Gamma)$ defines the selectively strongly star-Hurewicz property.

The relationships among the absolute and selective versions are given in Figure 2.

[1)] In [10], the authors employed an idea of Matveev to define, in a different general form, the absolute versions of star selection principles. Since part of the motivation for that general form was to give the selective version of the property (a), the word *selectively* was used as part of that terminology (see for instance Section 5 in [17]). Here we prefer to use a different terminology because of the introduction of the selective versions of star selection principles.

[2)] In private communication with Kočinac, they informed us that in [18] and [19] they have independently defined the principle absolutely $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ as *selectively* $(\mathcal{A}, \mathcal{B})$-$(a)_{\mathcal{K}}$-*space* and the principle selectively $SS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ as *strictly selectively* $(\mathcal{A}, \mathcal{B})$-$(a)_{\mathcal{K}}$-*space*.

[3)] In [18] and [19], Kočinac and Özçağ, call *selSSM*, *selSSH*, and *selSSR* spaces, respectively *Menger acc-spaces*, *Hurewicz acc-spaces*, and *Rothberger acc-spaces*.

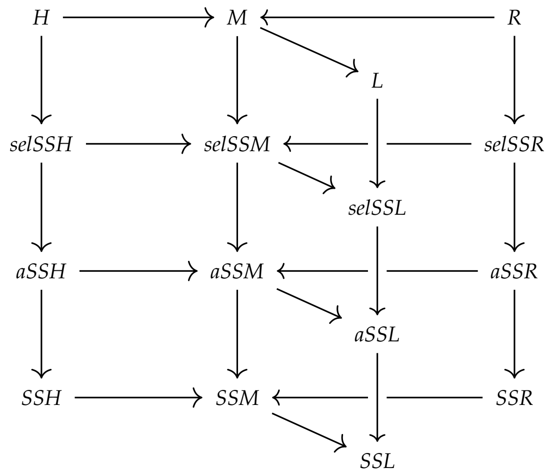


Figure 2: The absolute and selective versions of the classical star selection principles

Analogous to the question posed by Bonanzinga et al. in [7] and [11], whether there is an absolutely strongly star-Menger not selectively strongly star-Menger space, it is important to figure out whether the absolute and selective versions of the Rothberger and Hurewicz cases are equivalent, respectively.

It is natural to wonder whether similar results as Proposition 1.2, 1.3 and 1.4 also hold in the absolute and selective context. We provide an affirmative answer in a broader sense in Section 2.

1.4. Some Selective versions of Property (*a*)

In [12, Theorem 2.1.4, Theorem 2.1.5], van Douwen et.al. presented the proof that for Hausdorff spaces, countable compactness is equivalent to strongly starcompactness (they attribute this result to Fleischman [14]). Motivated by this equivalence, Matveev defined in [20] the absolute version of the strongly starcompact property:

Definition 1.11. ([20, Definition 1.1])A space X is absolutely countably compact (*acc*) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there is a finite set $F \subseteq D$ such that $St(F, \mathcal{U}) = X$.

Later, using a similar idea, the following interesting property was also introduced by Matveev in [21]:

Definition 1.12. ([21])A space X has property (a) if for every open cover $\mathcal{U}$ of X and each dense set D of X, there exists $C \subseteq D$ closed and discrete subset of X such that $St(C, \mathcal{U}) = X$.

Then, a selective version of property (*a*), called *selectively* (*a*), was given by Caserta, Di Maio and Kočinac in [10] using a general selection hypothesis. We mention the selectively (*a*) property in terms of our notation:

Definition 1.13. ([10])A space X is selectively (a) if satisfies $aSS^*_{cd}(\mathcal{O}, \mathcal{O})$.

In [10, Proposition 2.3], the authors pointed out that if X is a separable selectively (*a*) space, then every closed discrete subset of X is of size less than $\mathfrak{c}$. The general case was proved by da Silva in [26, Theorem 3.1] where he investigated the selectively (*a*) property in Ψ-spaces. In particular, he obtained the following result:

Theorem 1.14. ([26, Proposition 4.2]) *Let $\mathcal{A}$ be an infinite almost disjoint family on ω. Then*

1. *If the size of $\mathcal{A}$ is less than $\mathfrak{d}$, then $\Psi(\mathcal{A})$ is selectively* (*a*).
2. *Assume $\mathcal{A}$ is maximal. Then $\Psi(\mathcal{A})$ is selectively* (*a*) *if and only if $|\mathcal{A}| < \mathfrak{d}$.*

Variations of the previous notion will play an important role in sections 2 and 3. In particular, a stronger variation of the selectively (*a*) property which is mentioned in [24, Question 4.3] by Passos, Santana, and da Silva (called †), and also introduced in [11, Definition 1.1.6] and called selective strong property (*a*):

Definition 1.15. A space X is strongly selectively (*a*) if satisfies $selSS^*_{cd}(\mathcal{O},\mathcal{O})$.

It is worth mentioning that this property is currently being studied by Bonanzinga and Maesano in [7].

2. Star selection principles in small spaces

It is well-known that Lindelöf spaces of size less than $\mathfrak{d}$ are Menger and Lindelöf spaces of size less than $\mathfrak{b}$ are Hurewicz. Some star versions of these kind of results were presented in subsections 1.2 and 1.4. The goal of this section is to present a new way to bring all these results together.

subsectionGeneral theorems on small spaces

The notion of *star*-$\mathbb{P}$ was introduced in [23, Definition 3.1] (see also [22]) and its absolute version, namely, *absolutely star*-$\mathbb{P}$ in [34, Definition 1.2]. Paying attention to the following concepts turned out to be essential in our main results:

Definition 2.1. Given $\mathcal{K}$ a family of subsets of a space X, we call X:

- *Star*-$\sigma\mathcal{K}$ if for each open cover $\mathcal{U}$ of X there is $K \subseteq X$ so that K is a $\sigma\mathcal{K}$ kernel of X with respect to $\mathcal{U}$. That is, K is a countable union of subsets of X each one belonging to $\mathcal{K}$ and $St(K,\mathcal{U}) = X$.
- absolutely *Star*-$\sigma\mathcal{K}$, (abbreviated *aStar*-$\sigma\mathcal{K}$) if for each D dense subset of X and for each open cover $\mathcal{U}$ of X there is $K \subseteq D$ so that K is a $\sigma\mathcal{K}$ kernel of X with respect to $\mathcal{U}$. That is, K is a countable union of subsets of D each one belonging to $\mathcal{K}$ and $St(K,\mathcal{U}) = X$.

Remark 2.2. If $\mathcal{B}$ is either $\mathcal{O}$ or Γ and $\mathcal{K}$ is a family of subsets of a space X, then we have

$$selSS^*_{\mathcal{K}}(\mathcal{O},\mathcal{B}) \to aSS^*_{\mathcal{K}}(\mathcal{O},\mathcal{B}) \to aStar\text{-}\sigma\mathcal{K} \to Star\text{-}\sigma\mathcal{K}.$$

Theorem 2.3. *Given a space X of size less than $cov(\mathcal{M})$ and $\mathcal{K}$ a family of subsets of X, then*

1. *if X is Star-$\sigma\mathcal{K}$, then X is $SS^*_{\mathcal{K}}(\mathcal{O},\mathcal{O})$.*
2. *if X is aStar-$\sigma\mathcal{K}$, then X is $selSS^*_{\mathcal{K}}(\mathcal{O},\mathcal{O})$.*

Proof. We will prove item (2) as item (1) follows similarly. Hence, assume X is absolutely *Star*-$\sigma\mathcal{K}$ and $|X| < cov(\mathcal{M})$. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X and let $\{D_n : n \in \omega\}$ be any sequence of dense subsets of X. For $n \in \omega$, let $E_n \subseteq D_n$ be a $\sigma\mathcal{K}$ subset of X so that $St(E_n,\mathcal{U}_n) = X$. Thus, for each $n \in \omega$, $E_n = \bigcup_{m\in\omega} E^m_n$ where for each $n, m \in \omega$, $E^m_n \in \mathcal{K}$. Let us list X as $\{x_\alpha : \alpha < \kappa\}$ with $\kappa < cov(\mathcal{M})$. For each $n \in \omega$ and each $\alpha < \kappa$ let $f_\alpha(n) = min\{m \in \omega : x_\alpha \in St(E^m_n,\mathcal{U}_n)\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $cov(\mathcal{M})$, there exists $g \in \omega^\omega$ such that g guesses $\{f_\alpha : \alpha < \kappa\}$, i.e. for each $\alpha < \kappa$, $\{n \in \omega : g(n) = f_\alpha(n)\}$ is infinite. For each $n \in \omega$, let $C_n = E^{g(n)}_n$. Note that, for each $n \in \omega$, $C_n \subseteq D_n$ and $C_n \in \mathcal{K}$. Furthermore, $\{St(C_n,\mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Hence, X is $selSS^*_{\mathcal{K}}(\mathcal{O},\mathcal{O})$. □

Recall that an open cover $\mathcal{U}$ of a space X is *large* if each $x \in X$ belongs to infinitely many elements of $\mathcal{U}$ (see [25]); the family of all large open covers of X is denoted by $\mathcal{L}$. Observe that in the proof of Theorem 2.3, since for each $\alpha < \kappa$ there are infinitely many $n \in \omega$, so that $g(n) = f_\alpha(n)$, it actually holds that X is $selSS^*_{\mathcal{K}}(\mathcal{O},\mathcal{L})$. Theorem 2.3 can be visualized as Figure 3.

Theorem 2.4. *Given a space X of size less than $\mathfrak{d}$ and $\mathcal{K}$ a family of subsets of X which is closed under finite unions, then*

1. *if X is Star-$\sigma\mathcal{K}$, then X is $SS^*_{\mathcal{K}}(\mathcal{O},\mathcal{O})$.*
2. *if X is aStar-$\sigma\mathcal{K}$, then X is $selSS^*_{\mathcal{K}}(\mathcal{O},\mathcal{O})$.*

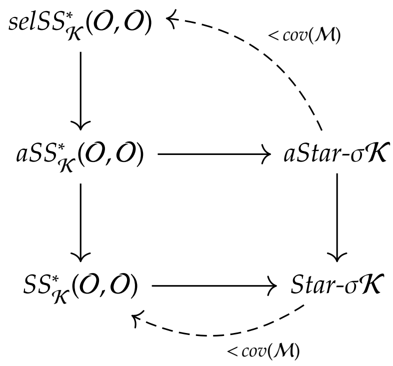


Figure 3: The dashed arrows hold if a space has size less than $cov(\mathcal{M})$

Proof. We will prove item (2) as item (1) follows similarly. Hence, assume X is absolutely *Star-*$\sigma\mathcal{K}$ and $|X| < \mathfrak{d}$. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of X and let $\{D_n : n \in \omega\}$ be any sequence of dense subsets of X. For $n \in \omega$, let $E_n \subseteq D_n$ be a $\sigma\mathcal{K}$ subset of X so that $St(E_n, \mathcal{U}_n) = X$. Thus, for each $n \in \omega$, $E_n = \bigcup_{m\in\omega} E_n^m$ where for each $n, m \in \omega$, $E_n^m \in \mathcal{K}$. Let us list X as $\{x_\alpha : \alpha < \kappa\}$ with $\kappa < \mathfrak{d}$. For each $n \in \omega$ and each $\alpha < \kappa$ let $f_\alpha(n) = min\{m \in \omega : x_\alpha \in St(E_n^m, \mathcal{U}_n)\}$. Since the collection $\{f_\alpha : \alpha < \kappa\}$ has size less than $\mathfrak{d}$, there exists $g \in \omega^\omega$ such that for every $\alpha < \kappa$, $g \not\leq^* f_\alpha$. For each $n \in \omega$, let $C_n = \bigcup_{m\leq g(n)} E_n^m$. For each $n \in \omega$, since $C_n \subseteq D_n$ is a finite union of elements of $\mathcal{K}$, and $\mathcal{K}$ is closed under finite unions, $C_n \in \mathcal{K}$ and $\{St(C_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X. Hence, X is $selSS^*_{\mathcal{K}}(\mathcal{O}, \mathcal{O})$. □

With similar ideas as in the proof of Theorem 2.4 it also holds true:

Theorem 2.5. *Given a space X of size less than $\mathfrak{b}$ and $\mathcal{K}$ a family of subsets of X which is closed under finite unions, then*

1. *if X is Star-$\sigma\mathcal{K}$, then X is $SS^*_{\mathcal{K}}(\mathcal{O}, \Gamma)$.*
2. *if X is aStar-$\sigma\mathcal{K}$, then X is $selSS^*_{\mathcal{K}}(\mathcal{O}, \Gamma)$.*

Remark 2.6. It would be interesting to investigate what happens when we change $\mathcal{O}$ or Γ in Theorems 2.3, 2.4 and 2.5 by other kind of subcollections of open covers (for instance, the ones mentioned in [25]).

Figure 4 summarizes the results presented in Theorem 2.4 and Theorem 2.5:

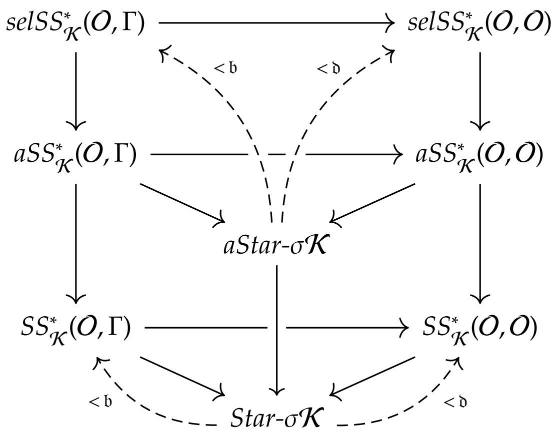


Figure 4: The dashed arrows hold if a space has countable extent, size less than the respective small cardinal invariant and $\mathcal{K}$ is closed under finite unions

2.1. Consequences of general theorems

Here, we collect some immediate consequences of Theorems 2.3, 2.4 and 2.5 proved in subsection 2.

Corollary 2.7. *Let X be any space of size less than* $cov(\mathcal{M})$. *Then*

1. *If X is SSL, then X is* $SS_1^*(\mathcal{O},\mathcal{O})$ *(i.e. SSR).*
2. *If X is aSSL, then X is* $selSS_1^*(\mathcal{O},\mathcal{O})$ *(i.e. SelSSR).*

Proof. To prove (1) and (2), let $\mathcal{K}$ be the family of all one-point subsets of X and observe that X is *SSL* if and only if it is *Star-*$\sigma\mathcal{K}$ and X is *aSSL* if and only if it is *aStar-*$\sigma\mathcal{K}$. Now apply Theorem 2.3 to get the result. □

Corollary 2.7 (1) is Proposition 1.4 and Corollary 2.7 (2) is the absolute version of it.

Corollary 2.8. *Let X be any space of size less than* $\mathfrak{d}$. *Then*

1. ([27, Proposition 1.7]) *If X is SSL, then X is* $SS_{fin}^*(\mathcal{O},\mathcal{O})$ *(i.e. SSM).*
2. *If X is aSSL, then X is* $selSS_{fin}^*(\mathcal{O},\mathcal{O})$ *(i.e. SelSSM).*[1)]
3. *If X is aStar-σ-cd, then X is* $selSS_{cd}^*(\mathcal{O},\mathcal{O})$ *(i.e. strongly selectively (a)).*

Proof. To prove (1) and (2), let $\mathcal{K} = fin$ be the family of all finite subsets of X and observe that X is *SSL* if and only if it is *Star-*$\sigma\mathcal{K}$ and X is *aSSL* if and only if it is *aStar-*$\sigma\mathcal{K}$. Since fin is closed under finite unions, apply Theorem 2.4 to get the result. To prove (3) let $\mathcal{K} = cd$ be the family of all closed discrete subsets of X and apply Theorem 2.4. □

Observe that Corollary 2.8 (1) is Sakai's Proposition 1.2. In addition, Corollary 2.8 (2) generalizes Proposition 9 in [6] where Bonanzinga, Cuzzupe and Sakai show that *aSSL* spaces of size less than $\mathfrak{d}$ are *selSSL*. Given that Ψ-spaces are *aStar-σ-cd*, Corollary 2.8 (3) generalizes da Silva's Theorem 1.14 (1).

Similarly to Corollary 2.8, we have the following for the Hurewicz case:

Corollary 2.9. *Let X be any space of size less than* $\mathfrak{b}$. *Then*

1. *If X is SSL, then X is* $SS_{fin}^*(\mathcal{O},\Gamma)$ *(i.e. SSH).*
2. *If X is aSSL, then X is* $selSS_{fin}^*(\mathcal{O},\Gamma)$ *(i.e. SelSSH).*
3. *If X is aStar-σ-cd, then X is* $selSS_{cd}^*(\mathcal{O},\Gamma)$*.*

Observe that Corollary 2.9 (1) is precisely Proposition 1.3. Given that Ψ-spaces are absolutely strongly star Lindelöf , Corollary 2.8 (2) and Corollary 2.9 (2) improve Song's [28, Remark 2.5], and [29, Remark 2.6] (see also [29, Remark 2.4]): in Ψ-spaces the properties strongly star-Menger and absolutely strongly star-Menger are equivalent and the properties strongly star-Hurewicz and absolutely strongly star-Hurewicz are equivalent. Furthermore, they allow us to provide the following results that can be seen as partial analogous of Theorems 1.5 and 1.6 for absolutely strongly star Lindelöf spaces:

Proposition 2.10. *Let X be a topological space of the form* $Y \cup Z$, *where* $Y \cap Z = \emptyset$, *Z is a σ-compact subspace and Y is a closed discrete set. If X is absolutely strongly star-Lindelöf and* $|Y| < \mathfrak{d}$, *then X is selectively strongly star-Menger.*

Proposition 2.11. *Let X be a topological space of the form* $Y \cup Z$, *where* $Y \cap Z = \emptyset$, *Z is a σ-compact subspace and Y is a closed discrete set. If X is absolutely strongly star-Lindelöf and* $|Y| < \mathfrak{b}$, *then X is selectively strongly star-Hurewicz.*

Remark 2.12. For Ψ-spaces and for the Niemytzki plane, the three properties *SSM*, *aSSM* and *selSSM* are equivalent, and the three properties *SSH*, *aSSH* and *selSSH* are equivalent.[2)]

Figure 5 sum things up for the Menger, Rothberger and Hurewicz cases:

[1)] In private communication with M. Bonanzinga, she informed the authors that, together with F. Maesano in [7], they obtained in a direct way a proof of Corollary 2.8 (2).

[2)] M. Bonanzinga also informed the authors that, together with F. Maesano in [7], they obtained the equivalences of *SSM*, *aSSM* and *selSSM* for Ψ-spaces.

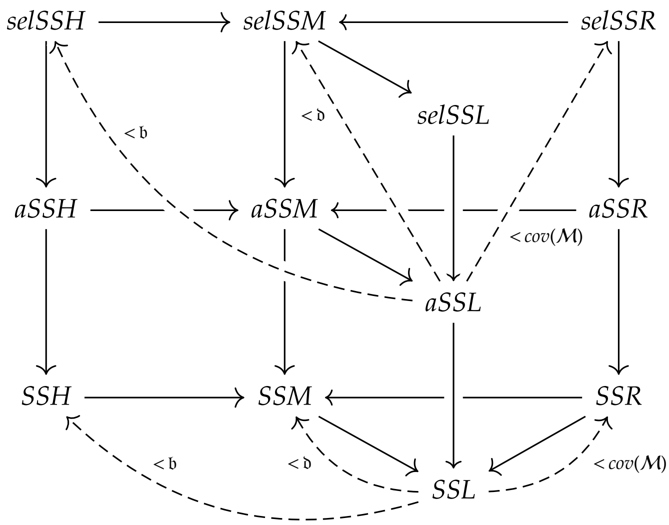


Figure 5: The dashed arrows hold if a space has size less than the respective small cardinal invariant

Remark 2.13. Recall that in T_1 spaces, finite sets are closed and discrete. Hence, if X a T_1 selectively strongly star-Menger space, then X is strongly selectively (a).

Remark 2.14. If X is selectively (a) then X is absolutely *Star*-σ-cd,

Remark 2.15. By Corollary 2.8 (2) and (3) and Remarks 2.13 and 2.14, for T_1 spaces Figure 6 holds.

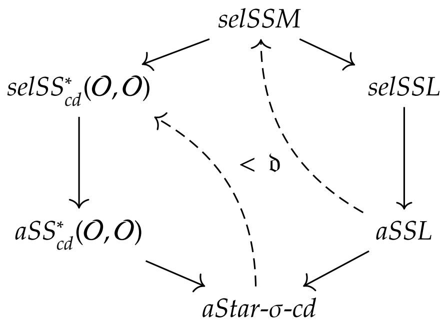


Figure 6: The dashed arrows hold if a space has size less than $\mathfrak{d}$

Other applications of Theorem 2.4 involve the following properties (see also [22]):
- [32] star-K-Menger.
- [33] σ-starcompact.
- [31] $\mathcal{L}$-starcompact.
- [30] $\mathcal{K}$-starcompact

Remark 2.16. Note that with the notation of definition 2.1 and the notion of *star*-$\mathbb{P}$ spaces, we have the following well-known implications:

star separable ⇔ star countable (SSL) ⇒ star compact ⇒ star σ-compact ⇒ star Lindelöf

Since the classes of σ-compact, Lindelöf and compact spaces are closed under finite unions, propositions 2.17, 2.18 and 2.19 below, hold.

Proposition 2.17. *Let X be a σ-starcompact space of size less than $\mathfrak{d}$. Then X is star-K-Menger.*

Proposition 2.18. *Let X be a $\mathcal{L}$-starcompact space of size less than $\mathfrak{d}$. Then X is star-$\mathcal{L}$-Menger.*

Proposition 2.19. *Let X be a K-starcompact space of size less than $\mathfrak{d}$. Then X is star-K-Menger.*

3. The extent of absolutely star selection principles

In the first part of this section we give a bound for the extent of absolutely star selection principles on separable spaces and, in the second part of the section, we provide some results regarding selectively (a)-type properties on small spaces with countable extent.

Sakai showed in [27, Theorem 2.1] that if X is a strongly star-Menger space and Y is a closed discrete subset of X, then $|Y| < cof(Fin(Y)^{\mathbb{N}})$ (in particular, $|Y| < \mathfrak{c}$). In addition, Caserta, Di Maio and Kočinac pointed out in [10, Proposition 2.3] that if X is a separable selectively-(a) space and $Y \subseteq X$ is closed and discrete, then $|Y| < \mathfrak{c}$ (see [26, Theorem 3.1] for a more general case). Since, absolutely strongly star-Menger spaces are selectively-(a), by this result we get that if X is a separable absolutely strongly star-Menger space, then all its closed and discrete subsets are of size less than the continuum. Corollary 3.3, shows that in fact, such subsets are of size less than the dominating number $\mathfrak{d}$.

We introduce the following definition that will be used in the proof of Theorem 3.2.

Definition 3.1. We say that a space X is *absolutely neighbourhood star-Menger* (*aNSM*) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers and each dense subset D of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of D such that for any open sets O_n with $F_n \subseteq O_n$, $n \in \omega$, $\{St(O_n, \mathcal{U}_n) : n \in \omega\}$ is an open cover of X.

Theorem 3.2. *Let X be a separable absolutely neighbourhood star-Menger space. If Y is a closed and discrete subset of X, then $|Y| < \mathfrak{d}$.*

Proof. Assume X is a separable absolutely neighbourhood star-Menger space. Then the set of isolated points is countable (otherwise it cannot be separable) and it has to be a subset of every dense set. Hence, without loss of generality, we can assume X has no isolated points. Now, if Y is a closed and discrete subset of X, given that X is separable and it has no isolated points, there is $E = \{e_n : n \in \omega\} \in [X \setminus Y]^{\omega}$ so that $cl_X(E) = X$.

Now, let us assume that $|Y| \geq \mathfrak{d}$. Let $\{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family in the strict sense (that is, for each $g \in \omega^\omega$ there is $\alpha < \mathfrak{d}$ such that $g \leq f_\alpha$). For each $\alpha < \mathfrak{d}$, choose distinct points $p_\alpha \in Y$ and let $P = \{p_\alpha : \alpha < \mathfrak{d}\}$. For each $\alpha < \mathfrak{d}$, each $n \in \omega$ and each $i \leq f_\alpha(n)$, define $O_n(p_\alpha)$ and $V_n^{i\alpha}$ open sets so that

1. $O_n(p_\alpha)$ is an open neighbourhood of p_α,
2. $O_n(p_\alpha) \cap Y = \{p_\alpha\}$,
3. $V_n^{i\alpha}$ is an open neighbourhood of e_i, and
4. $O_n(p_\alpha) \cap \bigcup_{i \leq f_\alpha(n)} V_n^{i\alpha} = \emptyset$.

For each $n \in \omega$ define $\mathcal{U}_n = \{O_n(p_\alpha) : \alpha < \mathfrak{d}\} \cup \{X \setminus P\}$. Observe that for each $n \in \omega$, $\mathcal{U}_n$ is an open cover of X. We will show that the sequence $\{\mathcal{U}_n : n \in \omega\}$ and the dense set E, witness X is not absolutely neighbourhood star-Menger. Let $\{F_n : n \in \omega\}$ be any sequence of finite subsets of E. For each $n < \omega$, let $g(n) = min\{m : F_n \subseteq \{e_0, e_1, \ldots, e_m\}\}$. Thus, there is $\alpha < \mathfrak{d}$ such that for each $n \in \omega$, $g(n) \leq f_\alpha(n)$. If for each $n \in \omega$ we let $W_n = \bigcup_{i \leq f_\alpha(n)} V_n^{i\alpha}$, then the sequence $(W_n : n \in \omega)$ satisfies that for each n, $F_n \subseteq W_n$ (here it is used that the family $\{f_\alpha : \alpha < \mathfrak{d}\}$ is dominating in the strict sense). It only remains to show that $p_\alpha \notin \bigcup\{St(W_n, \mathcal{U}_n) : n \in \omega\}$. Suppose the opposite, then there is $n \in \omega$ such that $p_\alpha \in St(W_n, \mathcal{U}_n)$. Since $O_n(p_\alpha)$ is the only element of $\mathcal{U}_n$ that contains p_α, then $O_n(p_\alpha) \cap W_n \neq \emptyset$. Then there is $i \leq f_\alpha(n)$ such that $O_n(p_\alpha) \cap V_n^i \neq \emptyset$, which is a contradiction. Hence, X is not absolutely neighbourhood star-Menger. □

Since every absolutely strongly star-Menger space is absolutely neighbourhood star-Menger, the following holds.

Corollary 3.3. *Let X be a separable absolutely strongly star-Menger space. If Y is a closed and discrete subset of X, then $|Y| < \mathfrak{d}$.*

The proof of Theorem 3.2 is a modification of the proof of [8, Proposition 2], where Bonanzinga and Matveev show that if an almost disjoint family $\mathcal{A}$ is so that the Mrówka-Isbell space $\Psi(\mathcal{A})$ is strongly star-Menger, then $|\mathcal{A}| < \mathfrak{d}$.

Problem 3.4. *Are aSSM and aNSM equivalent? If not, can we find a normal (Tychonoff) counterexample?*

Defining similarly the absolutely neighbourhood star-Hurewicz property, the Hurewicz case of Theorem 3.2 can also be proved:

Definition 3.5. We say that a space X is *absolutely neighbourhood star-Hurewicz* (*aNSH*) if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers and each dense subset D of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of D such that for any open sets O_n with $F_n \subseteq O_n$, $n \in \omega$, $\{St(O_n, \mathcal{U}_n) : n \in \omega\}$ is a γ-cover of X.

Theorem 3.6. *Let X be a separable absolutely neighbourhood star-Hurewicz space. If Y is a closed and discrete subset of X, then $|Y| < \mathfrak{b}$.*

Since every absolutely strongly star-Hurewicz space is absolutely neighbourhood star-Hurewicz, the following holds.

Corollary 3.7. *Let X be a separable absolutely strongly star-Hurewicz space. If Y is a closed and discrete subset of X, then $|Y| < \mathfrak{b}$.*

Similar to Problem 3.4 we have:

Problem 3.8. *Are aSSH and aNSH equivalent?*

Remark 3.9. Definitions 3.1 and 3.5 can be generalized to define the absolute and selective versions of neighbourhood star selection principles which were introduced in [16] and later studied in [5] (see Section 4 below).

Figure 7 shows the interplay between some selective versions of property (a) and the selective and absolute versions of the star selection principles.

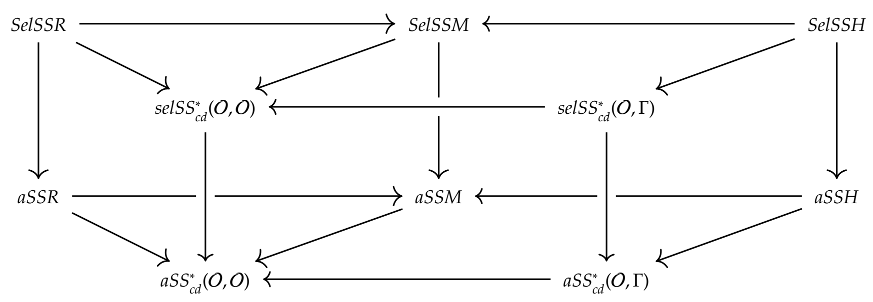


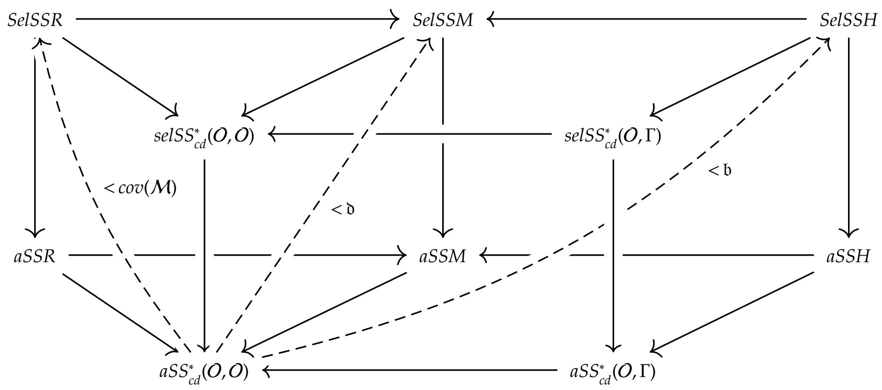
Figure 7: Natural relationships between selective versions of property (a) and the selective and absolute versions of the star selection principles

Lemma 3.10 says that spaces which are selectively (a) with countable extent are absolutely strongly star Lindelöf and therefore we can obtain other consequences of the general Theorems (showed in subsection 2) for spaces with some kind of selective version of property (a) and small cardinality.

Lemma 3.10. *If X is $aSS^*_{cd}(\mathcal{O}, \mathcal{O})$ with countable extent, then X is aSSL.*

Proof. Assume X is $aSS^*_{cd}(\mathcal{O},\mathcal{O})$ with countable extent. Let $\mathcal{U}$ be any open cover of X and let D be any dense subset of X. For each $n \in \omega$, let $\mathcal{U}_n = \mathcal{U}$, since X is $aSS^*_{cd}(\mathcal{O},\mathcal{O})$, for each $n \in \omega$, we can take $C_n \subseteq D$ closed discrete such that $\{St(C_n,\mathcal{U}_n) : n \in \omega\} = \{St(C_n,\mathcal{U}) : n \in \omega\}$ is a cover of X. Since X has countable extent, each C_n is countable and $X = St(\bigcup_{n\in\omega} C_n,\mathcal{U})$. Thus, X is absolutely strongly star-Lindelöf. □

Corollary 3.11. *The following diagram holds for any space X. The dashed arrows hold if X has countable extent and size less than the respective small cardinal invariant.*



Proof. Assume X is $aSS^*_{cd}(\mathcal{O},\mathcal{O})$ and has countable extent. By Lemma 3.10, we have that X is $aSSL$, then we apply Corollaries 2.7 (2), 2.8 (2) and 2.9 (2) to obtain the results. □

Remark 3.12. Corollary 3.11 give us nice characterizations on a small space X with countable extent:

1. If X has size less than $\mathfrak{d}$, then X is selectively (a) if and only if X is selectively strongly star-Menger.
2. If X has size less than $\mathfrak{b}$, then X is selectively (a) if and only if X is selectively strongly star-Hurewicz.
3. If X has size less than $cov(\mathcal{M})$, then X is selectively (a) if and only if X is selectively strongly star-Rothberger.

Remark 3.13. The hypothesis of the countable extent in Corollary 3.11 is necessary. Assuming $\omega_1 < \mathfrak{b}$ we have that the discrete space of size ω_1 is $selSS^*_{cd}(\mathcal{O},\Gamma)$ but is not $selSSM$.

4. Further study

The selective and absolute versions of the properties SSM, SSR and SSH were studied in this work. A different sort of these star selection principles which is closely related to the properties studied here is the neighbourhood version of the star selection principles. The definitions of these neighbourhood star selection principles were given in [16] (with different name) and studied in [5]. In this final section, using similar ideas as before, we introduce the absolute and selective versions of the neighbourhood star selection principles. Some further investigations on these kind of versions (which we have just begun to study) may be interesting.

We start by mentioning the definition of the absolute and selective versions of the neighbourhood star-Lindelöf property (the neighbourhood star-Lindelöf property was introduced in [5] and later studied by Song in [35] and [36]).

Definition 4.1. A space X is absolutely neighbourhood star-Lindelöf ($aNSL$) if for any open cover $\mathcal{U}$ of X and any dense subset D of X, there is a countable set $C \subseteq D$ such that for any open set O with $C \subseteq O$, $St(O,\mathcal{U}) = X$.

Definition 4.2. A space X is selectively neighbourhood star-Lindelöf (*selNSL*) if for any open cover $\mathcal{U}$ of X and any sequence $\{D_n : n \in \omega\}$ of dense sets of X, there are finite sets $F_n \subseteq D_n$, $n \in \omega$, such that for any open sets O_n in X with $F_n \subseteq O_n$, $n \in \omega$, $\{St(O_n, \mathcal{U}) : n \in \omega\}$ is an open cover of X.

Following the same notation and terminology of this article, we introduce general forms of two selection hypothesis which allows us to define the absolute and selective versions of the neighbourhood star selection properties.

Definition 4.3. Given a space X, the following selection hypothesis are defined:

absolutely $NSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ and each dense subset D of X, there exists a sequence $\{K_n : n \in \omega\} \subseteq \mathcal{K}$ with $K_n \subseteq D$, $n \in \omega$, such that for any open sets O_n with $K_n \subseteq O_n$, $n \in \omega$, $\{St(O_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

selectively $NSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n : n \in \omega\} \subseteq \mathcal{A}$ and each sequence $\{D_n : n \in \omega\}$ of dense sets of X, there exists a sequence $\{K_n : n \in \omega\} \subseteq \mathcal{K}$ with $K_n \subseteq D_n$, $n \in \omega$, such that for any open sets O_n with $K_n \subseteq O_n$, $n \in \omega$, $\{St(O_n, \mathcal{U}_n) : n \in \omega\} \in \mathcal{B}$.

For shortness, we write $aNSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ instead of absolutely $NSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ and, $selNSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ instead of selectively $NSS^*_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$.

Definition 4.4. By using the selection hypothesis given in 4.3, we introduce the following new properties:

- $aNSS^*_{fin}(\mathcal{O}, \mathcal{O})$ is named *absolutely neighbourhood star-Menger* (*aNSM*);
- $aNSS^*_1(\mathcal{O}, \mathcal{O})$ is named *absolutely neighbourhood star-Rothberger* (*aNSR*);
- $aNSS^*_{fin}(\mathcal{O}, \Gamma)$ is named *absolutely neighbourhood star-Hurewicz* (*aNSH*);
- $selNSS^*_{fin}(\mathcal{O}, \mathcal{O})$ is named *selectively neighbourhood star-Menger* (*selNSM*);
- $selNSS^*_1(\mathcal{O}, \mathcal{O})$ is named *selectively neighbourhood star-Rothberger* (*selNSR*);
- $selNSS^*_{fin}(\mathcal{O}, \Gamma)$ is named *selectively neighbourhood star-Hurewicz* (*selNSH*).

Obvious implications among these absolute and selective versions are shown in Figure 8.

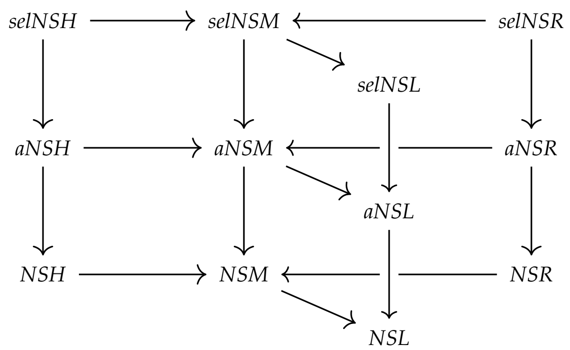


Figure 8: Absolute and selective versions of neighbourhood star selection principles

By including the selective and absolute versions of the strongly star principles, we get a general diagram in Figure 9 that involves all selective and absolute versions considered so far.

Notice that the selection principles absolutely $NSS^*_{cd}(\mathcal{A}, \mathcal{B})$ and selectively $NSS^*_{cd}(\mathcal{A}, \mathcal{B})$ are new selective versions of property (a) which study shall be interesting.

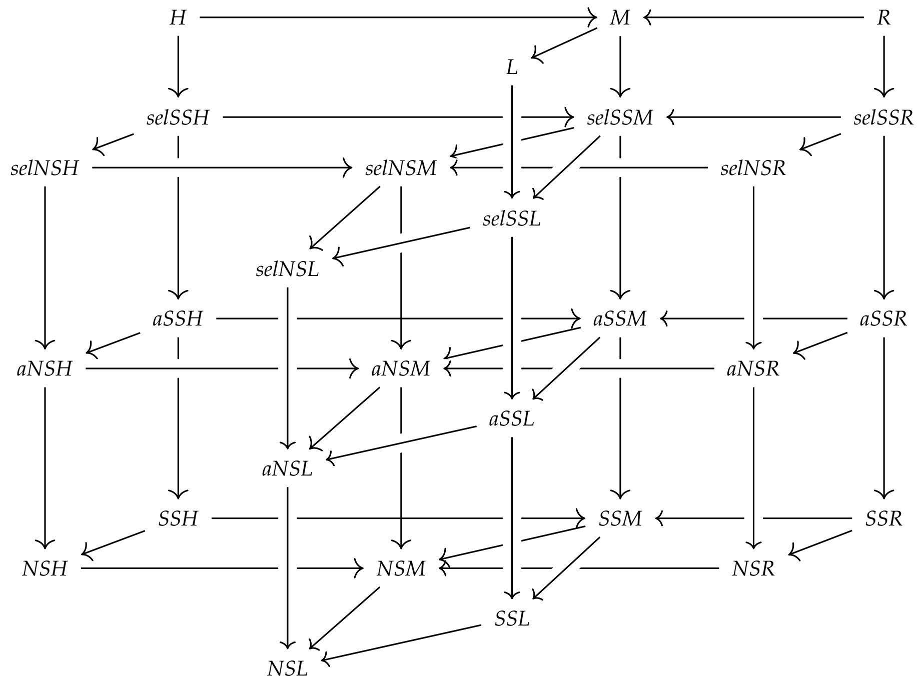


Figure 9: General diagram

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