Filomat 36:14 (2022), 4869–4876
https://doi.org/10.2298/FIL2214869W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Weak Stability of Ishikawa Iterations for Strongly Demicontractive Mappings in Hilbert Spaces

Chao Wang[a], **Honglei Fan**[a]

[a] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China.*

Abstract. In this article, we establish a weak stability theorem for Ishikawa iterations in Hilbert spaces. Moreover, a strongly demicontractive mapping is presented to illustrate that the Ishikawa iteration of T is weakly T-stable but not T-stable. Our results are new and extend several known results.

1. Introduction and Preliminaries

Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ be, respectively, the norm and inner product of a real Hilbert space H. C is a nonempty closed convex subset of H, $T : C \to C$ is a self mapping and $Fix(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T. A sequence $\{x_n\}$ which is generated by the Ishikawa iteration[5] of T if for arbitrary $x_0 \in C$,

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n \end{cases} \tag{1}$$

for all $n \geq 0$, where $\{a_n\}, \{b_n\} \subset [0, 1]$. In particular, if $b_n = 0$, then (1) is called the Mann iteration[8].

In general, if a small modification to the initial point will have a small impact (compared to the actual value) on the calculated value of a fixed point, then the iterative process of the fixed point is said to be numerically stable. The $T-$stability (see Definition 1.1) of several iterations for contractive mappings has been researched by many authors. Harder and Hicks[3] established some $T-$stability results for Picard iterations and Mann iterations with respect to some generalized contractions. Rhoades[12,13] obtained some generalized theorems for other classes of contractive mappings in normed linear spaces. Osilike[11] extended the results of [13] to complete metric spaces. And then, Osilike[10] estabilished some stability results for Ishikawa iterations in Banach spaces.

It is obvious that any $T-$stable iteration is weakly $T-$stable, but the converse statement is not necessarily true (see Definition 1.1 and Definition 1.2). Therefore, if an iteration procedure is not $T-$stable, then it is of great theoretical significance to study weak $T-$stability of the iteration procedure. Zhou et al.[17] obtained weak $T-$stability of the Ishikawa iteration (1) for Lipschitzian and $\phi-$hemicontractive mappings, but it needs a strictly condition:

$$\liminf_{n\to\infty} \frac{\phi(t)}{t} > 0.$$

2020 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10.
Keywords. Ishikawa iteration, strongly demicontraction mappings, weakly $T-$stable, $T-$stable
Received: 12 August 2021; Accepted: 20 December 2021
Communicated by Vasile Berinde
Research supported by the National Natural Science Foundation of China(11126290)
Email addresses: wangchaosx@nuist.edu.cn (Chao Wang), 20191215010@nuist.edu.cn (Honglei Fan)

Without the above condition, Huang[4] proved weak $T-$stability of the Ishikawa iteration (1) for $\phi-$hemi-contractive mappings. Timis and Berinde[15] studied the problem of weak stability of iteration prodedures for the common fixed points of some contractive type mappings, and also gave some numerical examples. Furthermore, Timis [14] considered a more weaker concept of stability which is called the weak ω^2-stable and studied some results of Picard iterations for the mappings which satisfy some contraction conditions.

Very recently, L. Maruster and St. Maruster[9] defined the concept of strongly demicontractive mappings (see Definition 1.3) and provided a $T-$ stability theorem of the Mann iteration. Wang et al. [16] obtained some formulas of error estimation of the Ishikawa iteration (1) and some $T-$ stability theorems are also proved. To the best of our knowledge, the weak stability of some iterations for strongly demicontractive mappings has not been studied. Moreover, Berinde [1] proposed an open problem: "It remains the task to identify, amongst the classes of operators for which a certain iteration is not T-stable, the ones for which the iteration is weakly T-stable." (see [1], page 165).

In this paper, motivated by [1,2,4,6-7,9,16-17], we consider weak $T-$stability of the Ishikawa iteration (1) for strongly demicontractive mappings. And an example is given to illustrate that a weakly $T-$stable iteration procedure is not always $T-$stable. Our results improve and generalize the corresponding theorems in [2,4,9,16-17] and the problem raised by [1] is partly solved.

Next, we recall some known definitions and results.

Definition 1.1. [1] *Let (X, d) be a metric space and $T : X \to X$ be a mapping. For arbitrary $x_0 \in X$, the sequence $\{x_n\}$ produced by*

$$x_{n+1} = f(T, a_n, x_n) \tag{2}$$

for all $n \geq 0$. Assume that $\{x_n\}$ converges to a fixed point p of T. For any sequence $\{y_n\} \subset X$, set

$$\varepsilon_n = d(y_{n+1}, f(T, a_n, y_n)) \tag{3}$$

for all $n \geq 0$. We say that the iteration procedure (2) is $T-$stable (or stable with respect to T) if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Longleftrightarrow \lim_{n\to\infty} y_n = p.$$

Definition 1.2. [17] *Let (X, d) be a metric space and $T : X \to X$ be a mapping. For arbitrary $x_0 \in X$, the sequence $\{x_n\}$ is defined by (2). Assume that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be any sequence in X and $\{\varepsilon_n\}$ be defined by (3) with $\varepsilon_n = \varepsilon'_n + \varepsilon''_n$. Suppose $\sum_{n=0}^{\infty} \varepsilon'_n < \infty$ and $\varepsilon''_n = o(a_n)$ implies that*

$$\lim_{n\to\infty} y_n = p.$$

Then we say that the iteration procedure (2) is weakly $T-$stable (or weakly stable with respect to T).

Definition 1.3. [9] *The mapping $T : C \to C$ is said to be strongly demicontractive if $Fix(T) \neq \emptyset$ and*

$$\|Tx - p\|^2 \leq \alpha\|x - p\|^2 + K\|Tx - x\|^2 \tag{4}$$

for all $x \in C$, where $p \in Fix(T)$, $\alpha \in (0, 1)$ and $K \geq 0$.

Remark 1.4. *It is obviously that if T is a strongly demicontractive mapping, then $Fix(T)$ is a singleton. And (4) is equivalent to the following inequality:*

$$\langle x - Tx, x - p\rangle \geq \frac{1-\alpha}{2}\|x - p\|^2 + \frac{1-K}{2}\|Tx - x\|^2. \tag{5}$$

Lemma 1.5. [1] *Let $\{\alpha_n\}$, $\{\beta_n\}$ be nonnegative real sequences satisfying*

$$\alpha_{n+1} \leq \theta\alpha_n + \beta_n$$

for all $n \geq 0$, where $\theta \in [0, 1)$. If $\lim_{n\to\infty} \beta_n = 0$, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.6. [1] *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\lambda_n\}$ *be nonnegative real sequences such that*

$$\alpha_{n+1} \leq (1-\lambda_n)\alpha_n + \beta_n\lambda_n + \gamma_n,$$

where

$$\lambda_n \in [0,1],\ \sum_{n=0}^{\infty}\lambda_n = \infty,\ \lim_{n\to\infty}\beta_n = 0 \text{ and } \sum_{n=0}^{\infty}\gamma_n < \infty.$$

Then

$$\lim_{n\to\infty}\alpha_n = 0.$$

2. Main results

In order to give a weak T−stability theorem of the Ishikawa iteration (1) for strongly demicontractive mappings, we first consider the following two lemmas.

Lemma 2.1. *Let* $T : C \to C$ *be L-Lipschitzian (i.e., for any* $x,\ y \in C$, *there exits* $L > 0$, *such that* $\|Tx - Ty\| \leq L\|x - y\|$ *) and strongly demicontractive with* $\alpha \in (0,1)$ *and* $K \geq 0$. *Assume that* $p \in Fix(T)$ *and* $\{x_n\}$ *is the sequence generated by the Ishikawa iteration (1).*
(i) If $K \leq 1$, *then*

$$\|x_{n+1} - p\| \leq \frac{1 + a_nLQ - a_n}{1 - \frac{1+\alpha}{2}a_n}\|x_n - p\|$$

for all $n \geq 0$, *where* $Q = (a_n + b_n)(1 + L) + a_nb_nL(L-1)$.
(ii) If $K > 1$ *and* $(K-1)(1+L)^2 + \alpha < 1$, *then*

$$\|x_{n+1} - p\| \leq \frac{1 + a_nLQ - a_n}{1 - Ma_n}\|x_n - p\|$$

for all $n \geq 0$, *where* $M = \frac{1+(K-1)(1+L)^2+\alpha}{2}$.

Proof. Since

$$x_{n+1} - p = a_n(Tx_{n+1} - x_{n+1}) + a_n(Ty_n - Tx_{n+1}) + (1-a_n)(x_n - p) + a_n(x_{n+1} - p).$$

We obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \langle x_{n+1} - p, x_{n+1} - p\rangle \\ &\leq a_n\langle Tx_{n+1} - x_{n+1}, x_{n+1} - p\rangle + a_nL\|y_n - x_{n+1}\| \cdot \|x_{n+1} - p\| + (1-a_n)\|x_n - p\| \cdot \|x_{n+1} - p\| \\ &\quad + a_n\|x_{n+1} - p\|^2, \end{aligned} \tag{6}$$

and

$$\begin{aligned}\|y_n - x_{n+1}\| &= \|[(1-b_n)x_n + b_nTx_n] - [(1-a_n)x_n + a_nTy_n]\| \\ &= \|b_n(Tx_n - x_n) + a_n(x_n - Ty_n)\| \\ &\leq b_n(1+L)\|x_n - p\| + a_n(\|x_n - p\| + L\|y_n - p\|) \\ &\leq [b_n(1+L) + a_n]\|x_n - p\| + a_nL\|(1-b_n)x_n + b_nTx_n - p\| \\ &\leq [a_n + b_n(1+L) + a_nL(1 - b_n + b_nL)]\|x_n - p\| \\ &= Q\|x_n - p\|, \end{aligned} \tag{7}$$

where $Q = a_n + b_n(1+L) + a_nL(1-b_n+b_nL) = (a_n+b_n)(1+L) + a_nb_nL(L-1)$. From (5), we have

$$\langle Tx_{n+1} - x_{n+1}, x_{n+1} - p\rangle \le \frac{K-1}{2}\|Tx_{n+1} - x_{n+1}\|^2 - \frac{1-\alpha}{2}\|x_{n+1} - p\|^2. \tag{8}$$

(i) Let $K \le 1$. From (8), we have

$$\langle Tx_{n+1} - x_{n+1}, x_{n+1} - p\rangle \le -\frac{1-\alpha}{2}\|x_{n+1} - p\|^2. \tag{9}$$

Plugging (7) and (9) into (6) results in

$$\begin{aligned}\|x_{n+1} - p\|^2 \le & -\frac{a_n(1-\alpha)}{2}\|x_{n+1} - p\|^2 + a_nLQ\|x_n - p\| \cdot \|x_{n+1} - p\| + (1-a_n)\|x_n - p\| \cdot \|x_{n+1} - p\| \\ & + a_n\|x_{n+1} - p\|^2.\end{aligned}$$

Without loss of generality, we may assume that $\|x_{n+1} - p\| > 0$ for all $n \ge 0$. Cancelling $\|x_{n+1} - p\| > 0$ on the both sides of the above inequality, we obtain

$$\|x_{n+1} - p\| \le [-\frac{a_n(1-\alpha)}{2} + a_n]\|x_{n+1} - p\| + [a_nLQ + (1-a_n)]\|x_n - p\|,$$

which implies

$$\|x_{n+1} - p\| \le \frac{1 + a_nLQ - a_n}{1 - \frac{1+\alpha}{2}a_n}\|x_n - p\|.$$

(ii) Let $K > 1$. Since $\|Tx_{n+1} - x_{n+1}\| \le (1+L)\|x_{n+1} - p\|$, from (8), we have

$$\langle Tx_{n+1} - x_{n+1}, x_{n+1} - p\rangle \le \frac{(K-1)(1+L)^2 - (1-\alpha)}{2}\|x_{n+1} - p\|^2. \tag{10}$$

Inserting (7) and (10) into (6) leads to

$$\begin{aligned}\|x_{n+1} - p\|^2 \le & \frac{a_n(K-1)(1+L)^2 - a_n(1-\alpha)}{2}\|x_{n+1} - p\|^2 + a_nLQ\|x_n - p\| \cdot \|x_{n+1} - p\| \\ & + (1-a_n)\|x_n - p\| \cdot \|x_{n+1} - p\| + a_n\|x_{n+1} - p\|^2.\end{aligned}$$

Without loss of generality, we may assume that $\|x_{n+1} - p\| > 0$ for all $n \ge 0$. Similarly, we have

$$\|x_{n+1} - p\| \le \frac{1 + (K-1)(1+L)^2 + \alpha}{2} \cdot a_n\|x_{n+1} - p\| + (1 + a_nLQ - a_n)\|x_n - p\|,$$

which implies

$$\|x_{n+1} - p\| \le \frac{1 + a_nLQ - a_n}{1 - Ma_n}\|x_n - p\|,$$

where $M = \frac{1+(K-1)(1+L)^2+\alpha}{2}$. □

Remark 2.2. *If $K \le 1$, then we do not need the condition "$(K-1)(1+L)^2 + \alpha < 1$".*

Remark 2.3. *Let*

$$\max\{\frac{1 + a_nLQ - a_n}{1 - \frac{1+\alpha}{2}a_n}, \frac{1 + a_nLQ - a_n}{1 - Ma_n}\} = N.$$

For any $K \ge 0$, if $(K-1)(1+L)^2 + \alpha < 1$, then we can always get that

$$\|x_{n+1} - p\| \le N\|x_n - p\|.$$

Lemma 2.4. *Suppose that all the conditions of Lemma 2.1 hold. Assume further that the sequences $\{a_n\}$ and $\{b_n\}$ of the Ishikawa iteration (1) satisfying*

$$\lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty} b_n = 0. \tag{11}$$

Then the sequence $\{\|x_n - p\|\}$ is bounded.

Proof. From Lemma 2.1, we need to consider the following two cases.

Case I: Suppose $K \leq 1$. Let $r = \|x_0 - p\|$. Now we will prove by induction that $\|x_n - p\| \leq r$ for all $n \geq 0$. Obviously, $\|x_0 - p\| \leq r$. Assume that $\|x_n - p\| \leq r$ for some $n \geq 1$, we will show that $\|x_{n+1} - p\| \leq r$. Suppose not, that is $\|x_{n+1} - p\| > r$. From the condition (11), there exits $N_1 \in \mathbb{N}$ such that

$$LQ = L[(a_n + b_n)(1 + L) + a_n b_n L(L - 1)] < \frac{1-\alpha}{2}$$

for all $n > N_1$. From Lemma 2.1(i), we have

$$r < \|x_{n+1} - p\| \leq \frac{1 + a_n LQ - a_n}{1 - \frac{1+\alpha}{2} a_n} \|x_n - p\| \leq r$$

for all $n > N_1$, which yields a contradiction. Therefore, $\{\|x_n - p\|\}$ is bounded.

Case II: Suppose $K > 1$. We also let $r = \|x_0 - p\|$. Similar to the proof of Case I, we can also get $\|x_n - p\| \leq r$ for all $n \geq 0$. Therefore, $\{\|x_n - p\|\}$ is bounded. □

Now, we give the main theorem of this paper.

Theorem 2.5. *Suppose that all the conditions of Lemma 2.4 hold and $\{x_n\}$ converges to a fixed point p of T. Assume that $\sum_{n=0}^{\infty} a_n = \infty$. Let $\{z_n\}$ be any sequence in C and $\{\varepsilon_n\}$ defined by*

$$\varepsilon_n = \|z_{n+1} - (1 - a_n)z_n - a_n T w_n\|,$$
$$w_n = (1 - b_n)z_n + b_n T z_n,$$

where $\varepsilon_n = \varepsilon_n' + \varepsilon_n''$, $\sum_{n=0}^{\infty} \varepsilon_n' < \infty$, and $\varepsilon_n'' = o(a_n)$. Then the sequence $\{x_n\}$ is weakly $T-$stable.

Proof. Let $z_{n+1} = (1 - a_n)z_n + a_n T w_n + \delta_n$. Then $\delta_n = z_{n+1} - (1 - a_n)z_n - a_n T w_n$. So,

$$\|\delta_n\| = \varepsilon_n = \varepsilon_n' + \varepsilon_n''.$$

Since

$$z_{n+1} - p = a_n(Tz_{n+1} - z_{n+1}) + a_n(Tw_n - Tz_{n+1}) + (1 - a_n)(z_n - p) + a_n(z_{n+1} - p) + \delta_n,$$

we have

$$\begin{aligned} \|z_{n+1} - p\|^2 \leq\ & a_n\langle Tz_{n+1} - z_{n+1}, z_{n+1} - p\rangle + a_n L\|w_n - z_{n+1}\| \cdot \|z_{n+1} - p\| + (1 - a_n)\|z_n - p\| \cdot \|z_{n+1} - p\| \\ & + a_n\|z_{n+1} - p\|^2 + \|\delta_n\| \cdot \|z_{n+1} - p\|, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|w_n - z_{n+1}\| &= \|[(1 - b_n)z_n + b_n T z_n] - [(1 - a_n)z_n + a_n T w_n + \delta_n]\| \\ &= \|b_n(Tz_n - z_n) + a_n(z_n - Tw_n) - \delta_n\| \\ &\leq b_n\|Tz_n - z_n\| + a_n\|z_n - Tw_n\| + \|\delta_n\|. \end{aligned} \tag{13}$$

From Lemma 2.4, we know that $\{\|z_n - p\|\}$ and $\{\|w_n - p\|\}$ are bounded. Notice that

$$\|z_n - Tz_n\| \leq \|z_n - p\| + \|Tz_n - p\| \leq (1 + L)\|z_n - p\|,$$
$$\|z_n - Tw_n\| \leq \|z_n - p\| + \|Tw_n - p\| \leq \|z_n - p\| + L\|w_n - p\|,$$

which imply that $\{\|z_n - Tz_n\|\}$ and $\{\|z_n - Tw_n\|\}$ are also bounded. Let

$$P = \max\{\ \sup\{\|z_n - p\|\},\ \sup\{\|w_n - p\|\},\ \sup\{\|z_n - Tz_n\|\},\ \sup\{\|z_n - Tw_n\|\}\ \}.$$

Then $P < \infty$. By (13), we have

$$\|w_n - z_{n+1}\| \leq (a_n + b_n)P + \|\delta_n\|. \tag{14}$$

From (5), we obtain

$$\langle Tz_{n+1} - z_{n+1}, z_{n+1} - p\rangle \leq \frac{K-1}{2}\|Tz_{n+1} - z_{n+1}\|^2 - \frac{1-\alpha}{2}\|z_{n+1} - p\|^2. \tag{15}$$

Now, we consider the following two cases:

Case I: Suppose $K \leq 1$. In this case, (15) becomes

$$\langle Tz_{n+1} - z_{n+1}, z_{n+1} - p\rangle \leq -\frac{1-\alpha}{2}\|z_{n+1} - p\|^2. \tag{16}$$

Inserting (14) and (16) into (12), we have

$$\begin{aligned}\|z_{n+1} - p\|^2 \leq{}& [-\frac{a_n(1-\alpha)}{2} + a_n] \cdot \|z_{n+1} - p\|^2 + (1-a_n)\|z_n - p\| \cdot \|z_{n+1} - p\| + [a_n(a_n + b_n)LP \\ &+ \|\delta_n\|] \cdot \|z_{n+1} - p\|.\end{aligned}$$

Without loss of generality, assume that $\|z_{n+1} - p\| > 0$ for all $n \geq 0$, we obtain

$$\|z_{n+1} - p\| \leq \frac{a_n(1+\alpha)}{2}\|z_{n+1} - p\| + (1-a_n)\|z_n - p\| + a_n(a_n + b_n)LP + \|\delta_n\|.$$

and it follows that

$$\begin{aligned}\|z_{n+1} - p\| &\leq \frac{(1-a_n)\|z_n - p\| + a_n(a_n + b_n)LP + \varepsilon_n' + \varepsilon_n''}{1 - \frac{a_n(1+\alpha)}{2}} \\ &\leq (1 - \frac{1 - \frac{1+\alpha}{2}}{1 - \frac{1+\alpha}{2}a_n}a_n)\|z_n - p\| + \frac{a_n(a_n + b_n)LP + \varepsilon_n' + \varepsilon_n''}{1 - \frac{1+\alpha}{2}} \\ &\leq [1 - (1 - \frac{1+\alpha}{2})a_n]\|z_n - p\| + \frac{2}{1-\alpha}[a_n(a_n + b_n)LP + \varepsilon_n' + \varepsilon_n''].\end{aligned} \tag{17}$$

Applying Lemma 1.6 to (17) yields that $\lim_{n\to\infty} z_n = p$.

Case II: Suppose $K > 1$. Since $\|Tz_{n+1} - z_{n+1}\| \leq (1+L)\|z_{n+1} - p\|$, from (15), we have

$$\langle Tz_{n+1} - z_{n+1}, z_{n+1} - p\rangle \leq \frac{(K-1)(1+L)^2 - (1-\alpha)}{2} \cdot a_n\|z_{n+1} - p\|^2. \tag{18}$$

Plugging (18) and (14) into (12), we have

$$\begin{aligned}\|z_{n+1} - p\|^2 \leq{}& \frac{(K-1)(1+L)^2 + 1 + \alpha}{2} \cdot a_n\|z_{n+1} - p\|^2 + (1-a_n)\|z_n - p\| \cdot \|z_{n+1} - p\| + [a_n(a_n + b_n)LP \\ &+ \|\delta_n\|] \cdot \|z_{n+1} - p\|.\end{aligned}$$

Assume that $\|z_{n+1} - p\| > 0$ for all $n \geq 0$, we have

$$\|z_{n+1} - p\| \leq [\frac{(K-1)(1+L)^2 + 1 + \alpha}{2}] \cdot a_n\|z_{n+1} - p\| + (1-a_n)\|z_n - p\| + [a_n(a_n + b_n)LP + \varepsilon_n' + \varepsilon_n''],$$

which implies

$$\|z_{n+1} - p\| \leq \frac{(1-a_n)\|z_n - p\| + [a_n(a_n + b_n)LP + \varepsilon_n' + \varepsilon_n'']}{1 - \frac{(K-1)(1+L)^2+1+\alpha}{2}a_n}.$$

Let $\beta = (K-1)(1+L)^2 + \alpha$. Since $(K-1)(1+L)^2 + \alpha < 1$ and $K > 1$, we know that $\beta \in (0,1)$. Similar to the proof of Case I, by Lemma 1.6, we have $\lim_{n\to\infty} z_n = p$. Therefore, the sequence $\{x_n\}$ is weakly $T-$stable. □

Example 2.6. *Let $C = [-\frac{3}{4}, \frac{3}{4}]$ and define $T : C \to C$ by*

$$Tx = 2x^3 - \frac{1}{2}x.$$

This function is L-Lipschitzian with $L = 2.875$ and strongly demicontractive with $\alpha = 0.243, K = 1.050$. T has a unique fixed point $p = 0$. Note that

$$(K-1)(1+L)^2 + \alpha = 0.994 < 1.$$

Set $a_0 = b_0 = 0$ and $a_n = b_n = \frac{1}{n}$ for all $n \geq 1$. Then

$$\lim_{n\to\infty} a_n = 0, \ \lim_{n\to\infty} b_n = 0 \ \text{ and } \ \sum_{n=0}^{\infty} a_n = \infty.$$

By the Ishikawa iteration (1), we have

$$y_n = (1 - \frac{1}{n})x_n + \frac{1}{n}(2x_n^3 - \frac{1}{2}x_n) = \frac{2}{n}x_n^3 + \frac{2n-3}{2n}x_n$$

and

$$\begin{aligned} x_{n+1} &= (1 - \frac{1}{n})x_n + \frac{1}{n}(2y_n^3 - \frac{1}{2}y_n) \\ &= (1 - \frac{1}{n})x_n + \frac{1}{n}[2(\frac{2}{n}x_n^3 + \frac{2n-3}{2n}x_n)^3 - \frac{1}{2}(\frac{2}{n}x_n^3 + \frac{2n-3}{2n}x_n)] \\ &= \frac{2}{n}(\frac{2}{n}x_n^3 + \frac{2n-3}{2n}x_n)^3 - \frac{1}{2n}(\frac{2}{n}x_n^3 + \frac{2n-3}{2n}x_n) + (1 - \frac{1}{n})x_n \end{aligned}$$

for all $n \geq 1$. It follows from the Matlab software that $\lim_{n\to\infty} x_n = 0$. Thus, the sequence $\{x_n\}$ is weakly $T-$stable by Theorem 2.5. However, take a sequence $\{z_n\}$: $z_n = \frac{1}{2}$, then

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|z_{n+1} - f(T, a_n, z_n)\| = 0.$$

Obviously, $\lim_{n\to\infty} z_n \neq 0$. From Definition 1.1, the sequence $\{x_n\}$ is not $T-$stable.

Remark 2.7. *In the above example, the Ishikawa iteration (1) is weakly $T-$stable but not $T-$stable in the framework of a strongly demicontractive mapping. Therefore, we partially solve the open problem in [1].*

References

[1] V. Berinde, Iterative Approximation of Fixed Points, Springer Verlag, Berlin Heidelberg, 2007.

[2] F. Gursoy, A. R. Khan, M. Erturk, V. Karakaya, Weak ω^2-stability and data dependence of Mann iteration method in Hilbert spaces, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 1 (2017) 11–20.

[3] A. M. Harder, T. L. Hicks, Stability results for fixed point iteration procedures, Math Japonica 33 (1988) 693–706.

[4] Z. Y. Huang, Weak stability of Mann and Ishikawa iterations with errors for $\phi-$hemicontractive operators, Applied Mathematics Letters 20 (2007) 470–475.

[5] S. Ishikawa, Fixed points by a new iteration method, Proceeding of the American Mathematical Society 1 (1974) 147–150.

[6] W. J. Liu, W. F. Zhao, On the stability of a laminated beam with structural damping and Gurti-Pipkin thermal law, Nonlinear Analysis-Modelling and Control 26 (2021) 396–418.
[7] Y. Luan, W. J. Liu, Well-posedness and asymptotic stability to a laminated beam in thermoelasticity of type III, Mathematical Methods in the Applied Sciences 43 (2020) 3148–3166.
[8] W. R. Mann, Mean value methods in iteration, Proceeding of the American Mathematical Society 4 (1953) 506–510.
[9] L. Maruster, St. Maruster, On the error estimation and T-stability of the Mann iteration, Journal of Computational and Applied Mathematics 276 (2015) 110–116.
[10] M. O. Osilike, Stable iteration procedures for nonlinear pseudocontractive and accretive operators in arbitrary Banach spaces, Indian Journal of Pure and Applied Mathematics 28 (1997) 1017–1029.
[11] M. O. Osilike, Some stability results for fixed point iteration procedures, Journal of the Nigerian Mathematical Society 14/15 (1995) 17–29.
[12] B. E. Rhoades, Some fixed point iteration procedures, International Journal of Mathematics and Mathematical Sciences 1 (1991) 1–16.
[13] B. E. Rhoades, Fixed point theorems and stability results for fixed point iteration procedures, Indian Journal of Pure and Applied Mathematics 24 (1993) 691–703.
[14] I. Timis, On the weak stability of Picard iteration for some contractive type mappings, Annals of the University of Craiova Mathematics and Computer Science Series 2 (2010) 106–114.
[15] I. Timis, V. Berinde. Weak stability of iterative procedures for some coincidence theorems, Creative Mathematics and Informatics 1 (2010) 85–95.
[16] C. Wang, X. L. Li, P. K. Huang, On the error estimation and T-stability of the Ishikawa iteration for strongly demicontractive mappings, Journal of Inequalities and Applications (2019) 75–86.
[17] H. Y. Zhou, S. S. Chang, Y. J. Cho, Weak stability of the Ishikawa iteration procedures for $\phi-$ hemicontractive and accretive operators, Applied Mathematics Letters 14 (2001) 949–954.