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Existence and Iteration for Fourth Order p-Laplacian Beam Equations with Integral Boundary Conditions

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Abstract. In this paper, we consider the fourth order p-Laplacian beam equation $(\Phi_p(u''(t)))'' = f(t, u(t), u''(t))$ with integral boundary conditions $u''(0) = u''(1) = 0, u(0)-\alpha u'(0) = \int_0^1 g_1(s)u(s)ds, u(1)+\beta u'(1) = \int_0^1 g_2(s)u(s)ds$. By using the contraction mapping principle, we establish the existence and uniqueness of solutions for the problem. The monotony of iterations is also considered. At last, some examples are presented to illustrate the main results.

1. Introduction

In this paper, we consider the fourth order p-Laplacian integral boundary value problem

$$\begin{cases} (\Phi_p(u''(t)))'' = f(t, u(t), u''(t)), \quad t \in (0, 1), \\ u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, \\ u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds, \quad u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where $\Phi_p(u) = |u|^{p-2}u$ is called one-dimensional p-Laplacian operator, constants $\alpha, \beta \geq 0$; $f \in C([0, 1]\times\mathbb{R}^2, \mathbb{R})$, $g_i \in C([0, 1], [0, 1))$ for $i = 1, 2$. Here $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\Phi_p^{-1}(u) = \Phi_q(u) = |u|^{q-2}u$.

In beam theory [1], this problem can describe the small deformation of elastic beam. Usually both ends are simply supported, or one end is simply supported and the other end is clamped by sliding clamps. Also vanishing moments at the ends of the attached beam motivate the boundary conditions (see [2] for more details). The special case of equation (1) with $p = 2$

$$u^{(4)}(t) = f(t, u(t), u''(t))$$

has been studied by several authors (see [3]-[7] and the references therein). For example, in 1986, Aftabizadeh [3] used the Schauder's fixed point theorem to obtain existence and uniqueness results to the problem mentioned above under the restriction that f is a bounded function. In 1997, by the monotone

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method in the presence of lower and upper solutions, Ma et al. [4] constructed two monotone sequences of functions converging to the extremal solutions of the problem under some monotone conditions of f. In 2019, Wei et al. [7] obtained the existence and uniqueness of solution to the problem by using the contraction mapping principle.

However, we know that $p = 2$ is the critical value of p-Laplacian operator, and the value of p directly affects its monotonicity. When $p \neq 2$, the operator $(\Phi_p(u''))''$ is nonlinear, the maximum principle and Fredholm alternative can not be applied. Meanwhile, the construction of upper and lower solutions to boundary value problems is complex and difficult. We can see some results on the existence of p-Laplacian boundary value problems, see [8]-[13], [17] and the references therein. In 2007, Zhang and Liu [8] used the method of upper and lower solutions and fixed point theorem to establish their main results on positive solutions of the following fourth-order p-Laplacian four-point boundary value problem

$$\begin{cases}(\Phi_p(u''(t)))'' = f(t, u(t)), \quad t \in (0, 1),\\ u(0) = u(1) - au(\xi) = u''(0) = u''(1) - bu''(\eta) = 0.\end{cases}$$

In 2011, Xu and Yang [9] studied the existence, multiplicity and uniqueness of positive solutions for the two point fourth order p-Laplacian boundary value problem. Based on a priori estimates achieved by utilizing properties of concave functions, they used the fixed point index theory to establish their main results. Recently, in 2019, by using a novel efficient iteration method, Bai et al. [13] obtained the existence and uniqueness of solution for the following boundary value problem

$$\begin{cases}(\Phi_p(u''(t)))'' = f(t, u(t), u''(t)), \quad t \in (0, 1),\\ u(0) = 0, u''(0) = 0,\\ u(1) = au(\xi), \Phi_p(u''(1)) = b\Phi_p(u''(\varsigma)),\end{cases}$$

where they only considered the case $1 < p \leq 2$.

Integral boundary conditions arise in thermal conduction problems [14], semiconductor problems [15], and hydrodynamic problems [16]. Integral boundary problems include two-point, three-point and multi-point boundary value problems as special cases. Zhang et al. [17] investigated a class of fourth-order p-Laplacian differential equations with integral boundary conditions of the following form

$$\begin{cases}(\Phi_p(u''(t)))'' = w(t)f(t, u(t)), \quad t \in (0, 1),\\ u(0) = u(1) = \int_0^1 g_1(s)u(s)ds,\\ \Phi_p(u''(0)) = \Phi_p(u''(1)) = \int_0^1 g_2(s)u(s)ds.\end{cases}$$

Their arguments were based upon a specially constructed cone and the fixed point theory for cones. They obtained the existence and nonexistence of symmetric positive solutions, but did not use numerical simulation to estimate the approximate value of the real solutions.

Motivated by the mentioned excellent works, in this paper, by using the contraction mapping principle, we obtain the existence and uniqueness of solutions of the fourth order integral boundary value problem with p-Laplacian operator. Different from reference [13], we consider the more general integral boundary conditions and both cases $1 < p \leq 2$ and $p > 2$. Also, we give an iterative method for solving the problem and prove the convergence of iterations and monotony of the iterative solution.

This paper is organized as follows. In Section 2, we give some notations and the related lemmas. In Section 3, we present our main results. At last, some examples are presented to illustrate the main results.

2. Preliminaries and lemmas

In this section, we shall present some notations and the related lemmas for later use. We set $E := C([0, 1], \mathbb{R})$ with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$.

Theorem A ([18]) If $\tilde{k}(t,s)$ is continuous on $[a,b]\times[a,b]$, $f(t)$ is continuous on $[a,b]$, and $|\lambda| < \frac{1}{\tilde{M}_k(b-a)}$, where $\tilde{M}_k = \max_{t,s\in[a,b]} |\tilde{k}(t,s)|$, then the Fredholm integral equation $u(t) = f(t) + \lambda \int_a^b \tilde{k}(t,s)u(s)ds$ has a unique continuous solution $u(t)$ on $[a,b]$, and $u(t)$ can be represented as the following convergence series

$$u(t) = f(t) + \sum_{n=1}^{\infty} \lambda^n \int_a^b \tilde{k}_n(t,s) f(s)ds,$$

where $\tilde{k}_1(t,s) = \tilde{k}(t,s)$ and $\tilde{k}_n(t,s) = \int_a^b \tilde{k}(t,z)\tilde{k}_{n-1}(z,s)dz, n = 2,3,\cdots$.

Define the auxiliary function

$$k(t,s) = \frac{(1+\beta-t)g_1(s) + (\alpha+t)g_2(s)}{1+\alpha+\beta}, t,s \in [0,1],$$

we can easily deduce

$$0 \le m_k := \min\{k(t,s)|t,s\in[0,1]\} \le \max\{k(t,s)|t,s\in[0,1]\} := M_k < 1.$$

Lemma 2.1. If function $y \in C([0,1],\mathbb{R})$, then the boundary value problem

$$\begin{cases} u''(t) = y(t), t\in(0,1), \\ u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, \\ u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds \end{cases} \tag{2}$$

has a unique solution of the following form

$$u(t) = \int_0^1 G_1(t,s)y(s)ds + \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)y(\tau)d\tau ds, \tag{3}$$

where $k_1(t,s) = k(t,s)$,

$$G_1(t,s) = \begin{cases} \frac{(\alpha+t)(s-1-\beta)}{1+\alpha+\beta}, 0 \le t \le s \le 1, \\ \frac{(\alpha+s)(t-1-\beta)}{1+\alpha+\beta}, 0 \le s \le t \le 1, \end{cases}$$

and

$$R(t,s) = \sum_{n=1}^{\infty} k_n(t,s), \;\; k_n(t,s) = \int_0^1 k(t,z)k_{n-1}(z,s)dz, \;\; n = 2,3,\cdots.$$

Proof: By using the method of constant variation, we see that the solution of (2) is equivalent to the continuous solution of the following integral equation

$$u(t) = \int_0^1 G_1(t,s)y(s)ds + \int_0^1 k(t,s)u(s)ds. \tag{4}$$

By using the condition of $g_i(s)$ and Theorem A, (4) has a unique continuous solution $u(t)$ on $[0,1]$, which can be represented as the following convergence series

$$u(t) = \int_0^1 G_1(t,s)y(s)ds + \sum_{n=1}^{\infty}\int_0^1 k_n(t,s)\int_0^1 G_1(s,\tau)y(\tau)d\tau ds.$$

The proof is complete. □

Remark 2.1. Since the function $g_i \in C([0,1],[0,1))$ for $i = 1,2$, each $k_n(t,s)$ is well defined and continuous in $[0,1]\times[0,1]$, and $0 \le k_n(t,s) \le (M_k)^n < 1, \;\; n = 1,2,\cdots$. Noting that the geometric series $\sum_{n=1}^{\infty}(M_k)^n$ is convergent, it can be obtained by Weierstraz criterion that the series $\sum_{n=1}^{\infty} k_n(t,s)$ is uniformly convergent with respect to $(t,s) \in [0,1] \times [0,1]$. Hence, for $(t,s) \in [0,1] \times [0,1]$, the sum function $R(t,s) = \sum_{n=1}^{\infty} k_n(t,s)$ in (3) is well defined and continuous, and $\frac{m_k}{1-m_k} \leq R(t,s) \leq \frac{M_k}{1-M_k}$.

Lemma 2.2. If $\eta \in C^2([0,1],\mathbb{R})$, then the following fourth order boundary value problem with p-Laplacian operator

$$(\Phi_p(u''(t)))'' = f(t,\eta(t),\eta''(t)), \quad t \in (0,1), \tag{5}$$

$$u''(0) = u''(1) = 0, \tag{6}$$

$$u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds \tag{7}$$

has a unique solution of the form

$$\begin{aligned} u(t) \quad &= \int_0^1 G_1(t,s)\Phi_q(\int_0^1 G_2(s,\tau)f(\tau,\eta(\tau),\eta''(\tau))d\tau)ds \\ &+ \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)f(x,\eta(x),\eta''(x))dx)d\tau ds, \end{aligned}$$

where $G_1(t,s)$ and $R(t,s)$ are defined as in Lemma 2.1, and

$$G_2(t,s) = \begin{cases} t(s-1), 0 \leq t \leq s \leq 1, \\ s(t-1), 0 \leq s \leq t \leq 1. \end{cases}$$

Proof: Let $\Phi_p(u'') = h$, then the boundary value problem (5)-(7) can be rewriten as

$$h''(t) = f(t,\eta(t),\eta''(t)), \quad t \in (0,1), \tag{8}$$

$$u''(t) = \Phi_q(h(t)), \qquad t \in (0,1), \tag{9}$$

$$h(0) = h(1) = 0, \tag{10}$$

$$u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)ds, u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)ds. \tag{11}$$

For the boundary value problem of second order ordinary differential equations (8) and (10), by using the method of constant variation, we can easily obtain the corresponding Green function $G_2(t,s)$. Furthermore, we can also obtain the unique solution of the boundary value problem (8) and (10) with the following form

$$h(t) = \int_0^1 G_2(t,\tau)f(\tau,\eta(\tau),\eta''(\tau))d\tau. \tag{12}$$

For the boundary value problem of second order ordinary differential equations (9) and (11), by using Lemma 2.1, we can obtain the corresponding kernel functions $G_1(t,s)$ and $R(t,s)$. At the same time, we can also obtain the unique solution of the boundary value problem (9) and (11) with the following form

$$u(t) = \int_0^1 G_1(t,s)\Phi_q(h(s))ds + \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(h(\tau))d\tau ds. \tag{13}$$

Combining (12) and (13), we can obtain the unique solution of the boundary value problem (5)-(7) with the following form

$$\begin{aligned} u(t) \quad &= \int_0^1 G_1(t,s)\Phi_q(\int_0^1 G_2(s,\tau)f(\tau,\eta(\tau),\eta''(\tau))d\tau)ds \\ &+ \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)f(x,\eta(x),\eta''(x))dx)d\tau ds. \end{aligned}$$

The proof is complete. □

Lemma 2.3. (Lemma 2.2, [19]) The following relations hold:

(1) If $1 < q \leq 2$, then
$$|\Phi_q(u+v) - \Phi_q(u)| \leq 2^{2-q}|v|^{q-1},$$
for all $u, v \in \mathbb{R}$;

(2) If $q > 2$, then
$$|\Phi_q(u+v) - \Phi_q(u) \leq (q-1)(|u|+|v|)^{q-2}|v|,$$
for all $u, v \in \mathbb{R}$.

Lemma 2.4. Given $\varphi \in C([0,1], \mathbb{R})$, let
$$v(t) = \Phi_q(\int_0^1 G_2(t,s)\varphi(s)ds),$$
$$u(t) = \int_0^1 G_1(t,\tau)v(\tau))d\tau + \int_0^1 R(t,\tau)\int_0^1 G_1(\tau,s)v(s))dsd\tau,$$
then we have
$$M_1 := \max_{0\leq t\leq 1}|\int_0^1 G_2(t,s)ds| = \frac{1}{8},$$
$$M_2 := \max_{0\leq t\leq 1}|\int_0^1 G_1(t,s)ds| = \frac{(\frac{1}{2}+\alpha)(\frac{1}{2}+\beta)(\frac{1}{2}+\alpha+\beta)}{(1+\alpha+\beta)^2},$$
$$\|v\| \leq (\frac{1}{8})^{q-1}\|\varphi\|^{q-1},$$
$$\|u\| \leq (\frac{1}{8})^{q-1}\|\varphi\|^{q-1}\cdot\frac{M_2}{1-M_k}.$$

Proof: According to Lemma 2.2, we have
$$\begin{aligned}
\max_{t\in[0,1]}|\int_0^1 G_2(t,s)ds| &= \max_{t\in[0,1]}\{|\int_0^t s(t-1)ds + \int_t^1 t(s-1)ds|\}\\
&= \max_{t\in[0,1]}\{|(t-1)\cdot\frac{t^2}{2} + t[(-1+\frac{1}{2}) - (\frac{t^2}{2}-t)]|\}\\
&= \max_{t\in[0,1]}\{|\frac{t^2}{2}-\frac{t}{2}|\} = \frac{1}{8}.
\end{aligned}$$
Similarly,
$$\begin{aligned}
\max_{t\in[0,1]}|\int_0^1 G_1(t,s)ds| &= \max_{t\in[0,1]}\{|\int_0^t \frac{(\alpha+s)(t-1-\beta)}{1+\alpha+\beta}ds + \int_t^1 \frac{(\alpha+t)(s-1-\beta)}{1+\alpha+\beta}ds|\}\\
&= \max_{t\in[0,1]}\{|\frac{1}{1+\alpha+\beta}[\frac{1+\alpha+\beta}{2}t^2 - t(\frac{1}{2}+\beta) - \alpha(\frac{1}{2}+\beta)]|\}\\
&= \frac{(\frac{1}{2}+\alpha)(\frac{1}{2}+\beta)(\frac{1}{2}+\alpha+\beta)}{(1+\alpha+\beta)^2}.
\end{aligned}$$
Considering that Φ_q is increasing, we have
$$v(t) = \Phi_q(\int_0^1 G_2(t,s)\varphi(s)ds) \leq \Phi_q(\int_0^1 G_2(t,s)\|\varphi\|ds) \leq \Phi_q(\frac{1}{8}\|\varphi\|) = (\frac{1}{8})^{q-1}\|\varphi\|^{q-1}.$$

Meanwhile, we have

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,\tau)\Phi_q(\int_0^1 G_2(\tau,s)\varphi(s)ds)d\tau + \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)\varphi(x)dx))d\tau ds \\ &\le (\frac{1}{8})^{q-1}\|\varphi\|^{q-1}\cdot\frac{(\frac{1}{2}+\alpha)(\frac{1}{2}+\beta)(\frac{1}{2}+\alpha+\beta)}{(1+\alpha+\beta)^2}\cdot(1+\int_0^1 R(t,s)ds) \\ &\le (\frac{1}{8})^{q-1}\|\varphi\|^{q-1}\cdot\frac{(\frac{1}{2}+\alpha)(\frac{1}{2}+\beta)(\frac{1}{2}+\alpha+\beta)}{(1+\alpha+\beta)^2}\cdot(1+\frac{M_k}{1-M_k}) \\ &= (\frac{1}{8})^{q-1}\|\varphi\|^{q-1}\cdot\frac{M_2}{1-M_k}. \end{aligned}$$

The proof is complete. □

3. Main results

For each number $M > 0$, we denote

$$B_0 := \max\{M^{q-2}\cdot 2^{3(1-q)},\ \ (q-1)\cdot M^{q-2}\cdot 2^{3(1-q)}\cdot 3^{q-2}\},$$

$$D_M := \{(t,u,v)\in\mathbb{R}^3 | 0\le t\le 1, |u|\le \frac{M_2}{1-M_k}\cdot(\frac{M}{8})^{q-1}, |v|\le(\frac{M}{8})^{q-1}\},$$

and $B[O,M]$, a closed ball centred at O with the radius M in the space of continuous functions $C([0,1],\mathbb{R})$.

Theorem 3.1 Suppose that there exist constants $M, L_1, L_2 \ge 0$ such that

(i) $|f(t,u,v)| \le M$, for $(t,u,v)\in D_M$;

(ii) $|f(t,u_2,v_2) - f(t,u_1,v_1)| \le L_1|u_2-u_1| + L_2|v_2-v_1|$, for $(t,u_i,v_i)\in D_M,\ \ i=1,2$;

(iii) $K := (\frac{L_1M_2}{1-M_k}+L_2)\cdot B_0 < 1$.

Then the boundary value problem (1) has a unique solution $u(t)\in C([0,1],\mathbb{R})$ such that

$$|u(t)| \le \frac{M_2}{1-M_k}\cdot(\frac{M}{8})^{q-1}, \quad |u''(t)| \le (\frac{M}{8})^{q-1}.$$

Proof: Firstly, for any function $\varphi(t)\in C([0,1],\mathbb{R})$, we consider the nonlinear operator $A : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$\begin{aligned} (A\varphi)(t) &= f(t,\int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)\varphi(x)dx))d\tau ds \\ &\quad + \int_0^1 G_1(t,\tau)\Phi_q(\int_0^1 G_2(\tau,s)\varphi(s)ds)d\tau, \Phi_q(\int_0^1 G_2(t,s)\varphi(s)ds). \end{aligned}$$

By the continuity of $G_1(t,s), G_2(t,s), R(t,s)$ and $f(t,u,v)$, it is easy to verify that the operator A is well defined and continuous. By using Lemma 2.2, we can prove that if $\varphi(t)$ is a fixed point of the operator A, then

$$u(t) = \int_0^1 G_1(t,\tau)\Phi_q(\int_0^1 G_2(\tau,s)\varphi(s)ds)d\tau + \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)\varphi(x)dx))d\tau ds$$

is a solution of the problem (1). On the contrary, if problem (1) has a solution $u(t)$, then $\varphi(t) = (\Phi_p(u''(t)))''$ is a fixed point of the operator A.

Secondly, we shall prove that the operator A maps $B[O,M]$ into itself. For any $\varphi(t)\in B[O,M]$, by Lemma 2.4, we have

$$\|v\| \le (\frac{M}{8})^{q-1},\ \|u\| \le (\frac{M}{8})^{q-1}\cdot(\frac{M_2}{1-M_k}).$$

For any $t \in [0,1]$, we have $(t, u(t), v(t)) \in D_M$. Then, from the assumption (i) of Theorem 3.1, it follows that

$$|A\varphi(t)| = |f(t, u(t), v(t))| \leq M, i.e., A\varphi \in B[O, M].$$

So the operator A maps $B[O, M]$ into itself.

Next, we have the following assertion:

For any $\varphi_1(t), \varphi_2(t) \in B[O, M]$, we have

$$\left|\Phi_q(\int_0^1 G_2(t,s)\varphi_2(s)ds) - \Phi_q(\int_0^1 G_2(t,s)\varphi_1(s)ds)\right| \leq B_0 \cdot \|\varphi_2 - \varphi_1\|. \tag{14}$$

In fact, by using Lemma 2.3, we can prove it as follows:

For $1 < q \leq 2$, we have

$$\begin{aligned}
&\left|\Phi_q(\int_0^1 G_2(t,s)\varphi_2(s)ds) - \Phi_q(\int_0^1 G_2(t,s)\varphi_1(s)ds)\right| \\
&= \left|\Phi_q(\int_0^1 G_2(t,s)(\varphi_1(s) + \varphi_2(s) - \varphi_1(s))ds) - \Phi_q(\int_0^1 G_2(t,s)\varphi_1(s)ds)\right| \\
&\leq 2^{2-q}|\int_0^1 G_2(t,s)(\varphi_2(s) - \varphi_1(s))ds|^{q-1} \\
&\leq 2^{2-q}(\frac{1}{8} \cdot 2M)^{q-2} \cdot \frac{1}{8} \cdot \|\varphi_2 - \varphi_1\| \\
&= 2^{3(1-q)} \cdot M^{q-2}\|\varphi_2 - \varphi_1\| \\
&\leq B_0 \cdot \|\varphi_2 - \varphi_1\|.
\end{aligned}$$

For $q > 2$, we have

$$\begin{aligned}
&\left|\Phi_q(\int_0^1 G_2(x,s)\varphi_2(s)ds) - \Phi_q(\int_0^1 G_2(x,s)\varphi_1(s)ds)\right| \\
&\leq (q-1)\left(|\int_0^1 G_2(x,s)\varphi_1(s)ds| + |\int_0^1 G_2(x,s)(\varphi_2(s) - \varphi_1(s))ds|\right)^{q-2} \left|\int_0^1 G_2(x,s)(\varphi_2(s) - \varphi_1(s))ds\right| \\
&\leq (q-1)\left(3R \cdot \frac{1}{8}\right)^{q-2} \cdot \frac{1}{8} \cdot \|\varphi_2 - \varphi_1\| \\
&\leq (q-1) \cdot (\frac{1}{8})^{q-1} \cdot (3R)^{q-2} \cdot \|\varphi_2 - \varphi_1\| \\
&\leq B_0 \cdot \|\varphi_2 - \varphi_1\|.
\end{aligned}$$

Thirdly, according to (14), we shall prove that A is a contraction operator in $B[O, M]$. Indeed, for any

$\varphi_1(t), \varphi_2(t) \in B[O, M]$, we have

$$
\begin{aligned}
&|A\varphi_2(t) - A\varphi_1(t)| \\
&= |f(t, u_2(t), v_2(t)) - f(t, u_1(t), v_1(t))| \\
&\le L_1|u_2(t) - u_1(t)| + L_2|v_2(t) - v_1(t)| \\
&= L_1|\int_0^1 G_1(t,\tau)\Phi_q(\int_0^1 G_2(\tau,s)\varphi_2(s)ds)d\tau + \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)\varphi_2(x)dx))d\tau ds \\
&- \int_0^1 G_1(t,\tau)\Phi_q(\int_0^1 G_2(\tau,s)\varphi_1(s)ds)d\tau - \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)\varphi_1(x)dx))d\tau ds| \\
&+L_2|\Phi_q(\int_0^1 G_2(t,s)\varphi_2(s)ds) - \Phi_q(\int_0^1 G_2(t,s)\varphi_1(s)ds)| \\
&\le L_1|\int_0^1 G_1(t,\tau)(\Phi_q(\int_0^1 G_2(\tau,s)\varphi_2(s)ds) - \Phi_q(\int_0^1 G_2(\tau,s)\varphi_1(s)ds))d\tau| \\
&+L_1|\int_0^1 R(t,s)\int_0^1 G_1(s,\tau)(\Phi_q(\int_0^1 G_2(\tau,x)\varphi_2(x)dx) - \Phi_q(\int_0^1 G_2(\tau,x)\varphi_1(x)dx))d\tau ds| \\
&+L_2|\Phi_q(\int_0^1 G_2(t,s)\varphi_2(s)ds) - \Phi_q(\int_0^1 G_2(t,s)\varphi_1(s)ds)| \\
&\le L_1B_0\frac{M_2}{1-M_k}\|\varphi_2 - \varphi_1\| + L_2B_0\|\varphi_2 - \varphi_1\| \\
&= (\frac{L_1M_2}{1-M_k} + L_2)B_0\|\varphi_2 - \varphi_1\| \\
&\le K\|\varphi_2 - \varphi_1\|.
\end{aligned}
$$

Hence, $|A\varphi_2(t) - A\varphi_1(t)| \le K\|\varphi_2 - \varphi_1\|, 0 < K < 1$. Thus, the operator $A : B[O, M] \to B[O, M]$ is a contraction mapping and it has a unique fixed point in $B[O, M]$.

The proof is complete. □

Now we consider the following iterative process:

(1) We choose $\varphi_0(t) = f(t, 0, 0), t \in [0, 1]$;

(2) For $k = 0, 1, \cdots$, let

$v_k(t) = \Phi_q(\int_0^1 G_2(t,s)\varphi_k(s)ds)$,

$u_k(t) = \int_0^1 G_1(t,\tau)v_k(\tau)d\tau + \int_0^1 R(t,\tau)\int_0^1 G_1(\tau,s)v_k(s)dsd\tau$,

$\varphi_{k+1}(t) = f(t, u_k, v_k)$.

From condition (i) of Theorem 3.1, we can easily know $\varphi_k(t) \in B[O, M], k = 0, 1, \cdots$. By using the Banach contracting mapping principle, the sequence $\varphi_n(t)$ converges with the rate of geometric progression to the fixed point of the operator A, denote it as $\varphi^*(t)$. Also, we have the estimation

$$\|\varphi_n - \varphi^*\| \le \frac{K^n}{1-K}\|\varphi_1 - \varphi_0\|.$$

Then, we can obtain an iterative sequence solution $\{u_n(t)\}$ of the boundary value problem (1)

$$u_n(t) = \int_0^1 G_1(t,s)\Phi_q(\int_0^1 G_2(s,\tau)\varphi_n(\tau)d\tau)ds + \int_0^1 R(t,s)\int_0^1 G_1(s,\tau)\Phi_q(\int_0^1 G_2(\tau,x)\varphi_n(x)dx))d\tau ds,$$

which converges to the unique solution $u^*(t)$ of the boundary value problem (1).

Finally, we present some examples to illustrate the feasibility of our results. Our first example is for the case $p > 2$ and we choose $p = 3$.

Example 3.1. Consider the following boundary value problem

$$\begin{cases} (\Phi_p(u''((t)))'' = \frac{1}{3}u(t)u''(t) + \frac{1}{8}u^2(t) - \frac{1}{4}(u''(t))^2 + 20t + 1, \quad t \in (0,1), \\ u(0) - \frac{1}{2}u'(0) = \frac{1}{2}\int_0^1 u(s)ds, \\ u(1) + \frac{1}{2}u'(1) = \frac{1}{4}\int_0^1 u(s)ds, u''(0) = u''(1) = 0, \end{cases} \tag{15}$$

where

$$f(t,u,v) = \frac{1}{3}uv + \frac{1}{8}u^2 - \frac{1}{4}v^2 + 20t + 1.$$

We can see that $p = 3, \alpha = \frac{1}{2}, \beta = \frac{1}{2}, g_1(t) = \frac{1}{2}, g_2(t) = \frac{1}{4}$. After a simple calculation, we can obtain $q = \frac{3}{2}, M_k = \frac{7}{16} < 1, M_2 = \frac{3}{8}, B_0 = \frac{1}{\sqrt{8M}}$,

$$k_1(t,s) = \frac{7-2t}{16}, k_n(t,s) = (\frac{3}{8})^{n-1} \cdot (\frac{7-2t}{16}), R(t,s) = \sum_{n=1}^{\infty} k_n(t,s) = \frac{7-2t}{10},$$

$$G_1(t,s) = \begin{cases} \frac{(\frac{1}{2}+t)(s-\frac{3}{2})}{2}, 0 \le t \le s \le 1, \\ \frac{(\frac{1}{2}+s)(t-\frac{3}{2})}{2}, 0 \le s \le t \le 1. \end{cases}$$

Clearly, for $(t,u,v) \in D_M$, there is

$$f(t,u,v) \le \frac{1}{3} \cdot \frac{2}{3} \cdot (\sqrt{\frac{M}{8}})^2 + \frac{1}{8} \cdot \frac{4}{9}(\sqrt{\frac{M}{8}})^2 + \frac{1}{4}(\sqrt{\frac{M}{8}})^2 + 21 < M$$

as soon as $M > 22.48$. Thus, we choose $M = 23$ and the condition (i) of Theorem 3.1 is satisfied in D_M.

On the other hand, for $(t,u,v) \in D_M$,

$$|f_u| = |\frac{1}{3}v + \frac{1}{4}u| \le |\frac{1}{3}\sqrt{\frac{M}{8}} + \frac{1}{4} \cdot \frac{2}{3} \cdot \sqrt{\frac{M}{8}}| \le \frac{1}{2}\sqrt{\frac{M}{8}},$$

$$|f_v| = |\frac{1}{3}u + \frac{1}{2}v| \le |\frac{1}{2}\sqrt{\frac{M}{8}} + \frac{1}{3} \cdot \frac{2}{3} \cdot \sqrt{\frac{M}{8}}| \le \frac{13}{18}\sqrt{\frac{M}{8}}.$$

So, we can choose $L_1 = \frac{1}{2}\sqrt{\frac{M}{8}}, L_2 = \frac{13}{18}\sqrt{\frac{M}{8}}$, and the condition (ii) of Theorem 3.1 is satisfied. Moreover,

$$K = (\frac{L_1 M_2}{1 - M_k} + L_2) \cdot B_0 = (\frac{2}{3} \cdot \frac{1}{2}\sqrt{\frac{M}{8}} + \frac{13}{18}\sqrt{\frac{M}{8}}) \cdot \frac{1}{\sqrt{8M}} < 1,$$

so the condition (iii) of Theorem 3.1 is satisfied. By Theorem 3.1, boundary value problem (15) has a unique solution. Let $\varphi_0(t) = 20t + 1$, see Figure 1 and Table 1 for the iterative process and we can find that the iterative method is very effective.

Figure 1: The approximation of the solution (15) for $p = 3$.

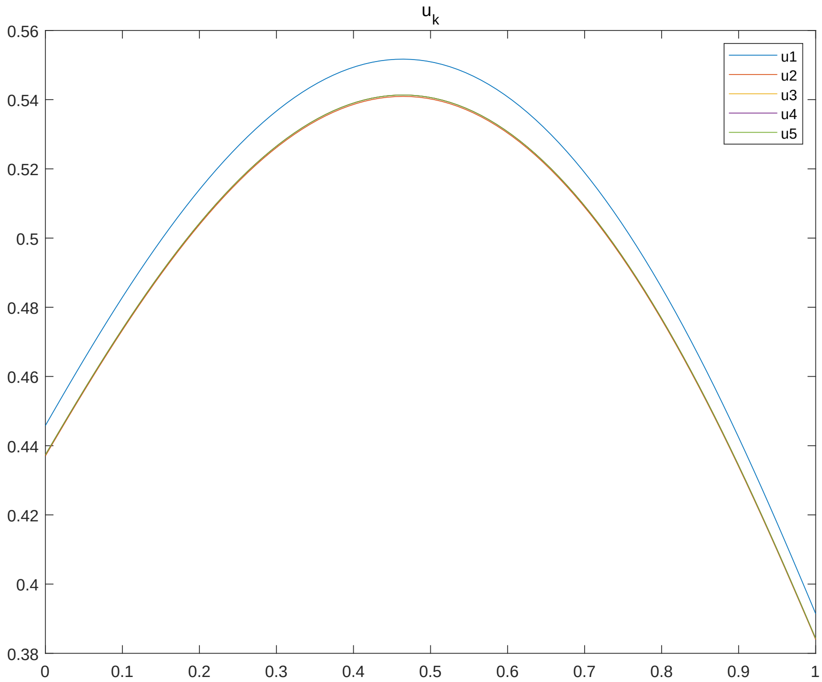

Table 1 The approximation of the solution (15) for $p = 3$.

t	0	0.1010101	0.2020202	0.3030303	0.4040404	0.5050505	0.6060606	0.7070707	0.8080808	0.9090909	1
u1	0.4458064	0.4832388	0.5145593	0.537246	0.5496579	0.5507146	0.5398311	0.5169339	0.482547	0.4379967	0.3914417
u2	0.4370199	0.4737051	0.5044146	0.5266806	0.5388937	0.5399895	0.5293848	0.5069951	0.4733202	0.4296486	0.3839808
u3	0.4373658	0.4740806	0.504814	0.5270963	0.5393168	0.5404104	0.5297942	0.5073842	0.4736811	0.4299749	0.3842725
u4	0.4373524	0.474066	0.5047985	0.5270801	0.5393003	0.5403941	0.5297783	0.5073691	0.4736671	0.4299622	0.3842612
u5	0.4373529	0.4740666	0.5047991	0.5270808	0.5393009	0.5403947	0.529779	0.5073697	0.4736676	0.4299627	0.3842616

Next, we consider the case $1 < p \leq 2$ and we choose $p = \frac{3}{2}$.

Example 3.2. Consider the following boundary value problem

$$\begin{cases} (\Phi_p(u''(t)))'' = \frac{3}{2}u(t) + u''(t)\cdot\sin(u(t)+u''(t)) + t + 1, \quad t \in (0,1), \\ u(0) - \frac{1}{2}u'(0) = \frac{1}{2}\int_0^1 u(s)ds, \\ u(1) + \frac{1}{2}u'(1) = \frac{1}{4}\int_0^1 u(s)(1-s)ds, u''(0) = u''(1) = 0, \end{cases} \tag{16}$$

where

$$f(t,u,v) = \frac{3}{2}u + v\sin(u+v) + t + 1.$$

We can see that $p = \frac{3}{2}, \alpha = \frac{1}{2}, \beta = \frac{1}{2}, g_1(s) = \frac{1}{4}, g_2(s) = \frac{1}{2}$. After a simple calculation, we can obtain $q = 3, M_k = \frac{7}{16} < 1, M_2 = \frac{3}{8}, B_0 = \frac{3M}{32}$,

$$k_1(t,s) = \frac{5+2t}{16}, k_n(t,s) = (\frac{3}{8})^{n-1}\cdot(\frac{5+2t}{16}), R(t,s) = \sum_{n=1}^{\infty} k_n(t,s) = \frac{5+2t}{10},$$

$$G_1(t,s) = \begin{cases} \frac{(\frac{1}{2}+t)(s-\frac{3}{2})}{2}, 0 \leq t \leq s \leq 1, \\ \frac{(\frac{1}{2}+s)(t-\frac{3}{2})}{2}, 0 \leq s \leq t \leq 1. \end{cases}$$

Clearly, for $(t, u, v) \in D_M$, there is

$$f(t,u,v) \le \frac{3}{2} \cdot \frac{2}{3} \cdot (\frac{M}{8})^2 + (\frac{M}{8})^2 + 2 < M.$$

as soon as $2.144 < M < 29.856$. Thus, we choose $M = 4$ and the condition (i) of Theorem 3.1 is satisfied.
On the other hand, for $(t, u, v) \in D_M$,

$$|f_u| = |\frac{3}{2} + v\cos(u+v)| \le \frac{3}{2} + (\frac{M}{8})^2 \le \frac{7}{4},$$

$$|f_v| = |\sin(u+v) + v\cos(u+v)| \le 1 + (\frac{M}{8})^2 \le \frac{5}{4}.$$

So we can choose $L_1 = \frac{7}{4}, L_2 = \frac{5}{4}$, and the condition (ii) of Theorem 3.1 is satisfied. Moreover,

$$K = (\frac{L_1 M_2}{1 - M_k} + L_2) \cdot B_0 = (\frac{2}{3} \cdot \frac{7}{4} + \frac{5}{4}) \cdot \frac{3M}{32} < 1.$$

So the condition (iii) of Theorem 3.1 is satisfied. By Theorem 3.1, boundary value problem (16) has a unique solution. Let $\varphi_0(t) = t + 1$, see Figure 2 and Table 2 for the iterative process. Obviously, in this case, the iterative method is also effective.

Figure 2: The approximation of the solution (16) for $p = \frac{3}{2}$.

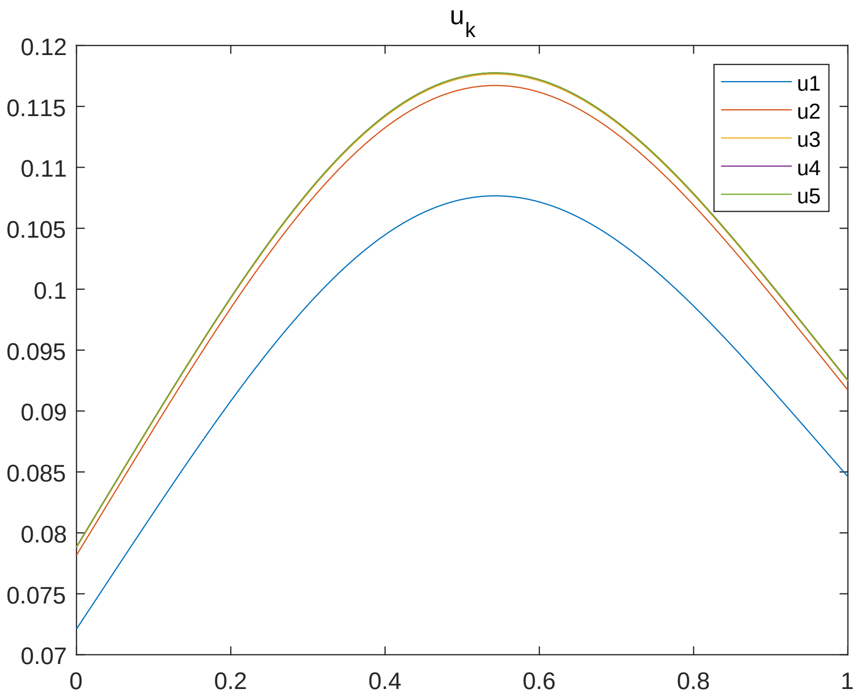

Table 2 The approximation of the solution (16) for $p = \frac{3}{2}$.

t	0	0.1010101	0.2020202	0.3030303	0.40404	0.505051	0.606061	0.707071	0.808081	0.909091	1
u1	0.072112	0.0817731	0.0909937	0.0989252	0.104639	0.107438	0.107049	0.103709	0.098108	0.091204	0.084614
u2	0.078172	0.0886447	0.098642	0.1072443	0.113443	0.116476	0.116049	0.11242	0.106343	0.098857	0.091715
u3	0.078797	0.0893541	0.0994317	0.1081035	0.114352	0.11741	0.116979	0.11332	0.107193	0.099647	0.092448
u4	0.078863	0.0894288	0.0995148	0.1081939	0.114448	0.117508	0.117077	0.113414	0.107283	0.09973	0.092526
u5	0.07887	0.0894366	0.0995236	0.1082034	0.114458	0.117519	0.117087	0.113424	0.107292	0.099739	0.092534

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