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Strongly I–Deferred Cesàro Summablity and μ-Deferred I–Statistically Convergence in Amenable Semigroups

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Abstract. The primary objective of this study is to extend the concept of strongly I-deferred Cesàro summability and μ-deferred I-statistical convergence on amenable semigroups. Furthermore, under few conditions, we also establish some inclusion-based results. After that, we introduce the μ-deferred I^*-statistical convergent, μ-deferred I-statistically pre-Cauchy, and μ-deferred I^*-statistically pre-Cauchy functions in amenable semigroups and prove results based on connection among them. Certain counter-examples are also presented to support our results.

1. Introduction

In the middle of the 20th century, Fast [8] and Steinhaus [23] individually worked on the concept of statistical convergence of a sequence. As an extended work on statistical convergence, the ideal convergence and statistical convergence via ideals came in theory. Using the concept of ideal defined on the set of natural numbers $\mathbb{N}$, I-convergence and I-statistical convergence of sequence was introduced by Kostyrko et al. [11] and Savas and Das [21], respectively. After that, further study on convergence via ideals by different authors came into literature, some references are [10, 19, 20], etc. Savas and Das [22] defined a new generalization of I-statistical convergent sequence and called it as I-statistically pre-Cauchy sequence. Baliarsingh [2] and Nayak et al. [15] defined notion of statistical deferred $A-$convergence for uncertain sequences and fuzzy sequences, respectively. Recently inspired by [7], Khan et al. [9] defined the notion of μ-deferred I–statistically convergence for real sequences by using concept of μ-deferred density defined on $(\mathbb{N}, \mathcal{L}, \mu)$, where $\mathcal{L}$ be the sigma-algebra of subsets of $\mathbb{N}$ and μ be the sigma finite measure on $\mathcal{L}$ with $\mu(\mathbb{N})=\infty$.
The natural density of $A \subset \mathbb{N}$ is defined by

$$d(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{a=1}^{n} \chi_A(a)$$

provided that the limit exists, where χ_A denotes the characteristic function of the set A.

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Definition 1.1. *[21] Let I be a non-trivial admissible ideal defined on* $\mathbb{N}$. *A sequence* $y = (y_k)$ *is said to be I-statistically convergent to L if for all* $\epsilon,\delta > 0$, *the following set*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \in \mathbb{N} : k \leq n, |y_k - L| \geq \epsilon\}\right| \geq \delta\right\} \in I.$$

We write it as I-st-$\lim y = L$. *Here* $|A|$ *denotes the cardinality of set A.*

Definition 1.2. *[22] Let I be a non-trivial admissible ideal defined on* $\mathbb{N}$. *A sequence* $y = (y_k)$ *is said to be I-statistically pre-Cauchy sequence if for all* $\epsilon,\delta > 0$, *the following set*

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\{(k,l) \in \mathbb{N} \times \mathbb{N} : k,l \leq n, |y_k - y_l| \geq \epsilon\}\right| \geq \delta\right\} \in I.$$

Here $|A|$ *denotes the cardinality of set A.*

Let $p = (p_n), q = (q_n)$ be any pair of increasing sequences of non-negative integers such that

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty. \tag{1}$$

Angew [1] gave a new generalization of Cesàro mean which is defined as for a real (or complex) sequence (y_k)

$$(D_{p,q}y)_n := \frac{1}{q_n - p_n}\sum_{k=p_n+1}^{q_n} y_k, \quad n \in \mathbb{N}.$$

and called it as deferred Cesàro mean of real (or complex) sequences (y_k).

Definition 1.3. *[9] Let I be a non-trivial admissible ideal defined on* $\mathbb{N}$. *A sequence* $y = (y_k)$ *is said to be strongly I–deferred Cesàro summable to a number* $L \in \mathbb{R}$ *if for all* $\epsilon > 0$, *the following set*

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\sum_{p_n}^{q_n} |y_k - L| \geq \epsilon\right\} \in I.$$

We write it as $[DC^I_{p,q}] - \lim y = L$.

Let $X = \mathbb{N}$ and Σ be a sigma-algebra of the subsets of X and μ be a sigma finite measure on Σ such that $\mu(X) = \infty$. Measure of any subset A of X which is in Σ will be denoted by $\mu(A) := \|A\|$. Note that here $\|A\|$ denote the sigma finite measure of set A.
The μ–deferred density of $A \subset \mathbb{N}$ is defined by

$$_{\mu}D(A) = \lim_{n\to\infty}\frac{\|A \cap I^*_{p,q}(n)\|}{\|I^*_{p,q}(n)\|} \tag{2}$$

provided that the limit exists, where $I^*_{p,q}(n) = [p_n, q_n] \cap \mathbb{N}$.

Definition 1.4. *[9] Let I be a non-trivial admissible ideal defined on* $\mathbb{N}$. *A sequence* $y = (y_k)$ *is said to be* μ*–deferred I–statistically convergent to a number* $L \in \mathbb{R}$ *if for all* $\epsilon,\delta > 0$, *the following set*

$$\left\{n \in \mathbb{N} : \frac{\|\{k \in \mathbb{N} : |y_k - L| \geq \epsilon \text{ for some } L \in \mathbb{R}\} \cap I^*_{p,q}(n)\|}{\|I^*_{p,q}(n)\|} \geq \delta\right\} \in I.$$

We write it as $[_{\mu}DS^I_{p,q}] - \lim y = L$. *Here* $\|A\|$ *denotes the* μ*-measure of set A.*

Lemma 1.5. *[24] A matrix $A = (a_{jn})_{j,n\in\mathbb{N}}$ is said to be regular iff the following conditions hold:*
(a) There exists $M > 0$ s.t. $\sum_n |a_{jn}| \le M$, $\forall j \in \mathbb{N}$,
(b) $\lim_{j\to\infty} a_{jn} = 0$, $\forall n \in \mathbb{N}$,
(c) $\lim_{j\to\infty} \sum_n a_{jn} = 1$.

In this paper, spaces of all bounded real functions and all real valued functions ϕ on H are denoted by $m(H)$ and $w(H)$, respectively. Here H is a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. With sup norm, $m(H)$ is Banach space. A real linear functional f on $m(H)$ is called a mean on $m(H)$ if

$$\inf\{\phi(h) : h \in H\} \le f(\phi) \le \sup\{\phi(h) : h \in H\}, \ \ \forall \phi \in m(H).$$

It is obvious that $f \ge 0$ and $f(e) = 1$, here e is unity element of H. If $f(\phi(lh)) = f(\phi(h))$, $\forall h, l \in H$ and $\forall \phi \in m(H)$ then we say mean f is left invariant mean. If $f(\phi(hl)) = f(\phi(h))$, $\forall h, l \in H$ and $\forall \phi \in m(H)$ then we say mean f is right invariant mean. If $\exists$ a right(left) invariant mean on $m(H)$ then semigroup H is called as right(left) amenable. If semigroup H is both right and left amenable then H is called amenable semigroup. If for every invariant left mean and every invariant right mean f, $f(\phi) = t$ then it is called as ϕ is almost convergent to t. For more information, we refer to [5]. In [14], it has been proven that $\exists$ a sequence $\{T_k\}$ of finite subsets of a discrete countable amenable group H such that $\{T_k\}$ have the following properties:
(1) $H = \bigcup_{k=1}^{\infty} T_k$,
(2) $T_k \subset T_{k+1}$, $k \in \mathbb{N}$,
(3) $\lim_{k\to\infty} \frac{|T_k h \cap T_k|}{|T_k|} = 1$, $\forall h \in H$.
Here $|S|$ denotes the cardinality of set S.
Any $\{T_k\}$ satisfying all three properties is known as Folner sequence for H. Folner seqeunce $\{T_k\} = \{0, 1, 2, ..., k-1\}$ gave rise to the classical Cesàro method of summability. The concept of summability in amenable semigroups introduced in [3, 5, 6, 12, 13]. The behaviour of convergence of the function defined on amenable semigroup depends upon the Folner sequence $\{T_k\}$. In [4], Douglas extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups. The behaviour of convergence of the function defined on amenable semigroup depends upon the Folner sequence $\{T_k\}$. In [16], Nuray introduced the notion of convergence and statistical convergence in amenable semigroups. In [17], he also defined the notion of almost statistical convergence in amenable semigroups. He [18] further introduced the concept of deferred Cesàro convergence and deferred statistical convergence in amenable semigroups.

Definition 1.6. *[16] Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for any Folner sequence $\{T_k\}$ for H,*

$$\lim_{k\to\infty} \frac{1}{|T_k|} \sum_{h\in T_k} |\phi(h) - t| = 0,$$

then we say $\phi \in w(H)$ is strongly summable to t for $\{T_k\}$. Here $|A|$ denotes the cardinality of set A.

Definition 1.7. *[16] Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for every $\epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\lim_{k\to\infty} \frac{1}{|T_k|} |\{h \in T_k : |\phi(h) - t| \ge \epsilon\}| = 0,$$

then we say $\phi \in w(H)$ is statistically convergent to t for $\{T_k\}$ and we write $\phi(h) \to t(S)$. Here $|A|$ denotes the cardinality of set A.

This study aims to extend the notions of strongly I–deferred Cesàro summability and μ-deferred I–statistically convergence to functions defined on discrete countable amenable semigroups. In addition, this study also introduces the concept of μ-deferred I–statistically pre-Cauchy condition on functions defined on discrete countable amenable semigroups.

2. Main results

Throughout the paper, we suppose H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H, (p, q) be a pair of sequences that satisfies (1) and I be a non-trivial admissible ideal of $\mathbb{N}$. Note that from here onwards, $\|A\|$ denotes the sigma finite measure (or μ-measure) of set A and $|A|$ denotes the cardinality of set A.

Let $X = H$ and $\mathcal{L}$ be a sigma-algebra of the subsets of X and μ be a sigma finite measure on $\mathcal{L}$ such that $\mu(X) = \infty$. μ-measure of any subset A of X which is in $\mathcal{L}$ will be denoted by $\mu(A) := \|A\|$. μ-deferred density of $A \subset H$ for any Folner sequence $\{T_k\}$ is defined as

$$\mu D(A) = \lim_{k\to\infty} \frac{\|A \cap (T_{q_{(k)}} \backslash T_{p_{(k)}})\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \tag{3}$$

provided that the limit exists. If $(p_k) = 0$ and $(q_k) = k$ for all $k \in \mathbb{N}$ then we have ${}_{\mu}D(A) = \lim_{k\to\infty} \frac{\|A\cap T_k\|}{\|T_k\|}$ and it is called as μ-natural density of set A. Clearly, if sigma finite measure μ is counting measure then definitions of μ-deferred density and μ-natural density reduce to definitions of deferred density and natural density, respectively.
Now by using the idea of μ-deferred density, we define our main definitions in upcoming subsections.

2.1. Strongly I–deferred Cesàro summablity and μ-deferred I–statistically convergence

Definition 2.1. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for every $\epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h\in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| \geq \epsilon\right\} \in I,$$

then we say $\phi \in w(H)$ is strongly I-deferred Cesàro summable to t for $\{T_k\}$ and we write $\phi(h) \to t(DC^I_{p,q})$.

Definition 2.2. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for each $\epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h\in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|^n \geq \epsilon\right\} \in I,$$

where $0 < n < \infty$, then we say $\phi \in w(H)$ is strongly I-deferred Cesàro n-summable to t for $\{T_k\}$.

Definition 2.3. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for any $\epsilon, \delta > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\left\{k \in \mathbb{N} : \frac{\|\{h \in T_k : |\phi(h) - t| \geq \epsilon\}\|}{\|T_k\|} \geq \delta\right\} \in I,$$

that is, μ-natural density of subset $\{h \in H : |\phi(h) - t| \geq \epsilon\}$ of H is I-convergent to zero, then we say $\phi \in w(H)$ is I-${}_{\mu}$statistically convergent to t for $\{T_k\}$ and we write $\phi(h) \to t({}_{\mu}S^I)$.

Definition 2.4. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for any $\epsilon, \delta > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\left\{k \in \mathbb{N} : \frac{\|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \geq \delta\right\} \in I,$$

that is, μ-deferred density of subset $\{h \in H : |\phi(h) - t| \geq \epsilon\}$ of H is I-convergent to zero, then we say $\phi \in w(H)$ is μ-deferred I-statistically convergent to t for $\{T_k\}$ and we write $\phi(h) \to t({}_{\mu}DS^I_{p,q})$.

Theorem 2.5. *If $\phi \in w(H)$ is strongly I-deferred Cesàro summable to t, then $\phi \in w(H)$ is μ-deferred I-statistically convergent to t. i.e., $\phi(h) \to t(DC^I_{p,q})$ implies $\phi(h) \to t({}_\mu DS^I_{p,q})$.*

Proof. Let $\phi \in w(H)$ is strongly deferred Cesàro summable to t, then $\forall\delta > 0$ and for any Folner sequence $\{T_k\}$ for H,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - t| \geq \delta\right\} \in I. \tag{4}$$

Now for any preassigned $\epsilon > 0$, $\exists$ a positive real number r such that

$$\begin{aligned}
\sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - t| &= \sum_{\substack{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} \\ |\phi(h)-t| \geq \epsilon}} |\phi(h) - t| + \sum_{\substack{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} \\ |\phi(h)-t| < \epsilon}} |\phi(h) - t| \\
&\geq \sum_{\substack{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} \\ |\phi(h)-t| \geq \epsilon}} |\phi(h) - t| \\
&\geq \frac{\epsilon}{r} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|.
\end{aligned}$$

So we get the following inequality

$$\frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - t| \geq \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \frac{\epsilon}{r} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|.$$

Hence, from (4)

$$\begin{aligned}
&\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\} \\
&\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - t| \geq \delta \frac{\epsilon}{r}\right\} \in I.
\end{aligned}$$

Therefore, $\phi \in w(H)$ is μ-deferred I-statistically convergent to t. □

Remark 2.6. *Converse part of Theorem 2.5 is not true in general.*

Example 2.7. *Let $H = \mathbb{Z}$, (p_k, q_k) is defined as in* (1) *and μ is counting measure. Take the Folner sequence*

$$\{T_k\} = \{h \in \mathbb{Z} : |h| \leq k\}$$

and function is

$$\phi(h) = \begin{cases} h, & \text{if } q_k - \sqrt{q_k} \leq h \leq q_k \\ 0, & \text{otherwise}. \end{cases}$$

Hence for any $\epsilon > 0$,

$$\frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - 0| \geq \epsilon\}\| \leq \frac{\sqrt{q_k}}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}.$$

Thus for any $\delta > 0$, we have

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - 0| \geq \epsilon\}\| \geq \delta\right\} \subseteq \left\{k \in \mathbb{N} : \frac{\sqrt{q_k}}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \geq \delta\right\}.$$

Since the set $\left\{k \in \mathbb{N} : \frac{\sqrt{q_k}}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|} \geq \delta\right\}$ *is a finite set, hence belongs to I, therefore* $\phi(h) \to 0({}_{\mu}DS^{I}_{p,q})$.
But from following inequality

$$\begin{aligned} \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - 0| &\leq \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{q_k - \sqrt{q_k}}^{q_k} h \\ &= \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \frac{2q_k^{\frac{3}{2}} - (q_k - \sqrt{q_k})}{2} \\ &\leq \frac{1}{2(q_k - p_k)} \frac{2q_k^{\frac{3}{2}} - p_k}{2}, \end{aligned}$$

we have

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - 0| \geq \frac{1}{8}\right\} \subseteq \left\{k \in \mathbb{N} : \frac{2q_k^{\frac{3}{2}} - p_k}{q_k - p_k} \geq \frac{1}{2}\right\} = \{c, c+1, c+2, ...\}$$

for some $c \in \mathbb{N}$ *and so belongs to* $\mathcal{F}(I)$ *as I is an admissible ideal. Hence* $\phi(h) \nrightarrow 0(DC^{I}_{p,q})$.

Theorem 2.8. *If a bounded* $\phi(h)$ *is* μ*-deferred I-statistically convergent to t, then* $\phi \in w(H)$ *is strongly I-deferred Cesàro summable to t.*

Proof. Let $\phi \in w(H)$ be μ-deferred I-statistically convergent to t. As $\phi(h)$ is bounded then $\exists D > 0$ such that $|\phi(h) - t| \leq D$, $\forall h \in H$. For a given $\epsilon > 0$, $\exists$ a positive real number r such that

$$\begin{aligned} \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - t| &= \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \left(\sum_{\substack{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} \\ |\phi(h)-t| \geq \frac{\epsilon}{2}}} |\phi(h) - t| + \sum_{\substack{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} \\ |\phi(h)-t| < \frac{\epsilon}{2}}} |\phi(h) - t| \right) \\ &\leq \frac{Dr}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon/2\}\| + \frac{\epsilon}{2}. \end{aligned}$$

So we get

$$\begin{aligned} &\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - t| \geq \epsilon\right\} \\ &\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \frac{\epsilon}{2Dr}\right\} \in I. \end{aligned}$$

Therefore, $\phi \in w(H)$ is strongly deferred Cesàro summable to t. □

Theorem 2.9. *If* $\liminf_k \frac{\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \neq 1$ *and* $\phi \in w(H)$ *is I-*${}_{\mu}$*statistically convergent to t, then* $\phi \in w(H)$ *is* μ*-deferred I-statistically convergent to t.*

Proof. Let $\liminf_k \frac{\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} = x(\neq 1)$, $\exists\ y > 0$ such that $\frac{\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \geq x + y$ for sufficiently large k. So we have

$$\frac{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \geq \frac{y}{x+y}. \tag{5}$$

As $\phi \in w(H)$ is I-${}_{\mu}$statistically convergent to t, then $\forall \epsilon, \delta > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\} \in I.$$

Therefore $\forall\epsilon, \delta > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\|}\|\{h \in T_{q_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\} \in I.$$

Since

$$\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\} \subseteq \{h \in T_{q_{(k)}} : |\phi(h) - t| \geq \epsilon\},$$

so we get

$$\begin{aligned} \frac{1}{\|T_{q_{(k)}}\|}\|\{h \in T_{q_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| &\geq \frac{1}{\|T_{q_{(k)}}\|}\|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \\ &= \frac{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \frac{\|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \\ &\geq \frac{y}{x + y} \frac{\|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}. \end{aligned}$$

Hence for any $\delta > 0$,

$$\begin{aligned} &\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}\|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\} \\ &\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\|}\|\{h \in T_{q_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \frac{\delta y}{x + y}\right\} \in I. \end{aligned}$$

Hence $\phi \in w(H)$ is μ-deferred I-statistically convergent to t. □

Theorem 2.10. *If* $\liminf_k \frac{\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \neq 1$ *and* $\phi \in w(H)$ *is I-strongly summable to t, then* $\phi \in w(H)$ *is strongly I-deferred Cesàro summable to t.*

Proof. Let $\liminf_k \frac{\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} = x(\neq 1)$, $\exists\, y > 0$ such that $\frac{\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \geq x + y$ for sufficiently large k. So we have

$$\frac{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|} \geq \frac{y}{x + y}. \tag{6}$$

As $\phi \in w(H)$ is I-strongly summable to t, then for all $\epsilon > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_k\|} \sum_{h \in T_k} |\phi(h) - t| \geq \epsilon\right\} \in I.$$

Therefore for all $\epsilon > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\|} \sum_{h \in T_{q_{(k)}}} |\phi(h) - t| \geq \epsilon\right\} \in I.$$

Since

$$\sum_{h \in T_{q_{(k)}}} |\phi(h) - t| \geq \sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|,$$

therefore we get

$$\begin{aligned}\frac{1}{\|T_{q_{(k)}}\|}\sum_{h\in T_{q_{(k)}}}|\phi(h)-t| &\geq \frac{1}{\|T_{q_{(k)}}\|}\sum_{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}}|\phi(h)-t| \\ &= \frac{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}{\|T_{q_{(k)}}\|}\frac{\sum_{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}}|\phi(h)-t|}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|} \\ &\geq \frac{y}{x+y}\frac{\sum_{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}}|\phi(h)-t|}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}.\end{aligned}$$

Hence for any $\epsilon > 0$,

$$\left\{k\in\mathbb{N}:\frac{1}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}\sum_{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}}|\phi(h)-t|\geq\epsilon\right\}$$
$$\subseteq\left\{k\in\mathbb{N}:\frac{1}{\|T_{q_{(k)}}\|}\sum_{h\in T_{q_{(k)}}}|\phi(h)-t|\geq\frac{\epsilon y}{x+y}\right\}\in I.$$

Hence $\phi\in w(H)$ is strongly I-deferred Cesàro summable to t. □

Theorem 2.11. *If* $\{p_{(k)}\}, \{q_{(k)}\}, \{r_{(k)}\}$ *and* $\{s_{(k)}\}$ *are sequences of non-negative integers such that* $\{p_{(k)}\}\leq\{r_{(k)}\}<\{s_{(k)}\}\leq\{q_{(k)}\}$ $\forall k\in\mathbb{N}$ *and*

$$\liminf_k \frac{\|T_{s_{(k)}}\|-\|T_{r_{(k)}}\|}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}>0,$$

then
(a) $\phi(h)\to t({}_{\mu}DS^I_{p,q}) \implies \phi(h)\to t({}_{\mu}DS^I_{r,s})$,
(b) $\phi(h)\to t(DC^I_{p,q}) \implies \phi(h)\to t(DC^I_{r,s})$.

Proof. Let $\liminf_k \frac{\|T_{s_{(k)}}\|-\|T_{r_{(k)}}\|}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}=x(>0)$, so there exists $y>0$ such that $\frac{\|T_{s_{(k)}}\|-\|T_{r_{(k)}}\|}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}\geq x+y$ for sufficiently large k.
(a) Let $\phi(h)\to t({}_{\mu}DS^I_{p,q})$, then for every $\epsilon,\delta>0$,

$$\left\{k\in\mathbb{N}:\frac{1}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}\|\{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}:|\phi(h)-t|\geq\epsilon\}\|\geq\delta\right\}\in I.$$

It is obvious that for any $\epsilon>0$,

$$\{h\in T_{s_{(k)}}\backslash T_{r_{(k)}}:|\phi(h)-t|\geq\epsilon\}\subset\{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}:|\phi(h)-t|\geq\epsilon\}.$$

So we have the following inequality

$$\begin{aligned}&\frac{1}{\|T_{s_{(k)}}\|-\|T_{r_{(k)}}\|}\|\{h\in T_{s_{(k)}}\backslash T_{r_{(k)}}:|\phi(h)-t|\geq\epsilon\}\| \\ &\leq\frac{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}{\|T_{s_{(k)}}\|-\|T_{r_{(k)}}\|}\frac{1}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}\|\{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}:|\phi(h)-t|\geq\epsilon\}\| \\ &\leq\frac{1}{x+y}\frac{1}{\|T_{q_{(k)}}\|-\|T_{p_{(k)}}\|}\|\{h\in T_{q_{(k)}}\backslash T_{p_{(k)}}:|\phi(h)-t|\geq\epsilon\}\|.\end{aligned}$$

Hence for any preassigned $\delta > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \|\{h \in T_{s_{(k)}} \backslash T_{r_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\}$$
$$\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta(x + y)\right\} \in I.$$

Therefore, $\phi(h) \to t({}_{\mu}DS^{I}_{r,s})$.

(b) Let $\phi(h) \to t(DC^{I}_{p,q})$, then for every $\epsilon > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| \geq \epsilon\right\} \in I.$$

From the inequality

$$\sum_{h \in T_{s_{(k)}} \backslash T_{r_{(k)}}} |\phi(h) - t| < \sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|,$$

we have the following inequality

$$\begin{aligned} \frac{1}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \sum_{h \in T_{s_{(k)}} \backslash T_{r_{(k)}}} |\phi(h) - t| &\leq \frac{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \frac{\sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \\ &\leq \frac{1}{x + y} \frac{\sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}. \end{aligned}$$

Hence for any given $\epsilon > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \sum_{h \in T_{s_{(k)}} \backslash T_{r_{(k)}}} |\phi(h) - t| \geq \epsilon\right\}$$
$$\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| \geq \epsilon(x + y)\right\} \in I.$$

Therefore, $\phi(h) \to t(DC^{I}_{r,s})$. □

Theorem 2.12. *If $\{p_{(k)}\}, \{q_{(k)}\}, \{r_{(k)}\}$ and $\{s_{(k)}\}$ be sequences of non-negative integers such that $\{p_{(k)}\} \leq \{r_{(k)}\} < \{s_{(k)}\} \leq \{q_{(k)}\}$, $\forall k \in \mathbb{N}$ and $\phi \in w(H)$ is such that $\phi(h) \to t({}_{\mu}DS^{I}_{p,r})$ and $\phi(h) \to t({}_{\mu}DS^{I}_{s,q})$, then $\phi(h) \to t({}_{\mu}DS^{I}_{p,q})$ and $\phi(h) \to t(DC^{I}_{p,q})$.*

Proof. Let $\phi(h) \to t({}_{\mu}DS^{I}_{r,s})$. For any $\epsilon > 0$,

$$\begin{aligned} \{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\} &= \{h \in T_{r_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\} \cup \{h \in T_{s_{(k)}} \backslash T_{r_{(k)}} : |\phi(h) - t| \geq \epsilon\} \\ &\cup \{h \in T_{q_{(k)}} \backslash T_{s_{(k)}} : |\phi(h) - t| \geq \epsilon\}. \end{aligned}$$

We get the following inequality

$$\begin{aligned} &\frac{\|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \\ &\leq \frac{\|\{h \in T_{r_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{r_{(k)}}\| - \|T_{p_{(k)}}\|} + \frac{\|\{h \in T_{s_{(k)}} \backslash T_{r_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \\ &+ \frac{\|\{h \in T_{q_{(k)}} \backslash T_{s_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{s_{(k)}}\|}. \end{aligned}$$

Hence for any $\delta > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\}$$
$$\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{r_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{r_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\}$$
$$\cup \left\{k \in \mathbb{N} : \frac{1}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \|\{h \in T_{s_{(k)}} \backslash T_{r_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\}$$
$$\cup \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{s_{(k)}}\|} \|\{h \in T_{q_{(k)}} \backslash T_{s_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\}.$$

Since the right handside set of above inclusion belongs to I, therefore $\phi(h) \to t({}_{\mu}DS^{I}_{p,q})$.
From the equality

$$\sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| = \sum_{h \in T_{r_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| + \sum_{h \in T_{s_{(k)}} \backslash T_{r_{(k)}}} |\phi(h) - t| + \sum_{h \in T_{q_{(k)}} \backslash T_{s_{(k)}}} |\phi(h) - t|,$$

we have the following inequality

$$\frac{\sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \leq \frac{\sum_{h \in T_{r_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t|}{\|T_{r_{(k)}}\| - \|T_{p_{(k)}}\|} + \frac{\sum_{h \in T_{s_{(k)}} \backslash T_{r_{(k)}}} |\phi(h) - t|}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} + \frac{\sum_{h \in T_{q_{(k)}} \backslash T_{s_{(k)}}} |\phi(h) - t|}{\|T_{q_{(k)}}| - |T_{s_{(k)}}\|}.$$

Hence for any $\epsilon > 0$,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| \geq \epsilon\right\}$$
$$\subseteq \left\{k \in \mathbb{N} : \frac{1}{\|T_{r_{(k)}}\| - \|T_{p_{(k)}}\|} \sum_{h \in T_{r_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - t| \geq \epsilon\right\}$$
$$\cup \left\{k \in \mathbb{N} : \frac{1}{\|T_{s_{(k)}}\| - \|T_{r_{(k)}}\|} \sum_{h \in T_{s_{(k)}} \backslash T_{r_{(k)}}} |\phi(h) - t| \geq \epsilon\right\}$$
$$\cup \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{s_{(k)}}\|} \sum_{h \in T_{q_{(k)}} \backslash T_{s_{(k)}}} |\phi(h) - t| \geq \epsilon\right\}.$$

Since the right handside set of above inclusion belongs to I, therefore $\phi(h) \to t(DC^{I}_{p,q})$. □

2.2. μ-deferred I^*–statistically convergence

Definition 2.13. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If ∃ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\lim_{\substack{k \to \infty \\ k \in M}} \frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \geq \epsilon\}\| = 0,$$

then we say $\phi \in w(H)$ is I^*–${}_{\mu}$statistically convergent to t for $\{T_k\}$ and we write $\phi(h) \to t({}_{\mu}S^{I^*})$.

Definition 2.14. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If ∃ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\lim_{\substack{k \to \infty \\ k \in M}} \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| = 0,$$

then we say $\phi \in w(H)$ is μ-deferred I^*–statistically convergent to t for $\{T_k\}$ and we write $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, q_k])$.

Theorem 2.15. *If $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, q_k])$, then $\phi(h) \to t({}_{\mu}DS^{I}_{p,q})$.*

Proof. Let a $\delta > 0$ is given and $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, q_k])$, then $\exists$ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,

$$\lim_{\substack{k\to\infty \\ k\in M}} \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| = 0.$$

Therefore $\exists k_0 \in \mathbb{N}$ such that for $k \geq k_0$, we have

$$\frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| < \delta$$

that is,

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\} \subset \{1, 2, 3, ...k_0 - 1\}.$$

Since I is an admissible ideal of $\mathbb{N}$, hence

$$\left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| \geq \delta\right\} \in I.$$

Therefore, $\phi(h) \to t({}_{\mu}DS^{I}_{p,q})$. □

Theorem 2.16. *If I satisfy property (AP) then, $\phi(h) \to t({}_{\mu}DS^{I}_{p,q})$ implies $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, q_k])$.*

Proof. We can prove it by applying the same method which is adapted in Theorem 2.28, hence proof is omitted. □

Theorem 2.17. *If $q_{(k)} = k$ $\forall k \in \mathbb{N}$, then $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, k])$ if and only if $\phi(h) \to t({}_{\mu}S^{I^*})$.*

Proof. Let $q_{(k)} = k$ $\forall k \in \mathbb{N}$ and $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, k])$, then $\exists$ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,

$$\lim_{\substack{k\to\infty \\ k\in M}} \frac{1}{\|T_k\| - \|T_{p_{(k)}}\|} \|\{h \in T_k \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\| = 0. \tag{7}$$

Assuming $p_{(k)} = k^{(1)}$, $p_{(k^{(1)})} = k^{(2)}$, $p_{(k^{(2)})} = k^{(3)}$,... . Hence we have

$$\{h \in T_k : |\phi(h) - t| \geq \epsilon\} = \{h \in T_{k^{(1)}} : |\phi(h) - t| \geq \epsilon\} \cup \{h \in T_k \setminus T_{k^{(1)}} : |\phi(h) - t| \geq \epsilon\}$$

$$\{h \in T_{k^{(1)}} : |\phi(h) - t| \geq \epsilon\} = \{h \in T_{k^{(2)}} : |\phi(h) - t| \geq \epsilon\} \cup \{h \in T_{k^{(2)}} \setminus T_{k^{(1)}} : |\phi(h) - t| \geq \epsilon\}$$

$$\{h \in T_{k^{(2)}} : |\phi(h) - t| \geq \epsilon\} = \{h \in T_{k^{(3)}} : |\phi(h) - t| \geq \epsilon\} \cup \{h \in T_{k^{(3)}} \setminus T_{k^{(2)}} : |\phi(h) - t| \geq \epsilon\}$$

$$\vdots$$

The above process run on until we get a $n \in \mathbb{N}$ which is depends on k, therefore

$$\{h \in T_{k^{(n-1)}} : |\phi(h) - t| \geq \epsilon\} = \{h \in T_{k^{(n)}} : |\phi(h) - t| \geq \epsilon\} \cup \{h \in T_{k^{(n)}} \setminus T_{k^{(n-1)}} : |\phi(h) - t| \geq \epsilon\}.$$

Here $k^{(n)} \geq 1$ and $k^{(n+1)} = 0$. Hence, it can be obtain that for every k,

$$\frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \geq \epsilon\}\| = \sum_{i=0}^{n} \frac{\|T_{k^{(i)}}\| - \|T_{k^{(i+1)}}\|}{\|T_k\|} . \frac{\|\{h \in T_{k^{(i+1)}} \setminus T_{k^{(i)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{k^{(i)}}\| - \|T_{k^{(i+1)}}\|}. \tag{8}$$

Now consider matrix (a_{ki}) which is defined as

$$a_{ki} = \begin{cases} \frac{\|T_{k^{(i)}}\| - \|T_{k^{(i+1)}}\|}{\|T_k\|}, & 0 \le i \le n \\ 0, & \text{otherwise} \end{cases}$$

here $k^{(0)} = k$.
From (8), it is obvious that the sequence

$$\left\{ \frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \ge \epsilon\}\| \right\}$$

is the (a_{ki}) transformation of the sequence

$$\left\{ \frac{\|\{h \in T_{k^{(i+1)}} \backslash T_{k^{(i)}} : |\phi(h) - t| \ge \epsilon\}\|}{\|T_{k^{(i)}}\| - \|T_{k^{(i+1)}}\|} \right\}.$$

Since $k^{(i)} > k^{(i+1)}, 0 \le i \le n$ and $k^{(n+1)} = 0$. For fixed j, $\frac{k^{(j)} - k^{(j+1)}}{k}$ is either zero or a fraction value in which the value of numerater is less than or eqaul to j and value of the denominator is k. So clearly, this transformation satisfies all the three properties (from Lemma 1.5), hence (a_{ki}) is a regular matrix. From (7), since for $M \in \mathcal{F}(I)$, the sequence

$$\left\{ \frac{\|\{h \in T_{k^{(i+1)}} \backslash T_{k^{(i)}} : |\phi(h) - t| \ge \epsilon\}\|}{\|T_{k^{(i)}}\| - \|T_{k^{(i+1)}}\|} \right\}.$$

is convergent to zero, so we have

$$\lim_{\substack{k \to \infty \\ k \in M}} \frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \ge \epsilon\}\| = 0.$$

Conversely, Let $\phi(h) \to t({}_{\mu}S^{I^*})$, then $\exists$ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,

$$\lim_{\substack{k \to \infty \\ k \in M}} \frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \ge \epsilon\}\| = 0. \tag{9}$$

Since

$$\{h \in T_k \backslash T_{p_{(k)}} : |\phi(h) - t| \ge \epsilon\} \subseteq \{h \in T_k : |\phi(h) - t| \ge \epsilon\},$$

we have the following inequality

$$\|\{h \in T_k \backslash T_{p_{(k)}} : |\phi(h) - t| \ge \epsilon\}\| \le \|\{h \in T_k : |\phi(h) - t| \ge \epsilon\}\|.$$

Hence

$$\frac{1}{\|T_k\| - \|T_{p_{(k)}}\|} \|\{h \in T_k \backslash T_{p_{(k)}} : |\phi(h) - t| \ge \epsilon\}\| \le \left(1 + \frac{\|T_{p_{(k)}}\|}{\|T_k\| - \|T_{p_{(k)}}\|}\right) \frac{1}{\|T_k\|} \|\{h \in T_k : |\phi(h) - t| \ge \epsilon\}\|.$$

Since $\left\{ \frac{\|T_{p_{(k)}}\|}{\|T_k\| - \|T_{p_{(k)}}\|} \right\}_{k \in \mathbb{N}}$ is bounded and from (9) we have

$$\lim_{\substack{k \to \infty \\ k \in M}} \frac{1}{\|T_k\| - \|T_{p_{(k)}}\|} \|\{h \in T_k \backslash T_{p_{(k)}} : |\phi(h) - t| \ge \epsilon\}\| = 0.$$

Hence $\phi(h) \to t({}_{\mu}DS^{I^*}[p_k, k])$. $\square$

2.3. μ-deferred I-statistically pre-Cauchy condition

Definition 2.18. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for any $\epsilon, \delta > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\left\{k \in \mathbb{N} : \frac{1}{||T_k||^2}||\{(h,l) \in T_k \times T_k : |\phi(h) - \phi(l)| \geq \epsilon\}|| \geq \delta\right\} \in I,$$

then we say $\phi \in w(H)$ is I-${}_\mu$statistically pre-Cauchy function for $\{T_k\}$.

Definition 2.19. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If for any $\epsilon, \delta > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\left\{k \in \mathbb{N} : \frac{1}{(||T_{q_{(k)}}|| - ||T_{p_{(k)}}||)^2}||\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}|| \geq \delta\right\} \in I,$$

then we say $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy function for $\{T_k\}$.

Theorem 2.20. *If $\phi \in w(H)$ is μ-deferred I-statistically convergent for any $\{T_k\}$, then $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy function for the same Folner sequence.*

Proof. Suppose $\phi(h) \to t(DS^I_{p,q})$ for any sequence $\{T_k\}$, then $\forall \epsilon, \delta > 0$,

$$M = \left\{k \in \mathbb{N} : \frac{1}{||T_{q_{(k)}}|| - ||T_{p_{(k)}}||}||\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \frac{\epsilon}{2}\}|| \geq \delta\right\} \in I.$$

So $\forall k \in M^c$,

$$\frac{1}{||T_{q_{(k)}}|| - ||T_{p_{(k)}}||}||\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| \geq \frac{\epsilon}{2}\}|| < \delta$$

i.e.,

$$\frac{1}{||T_{q_{(k)}}|| - ||T_{p_{(k)}}||}||\{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| < \frac{\epsilon}{2}\}|| > 1 - \delta, \ \forall k \in M^c.$$

Let $N = \{h \in T_{q_{(k)}} \backslash T_{p_{(k)}} : |\phi(h) - t| < \frac{\epsilon}{2}\}$, then for $h, l \in N$, we get

$$|\phi(h) - \phi(l)| \leq |\phi(h) - t| + |\phi(h) - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Consequently

$$N \times N \subset \{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| < \epsilon\}.$$

Hence we have the following inequality

$$\left[\frac{||N||^2}{(||T_{q_{(k)}}|| - ||T_{p_{(k)}}||)^2}\right] \leq \frac{||\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| < \epsilon\}||}{(||T_{q_{(k)}}|| - ||T_{p_{(k)}}||)^2}.$$

Therefore,

$$\frac{||\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| < \epsilon\}||}{(||T_{q_{(k)}}|| - ||T_{p_{(k)}}||)^2} \geq \left[\frac{||N||^2}{(||T_{q_{(k)}}|| - ||T_{p_{(k)}}||)^2}\right] > (1 - \delta)^2$$

i.e.,

$$\frac{||\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}||}{(||T_{q_{(k)}}|| - ||T_{p_{(k)}}||)^2} < 1 - (1 - \delta)^2.$$

Now for a preassigned $\delta' > 0$, we take $\delta > 0$ such that $\delta' > 1 - (1-\delta)^2$, hence

$$\frac{\|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} < \delta', \quad \forall k \in M^c.$$

Consequently,

$$\left\{k \in \mathbb{N} : \frac{\|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \delta'\right\} \subset M.$$

Since $M \in I$, we have

$$\left\{k \in \mathbb{N} : \frac{\|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \delta'\right\} \in I.$$

Therefore, $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy. □

Remark 2.21. *Converse of Theorem 2.20 is not true in general.*

Example 2.22. *Let $H = \mathbb{Z}$ and (p_k, q_k) is defined as in* (1)*. Take the Folner sequence*

$$\{T_k\} = \{h \in \mathbb{Z} : |h| \leq k\}$$

and for $s \in \mathbb{N}, h \in \mathbb{Z}$ such that $(s-1)! < |h| \leq s!$, set $\phi(h) = \sum_{i=1}^{s} \frac{1}{i}$ and for $h = 0$, $\phi(h) = 0$.
Clearly, ϕ is not μ-deferred I-statistically convergent.
But ϕ is μ-deferred I-statistically pre-Cauchy. For a given $\epsilon > 0$ and choose $s \in \mathbb{N}$ such that $\frac{1}{s} < \epsilon$.
Now note that if $s! < q_k - p_k \leq (s+1)!$ and $(s-1)! < |h|, |l| \leq q_k - p_k$, then $|\phi(h) - \phi(l)| \leq \frac{1}{s} < \epsilon$. Hence for $s! < q_k - p_k \leq (s+1)!$, we have the following inequality

$$\begin{aligned}
\frac{\|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| < \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} &\geq \frac{4[(q_k - p_k) - (s-1)!]^2}{4(q_k - p_k)^2} \\
&\geq \left[1 - \frac{(s-1)!}{s!}\right]^2 \\
&= \left[1 - \frac{1}{s}\right]^2.
\end{aligned}$$

So for any $\delta > 0$

$$\left\{k \in \mathbb{N} : \left|\frac{\|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| < \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} - 1\right| \geq \delta\right\} \subseteq \left\{s \in \mathbb{N} : \left|\left[1 - \frac{1}{s}\right]^2 - 1\right| \geq \delta\right\}.$$

In the above inclusion equation, the right-hand side set is finite so it belongs to I. Consequently,

$$\left\{k \in \mathbb{N} : \left|\frac{\|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| < \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} - 1\right| \geq \delta\right\} \in I.$$

Hence function ϕ is μ-deferred I-statistically pre-Cauchy.

Theorem 2.23. *Let $\phi \in w(H)$ be bounded function. $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy if and only if*

$$I - \lim_{k} \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \backslash T_{p_{(k)}}} |\phi(h) - \phi(l)| = 0. \tag{10}$$

Proof. Let (10) holds, then for any given $\epsilon > 0$ and $k \in \mathbb{N}$, we have the following inequality

$$\frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \geq \epsilon . \Big(\frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \Big)$$

Hence for any $\delta > 0$,

$$\Big\{ k \in \mathbb{N} : \frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \delta \Big\}$$

$$\subseteq \Big\{ k \in \mathbb{N} : \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \geq \delta\epsilon \Big\}.$$

Since (10) holds therefore in the above inclusion equation the right-hand side set belongs to I. Hence

$$\Big\{ k \in \mathbb{N} : \frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \delta \Big\} \in I.$$

Therefore, $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy.
For the converse part, suppose $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy. Since $\phi \in w(H)$ is bounded function then $\exists D > 0$ such that $|\phi(h)| \leq D$, $\forall h \in H$. For any given $\epsilon > 0$ and for each $k \in \mathbb{N}$, the following inequality holds

$$\frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \leq \frac{\epsilon}{2} + 2D \Big(\frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \frac{\epsilon}{2}\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \Big). \tag{11}$$

Since $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy so for any $\delta > 0$,

$$M = \Big\{ k \in \mathbb{N} : \frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \frac{\epsilon}{2}\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \delta \Big\} \in I$$

i.e.,

$$\frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \frac{\epsilon}{2}\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} < \delta, \ \ \forall k \in M^c.$$

So from equation(11), we get

$$\frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \leq \frac{\epsilon}{2} + 2D\delta.$$

For a preassigned $\delta' > 0$, choose $\epsilon, \delta > 0$ such that $\frac{\epsilon}{2} + 2D\delta < \delta'$. So we have

$$\frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \leq \delta', \ \forall k \in M^c.$$

That is,

$$\Big\{ k \in \mathbb{N} : \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \geq \delta' \Big\} \subset M.$$

Since $M \in I$, the following set

$$\left\{k \in \mathbb{N} : \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \sum_{h,l \in T_{q_{(k)}} \setminus T_{p_{(k)}}} |\phi(h) - \phi(l)| \geq \delta'\right\} \in I.$$

Hence (10) holds. □

Theorem 2.24. *Let $\phi \in w(H)$ be μ-deferred I-statistically pre-Cauchy function for any Folner sequence $\{T_k\}$. If $\exists M \subset H$ such that $\|M\| = \infty$ and ϕ is μ-deferred I-statistically convergent to t with respect to M and*

$$0 < I - \liminf_k \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : h \in M\}\| < \infty,$$

then $\phi \in w(H)$ is μ-deferred I-statistically convergent to t.

Proof. Since ϕ is μ-deferred I-statistically convergent to t with respect to M then for all $\epsilon > 0$, $\exists k_0 \in \mathbb{N}$ such that

$$|\phi(h) - t| < \epsilon, \ \ \forall m > k_0 \text{ and } h \in M \setminus T_m.$$

Suppose $P = \{h : m > k_0 \text{ and } h \in M \setminus T_m\}$ and $P(\epsilon) = \{h : |\phi(h) - t| \geq \epsilon\}$, so we get the following inequality

$$\frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \frac{\|\{h \in P : h \in T_{q_{(k)}} \setminus T_{p_{(k)}}\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \cdot \frac{\|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|}.$$

Since $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy function so for any $\delta > 0$,

$$Q = \left\{k \in \mathbb{N} : \frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \geq \delta\right\} \in I.$$

i.e.,

$$\frac{\|\{(h,l) \in (T_{q_{(k)}} \setminus T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\|}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} < \delta, \ \ \forall k \in Q^c.$$

Let $I - \liminf_k \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : h \in M\}\| = a > 0$, then

$$A = \left\{k \in \mathbb{N} : \frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : h \in M\}\| < \frac{a}{2}\right\} \in I.$$

i.e.,

$$\frac{1}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : h \in M\}\| \geq \frac{a}{2}, \ \forall k \in A^c.$$

Hence for all $k \in A^c \cup Q^c = (A \cup Q)^c$,

$$\frac{\|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} < \frac{2\delta}{a}.$$

For a preassigned $\delta' > 0$, choose $\delta > 0$ such that $\frac{2\delta}{a} < \delta'$. So we have for all $k \in (A \cup Q)^c$,

$$\frac{\|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} < \delta'.$$

Therefore, we conclude that

$$\left\{k \in \mathbb{N} : \frac{\|\{h \in T_{q_{(k)}} \setminus T_{p_{(k)}} : |\phi(h) - t| \geq \epsilon\}\|}{\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|} \geq \delta'\right\} \subset (A \cup Q) \in I.$$

Hence $\phi \in w(H)$ is μ-deferred I-statistically convergent to t. □

2.4. μ-deferred I^*-statistically pre-Cauchy condition

Definition 2.25. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If $\exists$ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\lim_{\substack{k\to\infty \\ k\in M}} \frac{1}{\|T_k\|^2} \|\{(h,l) \in T_k \times T_k : |\phi(h) - \phi(l)| \geq \epsilon\}\| = 0,$$

then we say $\phi \in w(H)$ is I^*-${}_\mu$statistically pre-Cauchy function for $\{T_k\}$.

Definition 2.26. *Let H be a discrete countable amenable semigroup with identity and both laws of cancellation (right and left) hold in H. If $\exists$ a set $M \in \mathcal{F}(I)$, $\forall \epsilon > 0$ and for any Folner sequence $\{T_k\}$ for H,*

$$\lim_{\substack{k\to\infty \\ k\in M}} \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| = 0,$$

then we say $\phi \in w(H)$ is μ-deferred I^*-statistically pre-Cauchy function for $\{T_k\}$.

Theorem 2.27. *If $\phi \in w(H)$ is μ-deferred I^*-statistically pre-Cauchy function, then $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy function.*

Proof. We can prove it by applying the same method which is adapted in Theorem 2.15, hence proof is omitted. □

Theorem 2.28. *If I satisfy property (AP) and $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy function, then $\phi \in w(H)$ is μ-deferred I^*-statistically pre-Cauchy function.*

Proof. Let I satisfy property (AP) and $\phi \in w(H)$ is μ-deferred I-statistically pre-Cauchy function, then we have for any $\epsilon, \delta > 0$ and for any Folner sequence $\{T_k\}$ for H,

$$\left\{k \in \mathbb{N} : \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| \geq \delta\right\} \in I.$$

Now for $a \in \mathbb{N}$, we define

$$P_a = \left\{k \in \mathbb{N} : \frac{1}{a+1} < \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| \leq \frac{1}{a}\right\}.$$

Now it is clear that $\{P_1, P_2, ...\}$ is a countable family of mutually disjoint sets belonging to I and therefore by the condition (AP) there is a countable family of sets $\{Q_1, Q_2, ...\}$ in I such that $P_i \triangle Q_i$ is a finite set for each $i \in \mathbb{N}$ and $Q = \cup_{i=1}^{\infty} Q_i$. Since $Q \in I$ so by definition of associate filter $\mathcal{F}(I)$ there is set $M \in \mathcal{F}(I)$ such that $M = \mathbb{N} \backslash Q$.

Now let $\eta > 0$ and choose a natural number b such that $\frac{1}{b} < \eta$, then we have the following inclusion

$$\left\{k \in \mathbb{N} : \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| \geq \eta\right\}$$
$$\subset \left\{k \in \mathbb{N} : \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| \geq \frac{1}{b}\right\} \subset \cup_{i=1}^{b+1} P_i.$$

Since $P_i \triangle Q_i$ is a finite set for each $i = 1, 2, .., b+1$, $\exists$ a $k_0 \in \mathbb{N}$ such that

$$(\cup_{i=1}^{b+1} Q_i) \cap \{k \in \mathbb{N} : k \geq k_0\} = (\cup_{i=1}^{b+1} P_i) \cap \{k \in \mathbb{N} : k \geq k_0\}.$$

If $k \geq k_0$ and $k \in M$, then $k \notin Q$. Consequently, $k \notin \cup_{i=1}^{b+1} Q_i$ and therefore $k \notin \cup_{i=1}^{b+1} P_i$. Hence $\forall k \geq k_0$ and $k \in M$, we have

$$\frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| < \eta.$$

Therefore, we conclude that

$$\lim_{\substack{k\to\infty \\ k\in M}} \frac{1}{(\|T_{q_{(k)}}\| - \|T_{p_{(k)}}\|)^2} \|\{(h,l) \in (T_{q_{(k)}} \backslash T_{p_{(k)}})^2 : |\phi(h) - \phi(l)| \geq \epsilon\}\| = 0.$$

□

Conclusion

In this research work we introduced and studied some new notions in amenable semigroups, that is, we presented strongly I-deferred Cesàro summability and μ-deferred I-statistical convergence on amenable semigroups. Also we explore relationships between them. After that we presented μ-deferred I^*-statistical convergent, μ-deferred I-statistically pre-Cauchy, and μ-deferred I^*-statistically pre-Cauchy functions in amenable semigroups and proved results based on connections among them. The results obtained in this paper are more unified and generalized and also yields novel tools to arrange and solve some problem of sequence convergence in numerous fields of science and engineering.

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