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Positive Solutions of Nonlinear Matrix Equations via Fixed Point Results in Relational Metric Spaces with w-Distance

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Abstract. We consider the non-linear matrix equation (NME) of the form $\mathcal{U} = Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U})\mathcal{A}_i$, where Q is an $n\times n$ Hermitian positive definite matrix, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ are $n\times n$ matrices, and $\hbar$ is a non-linear self-mapping of the set of all Hermitian matrices which are continuous in the trace norm. We discuss sufficient conditions ensuring the existence of a unique positive definite solution of the given NME. In order to do this, we introduce Θ_w-contractive conditions involving modified simulation functions in relational metric spaces and derive fixed points results based on them, followed by two suitable examples. In order to demonstrate the obtained conditions, we consider three different sets of matrices. Three different types of examples (including randomly generated matrix and a complex matrix) are given, together with convergence and error analysis, as well as average CPU time analysis with different dimensions bar graphs, and visualization of solutions in surface plot.

1. Introduction and preliminaries

1.1. *Positive definite solutions of NMEs*

The study of nonlinear matrix equations (NMEs) appeared first in the literature concerned with algebraic Riccati equations. These equations occur in large number of problems in control theory, dynamical programming, ladder network, stochastic filtering, queuing theory, statistics and many other applicable areas.

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n\times n$ Hermitian (resp. positive semi-definite, positive definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ the set of all $n\times n$ matrices over $\mathbb{C}$. In [16], Ran and Reurings discussed the existence of solutions of the equation

$$\mathcal{U} + \mathcal{B}^*\hbar(\mathcal{U})\mathcal{B} = Q \tag{1}$$

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in $\mathcal{K}(n)$, where $\mathcal{B} \in \mathcal{M}(n)$, $\mathcal{Q}$ is positive definite and $\hbar$ is a mapping from $\mathcal{K}(n)$ into $\mathcal{M}(n)$. Note that $\mathcal{U}$ is a solution of (1) if and only if it is a fixed point of the mapping $\mathcal{G}(\mathcal{U}) = \mathcal{Q} - \mathcal{B}^*\hbar(\mathcal{U})\mathcal{B}$. In [17], Ran and Reurings used the notion of partial ordering and established a modification of Banach Contraction Principle, which they applied for solving a class of NMEs of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^*\hbar(\mathcal{U})\mathcal{B}_i$ using the Ky Fan norm in $\mathcal{M}(n)$.

Theorem 1.1. [17] *Let $\hbar : \mathcal{H}(n) \to \mathcal{H}(n)$ be an order-preserving, continuous mapping which maps $\mathcal{P}(n)$ into itself and $\mathcal{Q} \in \mathcal{P}(n)$. If $\mathcal{B}_i, \mathcal{B}_i^* \in \mathcal{P}(n)$ and $\sum_{i=1}^{k} \mathcal{B}_i\mathcal{B}_i^* < M \cdot \mathcal{I}_n$ for some $M > 0$ ($\mathcal{I}_n$ – the unit matrix in $\mathcal{M}(n)$) and if $|\operatorname{tr}(\hbar(\mathcal{V}) - \hbar(\mathcal{U}))| \leq \frac{1}{M}|\operatorname{tr}(\mathcal{Y} - \mathcal{X})|$, for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(n)$ with $\mathcal{U} \leq \mathcal{V}$, then the equation $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^*\hbar(\mathcal{U})\mathcal{B}_i$ has a unique positive definite solution (PDS).*

In [21], Sawangsup and Sintunavarat studied the NME of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^*\hbar(\mathcal{U})\mathcal{B}_i$ using the spectral norm of a matrix, and applied a generalized contraction condition in metric spaces endowed with a transitive binary relation; they also tested numerically its approximate solutions. In the papers [2, 8, 9], the authors discussed PDSs of a pair of NMEs. Recently, in [5], Garai and Dey obtained sufficient conditions for the existence and uniqueness of solution for a system of NMEs, using common fixed point results in Banach spaces under conditions using a pair of altering distance functions.

1.2. Relational metric spaces

It is well-known that results of metrical fixed point theory can be applied for solving various nonlinear problems in different areas. These results use various generalized contractive conditions for operators acting in several kinds of generalized metric spaces. In this paper, we will consider so-called relational metric spaces with additional w-distance and contractive conditions formulated in terms of so-called simulation functions.

Throughout this article, the notations $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ have their usual meanings. We recall the following notions.

Definition 1.2. *Let X be a non-empty set and $\mathcal{R}$ be a binary relation defined on X.*

1. [14] *The relation $\mathcal{R}$ is said to be complete if for all $x, y \in X$, $[x, y] \in \mathcal{R}$, where $[x, y] \in \mathcal{R}$ means that either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$.*
2. [1] *The symmetric closure of $\mathcal{R}$ is defined by $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.*
3. [1] *A sequence $\{x_n\}$ in X is said to be $\mathcal{R}$-preserving if*

$$(x_n, x_{n+1}) \in \mathcal{R}, \ \forall n \in \mathbb{N} \cup \{0\}.$$

4. [20] *A subset E of X is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in E$ such that $(z, x) \in \mathcal{R}$ and $(z, y) \in \mathcal{R}$.*
5. [13] *For $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, \ldots, z_k\} \subset X$ satisfying the following conditions:*
   - *(i)* $z_0 = x$ *and* $z_k = y$,
   - *(ii)* $(z_i, z_{i+1}) \in \mathcal{R}$ *for each* i $(0 \leq i \leq k-1)$.

Definition 1.3. *Let X be a non-empty set, $\mathcal{R}$ be a binary relation defined on X, and let $\mathcal{T}$ be a self-map defined on X.*

1. [1] *The relation $\mathcal{R}$ is said to be $\mathcal{T}$-closed if $(x, y) \in \mathcal{R} \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in \mathcal{R}$.*
2. [10] *The relation $\mathcal{R}$ is said to be $\mathcal{T}$-*orbitally transitive *if it is transitive on $O(x; \mathcal{T})$ for all $x \in X$, where $O(x; \mathcal{T}) = \{\mathcal{T}^n x : n = 0, 1, 2, \ldots\}$ is the orbit of $\mathcal{T}$ at the point $x \in X$.*

Let X be a nonempty set. As has become standard, $(X, d, \mathcal{R})$ will be called a relational metric space if

(i) (X, d) is a metric space and
(ii) $\mathcal{R}$ is a binary relation on X.

Definition 1.4. *Let $(X, d, \mathcal{R})$ be a relational metric space, and let $\mathcal{T}$ be a self-map defined on X.*

1. [1] *The space (X, d) is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in X.*
2. [1] *The relation $\mathcal{R}$ is said to be d-self-closed if for every $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \to x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N} \cup \{0\}$.*
3. [1] *The mapping $\mathcal{T}$ is said to be $\mathcal{R}$-continuous at $x \in X$ if for every $\mathcal{R}$-preserving sequence $\{x_n\}$ converging to x, we have*
$$\mathcal{T}x_n \to \mathcal{T}x \text{ as } n \to \infty.$$
4. [10] *The mapping $\mathcal{T}$ is said to be orbitally $\mathcal{R}$-continuous at a point z in X if for any $\mathcal{R}$-preserving sequence $\{x_n\} \subset O(x; \mathcal{T})$ (for some $x \in X$), $x_n \to z$ as $n \to \infty$ implies $\mathcal{T}x_n \to \mathcal{T}z$ as $n \to \infty$.*
5. [22] *The mapping $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be $\mathcal{R}$-lower semicontinuous ($\mathcal{R}$-LSC, for short) at x if for every $\mathcal{R}$-preserving sequence $\{x_n\}$ converging to x, we have*
$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

Remark 1.5.

1. [10] Transitivity $\Rightarrow$ $\mathcal{T}$-orbital transitivity; the converse is not true.
2. A path of length k involves $k + 1$ elements of X, although they are not necessarily distinct.
3. [10] The following implications are obvious:
$$\begin{array}{ccc} \text{Continuity} & \Longrightarrow & \text{orbital continuity} \\ \Downarrow & & \Downarrow \\ \mathcal{R}\text{-continuity} & \Longrightarrow & \text{orbital } \mathcal{R}\text{-continuity.} \end{array}$$
4. [22] Every lower semi-continuous function is $\mathcal{R}$-LSC, but the converse is not true. If $\mathcal{R}$ is the universal relation, then these two notions coincide.

We shall also need the following notions.

Definition 1.6. *Let $(X, d, \mathcal{R})$ be a relational metric space, and let $\mathcal{T}$ be a self-map defined on X.*

1. *The relation $\mathcal{R}$ is said to be $\mathcal{T}$-orbitally closed at $z \in X$ if $(x, y) \in \mathcal{R} \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in \mathcal{R}$, for all $x, y \in O(z; \mathcal{T})$.*
2. *The space $(X, d, \mathcal{R})$ is said to be $\mathcal{T}$-orbitally $\mathcal{R}$-complete at $x \in X$ if every $\mathcal{R}$-preserving Cauchy sequence contained in $O(x; \mathcal{T})$ converges in X.*

Remark 1.7.

1. Every $\mathcal{T}$-closed relation is $\mathcal{T}$-orbitally closed at each point, but the converse is not true.
2. Every complete relational metric space is $\mathcal{T}$-orbitally complete for any $\mathcal{T}$, and every $\mathcal{T}$-orbitally complete space is $\mathcal{T}$-orbitally $\mathcal{R}$-complete, but the converses are not true.

Example 1.8. Let $X = [0, 1]$ be equipped with the standard metric d and let the relation $\mathcal{R}$ be defined on X by

$$(x, y) \in \mathcal{R} \iff x, y > 0 \ \vee \ (x, y) \in \left\{(0, 0), (0, \frac{1}{5})\right\} \cup \left\{(0, \frac{1}{5^n}) \mid n \geq 3\right\}.$$

Consider the self-mapping $\mathcal{T}$ on X be given by $\mathcal{T}x = \frac{x}{5}$. Take $x_0 = \frac{1}{5}$. Then

$$O(x_0; \mathcal{T}) = \left\{\frac{1}{5^n} \mid n \in \mathbb{N}\right\}, \quad \overline{O(x_0; \mathcal{T})} = O(x_0; \mathcal{T}) \cup \{0\} \subset [0, \frac{1}{5}].$$

Then $\mathcal{R}$ is $\mathcal{T}$-orbitally closed, and $\mathcal{R}$ is $\mathcal{T}$-orbitally transitive but it is neither $\mathcal{T}$-closed nor transitive. To see this, observe that

$$(x, y) = (0, \frac{1}{5}) \in \mathcal{R} \text{ but } (\mathcal{T}x, \mathcal{T}y) = (0, \frac{1}{25}) \notin \mathcal{R}$$

and

$$(0, \frac{1}{5}), (\frac{1}{5}, \frac{1}{25}) \in \mathcal{R} \text{ but } (0, \frac{1}{25}) \notin \mathcal{R}.$$

Also, $(X, d, \mathcal{R})$ is $\mathcal{T}$-orbitally $\mathcal{R}$-complete at x_0.

We are going to use the following notations:

(i) $F(\mathcal{T})$:= the set of all fixed points of $\mathcal{T}$,
(ii) $\mathcal{R}(\mathcal{T}) := \{x \in X : (x, \mathcal{T}x) \in \mathcal{R}$ and $(\mathcal{T}x, x) \in \mathcal{R}\}$,
(iii) $\mathcal{Y}(x, y, E, \mathcal{R})$:= the class of all $\mathcal{R}$-paths in E from x to y, where $E \subseteq X$.

A new type of control functions, named as simulation functions has been designed by Khojasteh et al. [12] and later slightly modified and enlarged by Roldán-Lopez-de-Hierro et al. [19]. Very recently, Hazarika et al. [7] modified this notion introducing the notion of modified simulation function.

Definition 1.9. [7] *The set of modified simulation functions Θ is a class of functions $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following conditions:*

(θ_1) $\theta(\xi, \zeta) < \zeta - \xi$, *for all* $\zeta, \xi > 0$;

(θ_2) *if* $\{\xi_n\}$, $\{\zeta_n\}$ *are sequences in* $(0, +\infty)$ *such that* $\lim_{n\to\infty} \xi_n = \alpha > 0$ *and* $\lim_{n\to\infty} \zeta_n = \beta > 0$, *then* $\limsup_{n\to\infty} \theta(\xi_n, \zeta_n) < \beta - \alpha$.

In Section 2 of this paper, we consider relational metric spaces endowed with additional w-distance. Θ_w-contractive conditions involving modified simulation functions in these spaces are introduced and fixed points results, based on them, are obtained. Several special cases are considered in Section 3, together with suitable examples, illustrating the obtained results.

Applications of the obtained results to non-linear matrix equations are considered in Section 4. We discuss sufficient conditions ensuring the existence of a unique positive definite solution of the NMEs of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i$, where $\mathcal{Q}$ is an $n \times n$ Hermitian positive definite matrix, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are $n \times n$ matrices, and $\hbar$ is a non-linear self-mapping of the set of all Hermitian matrices which are continuous in the trace norm.

In order to demonstrate the obtained results, we consider three different types of matrices in Section 5. This includes convergence and error analysis, as well as average CPU time analysis with different dimensions bar graphs, and visualization of solutions in surface plot.

2. Relational metric spaces with w-distance

In 1996, Kada et al. [11] introduced the concept of w-distance on a metric space and proved a generalized Caristi fixed point theorem, Ekeland's ϵ-variational principle and the non-convex minimization theorem, according to Mizoguchi and Takahashi [15]. Senapati and Dey [22] presented a modified version of w-distance function. The corresponding definitions and lemmas, in the setting of metric spaces endowed with an arbitrary binary relation $\mathcal{R}$, are as follows:

Definition 2.1. [22] *Let $(X, d, \mathcal{R})$ be a relational metric space. A function $w\colon X \times X \to [0, +\infty)$ is called a w-distance on X if it satisfies the following properties:*

(W1) $w(x, z) \le w(x, y) + w(y, z)$ *for any* $x, y, z \in X$;
(W2′) *w is $\mathcal{R}$-LSC in its second variable; i.e., if $x \in X$ and $y_n \to y \in X$ such that $y_n \mathcal{R} y_{n+1}$, then* $w(x, y) \le \liminf_{n\to\infty} w(x, y_n)$;
(W3) *for each $\epsilon > 0$, there exists a $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$ imply $d(x, y) \le \epsilon$.*

The following lemma is a modified version of Kada et al. [11], due to Senapati and Dey [22].

Lemma 2.2. [22] *Let $(X, d, \mathcal{R})$ be a relational metric space and let w be a w-distance on X. Suppose that $\{x_n\}$ and $\{y_n\}$ are $\mathcal{R}$-preserving sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in X$. Then the following assertions hold:*

(i) if $w(x_n, y) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for all $n \in \mathbb{N}$, then $y = z$, particularly, if $w(x, y) = w(x, z) = 0$, then $y = z$,
(ii) if $w(x_n, y_n) \le \alpha_n$ and $w(x_n, y) \le \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to y,

(iii) if $w(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence,
(iv) if $w(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.3. [11, 23]. *Let w be a w-distance on a metric space (X, d) and $\{x_n\}$ be a sequence in X such that for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $m > n > N_\epsilon$ implies $w(x_n, x_m) < \epsilon$, i.e., $\lim_{m,n\to\infty} w(x_n, x_m) = 0$. Then $\{x_n\}$ is a Cauchy sequence.*

Definition 2.4. *Let $(X, d, \mathcal{R})$ be a relational metric space with w-distance w and $\mathcal{T}: X \to X$ be a given mapping. We say that $\mathcal{T}$ is a Θ_w-contractive mapping, if there exists a function $\theta \in \Theta$ such that*

$$\theta(w(\mathcal{T}x, \mathcal{T}y), \max\{w(x, y), w(x, \mathcal{T}x), w(y, \mathcal{T}y)\}) \geq 0, \tag{2}$$

for all $(x, y) \in \mathcal{R}$.

If (2) is satisfied for $x, y \in \overline{O(x_0; \mathcal{T})}$ (for some $x_0 \in X$), we say that $\mathcal{T}$ is an orbitally Θ_w-contractive mapping (at x_0).

Now, we are equipped to state and prove our first main result as follows:

Theorem 2.5. *Let $(X, d, \mathcal{R})$ be a relational metric space with w-distance w and $\mathcal{T}: X \to X$. Suppose that the following conditions hold:*

(i) there exists an $x_0 \in \mathcal{R}(\mathcal{T})$;
(ii) $\mathcal{R}$ is $\mathcal{T}$-orbitally closed and $\mathcal{T}$-orbitally transitive;
(iii) $(X, d, \mathcal{R})$ is $\mathcal{T}$-orbitally $\mathcal{R}$-complete at x_0;
(iv) $\mathcal{T}$ is an orbitally Θ_w-contractive mapping;
(v) $\mathcal{T}$ is orbitally $\mathcal{R}$-continuous.

Then there exists a point $u \in F(\mathcal{T})$. In addition, $w(u, u) = 0$.

Proof. Let $x_0 \in \mathcal{R}(\mathcal{T})$ be a point as given in (i). If $\mathcal{T}^n x_0 = \mathcal{T}^{n+1} x_0$ for some $n \in \mathbb{N} \cup \{0\}$, then there is nothing to prove. Construct the sequence $\{x_n\}$ of Picard iterates $x_n = \mathcal{T}^n(x_0)$ for all $n \in \mathbb{N} \cup \{0\}$.

Using (i)-(ii), we have that $(\mathcal{T}x_0, \mathcal{T}^2 x_0) \in \mathcal{R}$. Continuing this process inductively, we obtain

$$(\mathcal{T}^n x_0, \mathcal{T}^{n+1} x_0) \in \mathcal{R} \tag{3}$$

for any $n \in \mathbb{N} \cup \{0\}$. Hence, $\{x_n\}$ is an $\mathcal{R}$-preserving sequence.

Next, we show that

$$\lim_{n\to\infty} w\left(\mathcal{T}^n x_0, \mathcal{T}^{n+1} x_0\right) = 0. \tag{4}$$

Denote $\Lambda_n = w\left(\mathcal{T}^n x_0, \mathcal{T}^{n+1} x_0\right)$ for all $n \in \mathbb{N} \cup \{0\}$. Now, observe that

$$\begin{aligned} 0 &\leq \theta(w(\mathcal{T}^n x_0, \mathcal{T}^{n+1} x_0), \\ &\qquad \max\{w(\mathcal{T}^{n-1} x_0, \mathcal{T}^n x_0), w(\mathcal{T}^{n-1} x_0, \mathcal{T}^n x_0), w(\mathcal{T}^n x_0, \mathcal{T}^{n+1} x_0)\}) \\ &= \theta(\Lambda_n, \max\{\Lambda_{n-1}, \Lambda_n\}). \end{aligned} \tag{5}$$

We shall show that $\{\Lambda_n\}$ is a nonincreasing sequence. Indeed, if $\Lambda_{n-1} < \Lambda_n$ for some $n \in \mathbb{N}$, then (5) would imply that

$$0 \leq \theta(\Lambda_n, \Lambda_n) < \Lambda_n - \Lambda_n = 0,$$

a contradiction. Therefore, $\{\Lambda_n\}$ is a nonincreasing sequence of positive real numbers. Hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} \Lambda_n = \lim_{n\to\infty} w(\mathcal{T}^n x_0, \mathcal{T}^{n+1} x_0) = r.$$

If $r > 0$, then, using the condition (θ_2), we would have

$$0 \leq \limsup_{n\to\infty} \theta(\Lambda_n, \Lambda_{n-1}) < r - r = 0,$$

a contradiction. Thus, we conclude that $\lim_{n\to\infty} \Lambda_n = 0$, which establishes (4).

Similarly, from $(\mathcal{T}x_0, x_0) \in \mathcal{R}$ and using condition (ii), we get $(x_{n+1}, x_n) \in \mathcal{R}$ for all $n \in \mathbb{N}^*$. Using this conclusion and the above arguments, it can be shown that

$$\lim_{n\to\infty} w(\mathcal{T}^{n+1}x_0, \mathcal{T}^n x_0) = 0. \tag{6}$$

Next, we show that $\{\mathcal{T}^n x_0\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{T})$. Suppose that this is not the case – then, by Lemma 2.3, the relation

$$\lim_{m,n\to\infty} w(\mathcal{T}^n x_0, \mathcal{T}^m x_0) = 0 \tag{7}$$

does not hold. It follows that we can find a $\delta > 0$ and increasing sequences $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ of positive integers with $m_k > n_k$ such that

$$w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0) \geq \delta, \text{ for all } k \in \{1, 2, 3, \cdots\}. \tag{8}$$

By (4), there exists a $k_0 \in \mathbb{N}$, such that $n_k > k_0$ implies that

$$w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{n_k+1}x_0) < \delta.$$

In view of the two last inequalities, we observe that $m_k \neq n_{k+1}$. We may assume that m_k is the minimal index such that (8) holds, so that

$$w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{r}x_0) < \delta, \text{ for } r \in \{n_{k+1}, n_{k+2}, \ldots, m_k - 1\}.$$

Now, making use of (8), we get

$$\begin{aligned} 0 < \delta \leq w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0) &\leq w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k-1}x_0) + w(\mathcal{T}^{m_k-1}x_0, \mathcal{T}^{m_k}x_0) \\ &< \delta + w(\mathcal{T}^{m_k-1}x_0, \mathcal{T}^{m_k}x_0). \end{aligned}$$

Thus,

$$\lim_{k\to\infty} w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0) = \delta. \tag{9}$$

Using the triangle inequality, we have

$$\begin{aligned} w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0) &\leq w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k+1}x_0) + w(\mathcal{T}^{m_k+1}x_0, \mathcal{T}^{m_k}x_0) \\ &\leq w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{n_k+1}x_0) + w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0) + w(\mathcal{T}^{m_k+1}x_0, \mathcal{T}^{m_k}x_0). \end{aligned}$$

Taking the limit on both sides and making use of (4), (6) and (9), we obtain

$$\lim_{k\to\infty} w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0) \geq \delta. \tag{10}$$

Again, using the triangle inequality, we have

$$\begin{aligned} w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0) &\leq w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{n_k}x_0) + w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k+1}x_0) \\ &\leq w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{n_k}x_0) + w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0) + w(\mathcal{T}^{m_k}x_0, \mathcal{T}^{m_k+1}x_0). \end{aligned}$$

Taking the limit on both sides and making use of (4), (6) and (9), we obtain

$$\lim_{k\to\infty} w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0) \leq \delta. \tag{11}$$

Combining (10) and (11), we have

$$\lim_{k\to\infty} w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0) = \delta. \tag{12}$$

Now, since $\mathcal{R}$ is $\mathcal{T}$-orbitally transitive and since $\{x_n\} \subseteq O(x_0;\mathcal{T})$, therefore we must have $(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0) \in \mathcal{R}$, for all $r \in \mathbb{N}$. Denote $\xi_k = w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0)$ and $\zeta_k = \max\{w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0), w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{n_k+1}x_0), w(\mathcal{T}^{m_k}x_0, \mathcal{T}^{m_k+1}x_0)\}$. Applying condition (2), we get

$$0 \le \theta\Big(w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0), \max\{w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0), w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{n_k+1}x_0), w(\mathcal{T}^{m_k}x_0, \mathcal{T}^{m_k+1}x_0)\Big).$$

Taking the limit on both sides and using (9), (12) and (θ_2), we have

$$\begin{aligned} 0 &\le \limsup_{k\to\infty} \theta\Big(w(\mathcal{T}^{n_k+1}x_0, \mathcal{T}^{m_k+1}x_0), \max\{w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{m_k}x_0), w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{n_k+1}x_0), w(\mathcal{T}^{m_k}x_0, \mathcal{T}^{m_k+1}x_0)\Big) \\ &< \delta - \delta = 0, \end{aligned}$$

a contradiction. Hence, $\{\mathcal{T}^n x_0\}$ must be a Cauchy sequence in $O(x_0;\mathcal{T})$.

Since $(X, d, \mathcal{R})$ is $\mathcal{T}$-orbitally $\mathcal{R}$-complete, there exists a point $u \in X$ such that $\lim_{n\to\infty} \mathcal{T}^n x_0 = u$. We shall show that u is a fixed point of $\mathcal{T}$.

Using the orbital $\mathcal{R}$-continuity of $\mathcal{T}$ (due to the condition (v)), we have $\lim_{n\to\infty} \mathcal{T}\mathcal{T}^n x_0 = \mathcal{T}u$. Owing to the uniqueness of the limit, we obtain $\mathcal{T}u = u$.

Finally, assume that $w(u,u) > 0$. Then, putting $x = y = u$ in (2), we have

$$\begin{aligned} 0 &\le \theta(w(u,u), \max\{w(u,u), w(u,u), w(u,u)\}) \\ &= \theta(w(u,u), w(u,u)) < w(u,u) - w(u,u) = 0, \end{aligned}$$

a contradiction. Therefore, $w(u,u) = 0$. □

Next, we have the following result.

Theorem 2.6. *The conclusion of Theorem 2.5 remains true if the condition* (v) *is replaced by the following one:*

(v′) for every $y \in X$ with $y \ne \mathcal{T}y$, $\inf\{w(x,y) + w(x,\mathcal{T}x) \mid x \in X\} > 0$.

Proof. Following the proof of Theorem 2.5, we observe that the sequence $\{\mathcal{T}^n x_0\}$ is a Cauchy sequence, and so there exists a point u in X such that $\lim_{n\to\infty} \mathcal{T}^n x_0 = u$. Since $\lim_{m,n\to\infty} w(\mathcal{T}^n x_0, \mathcal{T}^m x_0) = 0$, for each $\epsilon > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon$ implies $w(\mathcal{T}^{N_\epsilon}x_0, \mathcal{T}^n x_0) < \epsilon$. Since $\lim_{n\to\infty} \mathcal{T}^n x_0 = u$ and $w(x,\cdot)$ is lower semi-continuous,

$$w(\mathcal{T}^{N_\epsilon}x_0, u) \le \liminf_{n\to\infty} w(\mathcal{T}^{N_\epsilon}x_0, \mathcal{T}^n x_0) < \epsilon.$$

Therefore, $w(\mathcal{T}^{N_\epsilon}x_0, u) \le \epsilon$. Set $\epsilon = 1/k$, $N_\epsilon = n_k$ so that

$$\lim_{k\to\infty} w(\mathcal{T}^{n_k}x_0, u) = 0.$$

Assume that $\mathcal{T}u \ne u$. Then, by the hypothesis (v′), we have

$$\begin{aligned} 0 &< \inf\{w(x,u) + w(x,\mathcal{T}x) \mid x \in X\} \\ &\le \inf\{w(\mathcal{T}^{n_k}x_0, u) + w(\mathcal{T}^{n_k}x_0, \mathcal{T}^{n_k+1}x_0) \mid n \in \mathbb{N}\} \to 0, \end{aligned}$$

which contradicts our assumption. Therefore, $\mathcal{T}u = u$.

The last conclusion is derived as in the proof of Theorem 2.5. □

In what follows, we give various sufficient conditions for the uniqueness of the fixed point in Theorems 2.5 and 2.6.

Theorem 2.7. *In addition to the hypotheses of Theorem 2.5 (or Theorem 2.6), if any of the following conditions is fulfilled:*

(I) for all $u, v \in X$, there exists a $z \in X$ such that,

$$\{(z, \mathcal{T}z), (z, u), (z, v)\} \subseteq \mathcal{R}; \tag{13}$$

(II) the set $\mathcal{T}(X)$ is $\mathcal{R}$-directed;

(III) $\mathcal{R}|_{\mathcal{T}X}$ is complete;

(IV) $\mathcal{Y}(u, v, F(\mathcal{T}), \mathcal{R}^s)$ is nonempty, for each $u, v \in F(\mathcal{T})$,

then $\mathcal{T}$ has a unique fixed point.

Proof. In view of Theorem 2.5 (or Theorem 2.6), $F(\mathcal{T}) \neq \emptyset$ and $w(u, u) = 0$ for each $u \in F(\mathcal{T})$.

- **Assume (I)**. Suppose there exist distinct fixed points u and v of $\mathcal{T}$. We will consider the following two cases.

  ♣ **Case (A)**: u and v are $\mathcal{R}$-comparable. Then $\mathcal{T}^n u = u$ and $\mathcal{T}^n v = v$ are comparable for $n = 0, 1, \ldots$. Therefore, using condition (2),

$$\begin{aligned} 0 &\leq \theta(w(\mathcal{T}^n u, \mathcal{T}^n v), \max\left\{w(\mathcal{T}^{n-1}u, \mathcal{T}^{n-1}v), w(\mathcal{T}^{n-1}u, \mathcal{T}^n u), w(\mathcal{T}^{n-1}v, \mathcal{T}^n v)\right\}) \\ &= \theta(w(u, v), w(u, v)), \end{aligned}$$

  since u and v are fixed points of $\mathcal{T}$. This implies that

$$0 \leq \theta(w(u, v), w(u, v))$$

  which is possible only if $w(u, v) = 0$. Since $w(u, u) = 0$, by using Lemma 2.2, we have $u = v$; i.e., the fixed point of $\mathcal{T}$ is unique.

  ♣ **Case (B)**: By the assumption (I), there exists a $z \in X$, satisfying condition (13). Due to $\mathcal{T}$-closedness of $\mathcal{R}$, we get

$$(\mathcal{T}^{n-1}z, u) \in \mathcal{R}, \quad (\mathcal{T}^{n-1}z, v) \in \mathcal{R},$$

  and, using (2), it follows that

$$0 \leq \theta(w(\mathcal{T}^n z, u), \max\{w(\mathcal{T}^{n-1}z, u), w(\mathcal{T}^{n-1}z, \mathcal{T}^n z), w(u, \mathcal{T}u)\}). \tag{14}$$

  Using $(z, \mathcal{T}z) \in \mathcal{R}$, similarly as in the proof of Theorem 2.5, it can be shown that $w(\mathcal{T}^{n-1}z, \mathcal{T}^n z) \to 0$ as $n \to \infty$. Therefore, for n sufficiently large,

$$\max\{w(\mathcal{T}^{n-1}z, u), w(\mathcal{T}^{n-1}z, \mathcal{T}^n z), w(u, \mathcal{T}u)\} = w(\mathcal{T}^{n-1}z, u)$$

  and, from (14), we have

$$0 \leq \theta(w(\mathcal{T}^n z, u), w(\mathcal{T}^{n-1}z, u)).$$

  As in the proof of Theorem 2.5, it can be shown that $w(\mathcal{T}^n z, u) \leq w(\mathcal{T}^{n-1}z, u)$. It follows that the sequence $\{w(\mathcal{T}^n z, u)\}$ is nonincreasing. As earlier, we have

$$\lim_{n\to\infty} w(\mathcal{T}^n z, u) = 0.$$

  Also, since $(z, v) \in \mathcal{R}$, proceeding as earlier, we can prove that

$$\lim_{n\to\infty} w(\mathcal{T}^n z, v) = 0,$$

  and by using Lemma 2.2 we infer that $u = v$; i.e., the fixed point of $\mathcal{T}$ is unique.

- **Assume (II)**. For any two fixed points u, v of $\mathcal{T}$, there must be an element $z \in \mathcal{T}(X)$, such that

$$(z, u) \in \mathcal{R} \text{ and } (z, v) \in \mathcal{R}.$$

As $\mathcal{R}$ is $\mathcal{T}$-closed, so for all $n \in \mathbb{N} \cup \{0\}$,

$$(\mathcal{T}^n z, u) \in \mathcal{R} \text{ and } (\mathcal{T}^n z, v) \in \mathcal{R}.$$

In the line of proof of Case(B) (I), we obtain $u = v$, i.e., $\mathcal{T}$ has a unique fixed point.

- **Assume (III)**. Suppose u, v are two fixed points of $\mathcal{T}$ with $u \neq v$. Then, we must have $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$. For $(u, v) \in \mathcal{R}$, since $\mathcal{T}u = u$, $\mathcal{T}v = v$, $w(u, u) = 0$ and $w(v, v) = 0$, we obtain

$$\begin{aligned} 0 &\leq \theta(w(\mathcal{T}u, \mathcal{T}v), \max\{w(u, v), w(u, \mathcal{T}u), w(v, \mathcal{T}v)\}) \\ &= \theta(w(u, v), w(u, v)) < 0, \end{aligned}$$

a contradiction. Hence, we must have $u = v$.

In a similar way, if $(v, u) \in \mathcal{R}$, we have $u = v$.

- **Assume (IV)**. Suppose u, v are two fixed points of $\mathcal{T}$. Let $\{z_0, z_1, \dots, z_k\}$ be an $\mathcal{R}^s$-path in $F(\mathcal{T})$ connecting u and v. As in Case (I,A), It must be $z_{i-1} = z_i$ for each $i = 1, 2, \dots, k$, and it follows that $u = v$.

□

3. Some consequences and examples

Some fixed point results can be derived using the condition (2) of Theorems 2.5–2.7, with various forms of function $\theta \in \Theta$. We state just a few examples as corollaries out of which some of them are new and the rest include existing results in the literature.

To simplify the notation, in this section we denote

$$M(x, y) = \max\{w(x, y), w(x, \mathcal{T}x), w(y, \mathcal{T}y)\}.$$

Corollary 3.1. [Generalization of [4]]. *Under the conditions of Theorem 2.7, except that (iv) is replaced by*

$$w(\mathcal{T}x, \mathcal{T}y) \leq \lambda M(x, y), \quad \textit{for all } x, y \in X, \tag{15}$$

where $\lambda \in (0, 1)$, similar conclusions hold for the mapping $\mathcal{T}$.

Proof. Taking $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as $\theta(\xi, \zeta) = \lambda\,\zeta - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$ in (2), we obtain the conclusion. □

Corollary 3.2. [Generalization of [18, 24]] *Under the conditions of Theorem 2.7, except that (iv) is replaced by*

$$w(\mathcal{T}x, \mathcal{T}y) \leq M(x, y) - \varphi(M(x, y)), \quad \textit{for all } x, y \in X,$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi-continuous function such that $\varphi(\xi) = 0$ if and only if $\xi = 0$, similar conclusions hold for the mapping $\mathcal{T}$.

Proof. Taking $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as $\theta(\xi, s) = \zeta - \varphi(\zeta) - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$, we obtain the desired conclusions. □

Corollary 3.3. *Let all conditions of Theorem 2.7 be satisfied, except that (iv) is replaced by*

$$\psi(w(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(M(x, y)), \quad \textit{for all } x, y \in X,$$

where $\psi, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) < t \leq \psi(t)$ for all $t > 0$. Then the same conclusions hold for the mapping $\mathcal{T}$.

Proof. If in equation (2), we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $\theta(\xi, \zeta) = \varphi(\zeta) - \psi(\xi)$ for all $\xi, \zeta \in \mathbb{R}_+$, we obtain the conclusions. □

Corollary 3.4. [Generalization of [3]] *Let all conditions of Theorem 2.7 be satisfied, except that (iv) is replaced by*

$$w(\mathcal{T}x, \mathcal{T}y) \le \varphi(M(x, y)), \quad \text{for all } x, y \in X, \tag{16}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an upper semi-continuous function with* $\varphi(s) < s$ *for all* $s > 0$ *and* $\varphi(s) = 0$ *if and only if* $s = 0$. *Then the same conclusions hold for the mapping* $\mathcal{T}$.

Proof. If we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $\theta(\xi, \zeta) = \varphi(\zeta) - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$, then (2) is converted to (16), and the result follows from Theorem 2.7. □

Corollary 3.5. [Generalization of [6]] *Let all conditions of Theorem 2.7 be satisfied, except that (iv) is replaced by*

$$w(\mathcal{T}x, \mathcal{T}y) \le M(x, y)\varphi(M(x, y)), \quad \text{for all } x, y \in X, \tag{17}$$

where $\varphi : \mathbb{R}_+ \to [0, 1)$ *is a function with* $\limsup_{t \to \tau^+} \varphi(t) < 1$ *for all* $\tau > 0$. *Then the same conclusions hold for the mapping* $\mathcal{T}$.

Proof. If we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by $\theta(\xi, \zeta) = \zeta\varphi(\zeta) - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$, then (2) becomes (17), and we reach the conclusion. □

Example 3.6. Let $X, d, \mathcal{R}, \mathcal{T}$ and x_0 be as in Example 1.8, and let a w-distance on X be given by $w(x, y) = y$ for all $x, y \in X$. In order to show that all conditions of Corollary 3.1 are satisfied, just the condition (15) has to be checked.

Take $x, y \in \overline{O(x_0; \mathcal{T})}$ with $(x, y) \in \mathcal{R}$, and so $0 \le x, y \le \frac{1}{5}$. Consider two cases:

Case 1. If $x = 0$ and $y = 1/5^n$, $n \in \mathbb{N}$, then (15) reduces to $1/5^{n+1} \le \lambda \cdot 1/5^n$ and is fulfilled for $\lambda = 1/5$. If $y = 0$ and $x = 1/5^n$, $n \in \mathbb{N}$ or $x = 0$, then (15) holds trivially.

Case 2. Let $x, y \in \{1/5^n \mid n \in \mathbb{N}\}$ with $0 < y < x$, i.e., $y \le x/5$. Then we have

$$w(\mathcal{T}x, \mathcal{T}y) = w\left(\frac{x}{5}, \frac{y}{5}\right) = \frac{y}{5}$$

and

$$M(x, y) = \max\left\{y, \frac{x}{5}, \frac{y}{5}\right\} = \frac{x}{5}.$$

Thus, (15) reduces to

$$\frac{y}{5} \le \lambda \frac{x}{5}$$

i.e., to $y \le \lambda x$. If we take $\lambda = \frac{1}{5} < 1$, then in all cases the contractive condition (15) holds true. Hence, $\mathcal{T}$ is an orbitally $\mathcal{R}$-contractive mapping.

Therefore, all the conditions of Corollary 3.1 are fulfilled and $x = 0$ is the unique fixed point of T in $\overline{O(x_0; T)}$.

Example 3.7. Consider the set $X = [0, +\infty)$ with the usual metric d. Define a w-distance $w \colon X \times X \to [0, \infty)$ by $w(x, y) = x^2 + y^2$ for all $x, y \in X$ and the binary relation $\mathcal{R}$ by

$$(x, y) \in \mathcal{R} \iff x, y > 0 \text{ or } (x, y) \in \left\{(0, 0), (0, \frac{1}{7})\right\} \cup \left\{(0, \frac{1}{7^{2n+1}}) : n \ge 2\right\}.$$

Consider the self-mapping $\mathcal{T}$ on X given by

$$\mathcal{T}x = \begin{cases} \dfrac{x^2}{7}, & x \in [0,1] \\ x - \frac{2}{7}, & x > 1. \end{cases}$$

Take $x_0 = 1$. It is simple to show that

$$O(x_0;\mathcal{T}) \subset \left\{ \frac{1}{7^k} \mid k \in \mathbb{N} \cup \{0\} \right\} \text{ and } \overline{O(x_0;\mathcal{T})} = O(x_0;\mathcal{T}) \cup \{0\},$$

and that (X, w) is $\mathcal{T}$-orbitally complete at x_0.

Then $\mathcal{R}$ is $\mathcal{T}$-orbitally closed, and $\mathcal{R}$ is $\mathcal{T}$-orbitally transitive (which is not $\mathcal{T}$-closed and transitive – similarly as in Example 1.8). Also (X, w) is $\mathcal{T}$-orbitally $\mathcal{R}$-complete at x_0.

Take $x, y \in \overline{O(x_0;\mathcal{T})}$ with $(x, y) \in \mathcal{R}$, and so $0 \le x, y \le 1$. Consider two cases:

Case 1. If $x = 0$ and $y = 1/7^{2k+1}$, $k \in \mathbb{N}$, then (15) reduces to

$$\frac{1}{7^{8k+6}} \le \lambda\Big(\frac{1}{7^{4k+2}} + \frac{1}{7^{8k+6}}\Big),$$

and is fulfilled, e.g., for $\lambda = 1/7$. If $y = 0$ and $x = 1/7^{2k+1}$, $k \in \mathbb{N}$ or $x = 0$, then (15) holds trivially.

Case 2. Let $x, y \in \{1/7^k \mid k \in \mathbb{N} \cup \{0\}\}$ with $0 < y < x \le 1$. Then

$$w(\mathcal{T}x, \mathcal{T}y) = w\left(\frac{x^2}{7}, \frac{y^2}{7}\right) = \frac{x^4}{49} + \frac{y^4}{49}$$

and

$$M(x, y) = \max\left\{x^2 + y^2, x^2 + \frac{x^4}{49}, y^2 + \frac{y^4}{49}\right\}.$$

Case 2a. Let $y \ge x^2/7$. Then $x^2 + y^2 \ge x^2 + (x^4/49)$ and $M(x, y) = x^2 + y^2$. Therefore, the condition (15) reduces to

$$\frac{x^4}{49} + \frac{y^4}{49} \le \lambda(x^2 + y^2).$$

Case 2b. If $y \le x^2/7$, then $x^2 + y^2 \le x^2 + (x^4/49)$ and $M(x, y) = x^2 + (x^4/49)$. Therefore, (15) reduces to

$$\frac{x^4}{49} + \frac{y^4}{49} \le \lambda\left(x^2 + \frac{x^4}{49}\right),$$

It can be easily checked that the above cases hold true for $\lambda = 4/7$.

Thus $\mathcal{T}$ is orbitally $\mathcal{R}$-contractive mapping.

Finally, we will show that the condition (v′) of Theorem 2.6 holds true. Indeed, for any $n \in \mathbb{N}$ we have

$$\frac{1}{7^n} \ne \mathcal{T}\Big(\frac{1}{7^n}\Big),$$

and for arbitrary $n \in \mathbb{N}$ we get

$$\inf\left\{w\left(\frac{1}{7^m}, \frac{1}{7^n}\right) + w\left(\frac{1}{7^m}, \frac{1}{7^{m+1}}\right) \mid m \in \mathbb{N}\right\} = \frac{1}{7^{2n}} > 0.$$

Therefore, all the conditions of Theorem 2.6 (with the condition (13) of Theorem 2.7) are satisfied (with $\theta(\xi, \zeta) = \lambda\,\zeta - \xi$) and $x = 0$ is the unique fixed point of $\mathcal{T}$ in $\overline{O(x_0;\mathcal{T})}$.

4. Application to nonlinear matrix equations

Denote by $s(\mathcal{U})$ any singular value of a matrix $\mathcal{U}$, and its trace norm by $s^+(\mathcal{U}) = \|\mathcal{U}\|$. For $\mathcal{C}, \mathcal{D} \in \mathcal{H}(n)$, $\mathcal{C} \geq \mathcal{D}$ (resp. $\mathcal{C} > \mathcal{D}$) will mean that the matrix $\mathcal{C} - \mathcal{D}$ is positive semi-definite (resp. positive definite).

Theorem 4.1. *Consider the equation*

$$\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i, \tag{18}$$

where $\mathcal{Q} \in \mathcal{P}(n)$, $\mathcal{A}_i \in M(n)$, $i = 1, \ldots, k$, and the operator $\hbar : \mathcal{P}(n) \to \mathcal{P}(n)$ is continuous in the trace norm. Let, for some $M, N_1 \in \mathbb{R}$, and for any $\mathcal{U} \in \mathcal{P}(n)$ with $\|\mathcal{U}\| \leq M$, $s(\hbar(\mathcal{U})) \leq N_1$ hold for all singular values of $\hbar(\mathcal{U})$. Assume that:

(I) $\|\mathcal{Q}\| \leq M - NN_1 n$, where $\sum_{i=1}^{k} \|\mathcal{A}_i^*\| \, \|\mathcal{A}_i\| = N$;
(II) for any $\mathcal{W} \in \mathcal{P}(n)$ with $\|\mathcal{W}\| \leq M$, $\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{W}) \mathcal{A}_i \geq O$ holds;
(III) for any $\mathcal{W} \in \mathcal{P}(n)$ with $\|\mathcal{W}\| \leq M$, $\mathcal{W} \leq \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{W}) \mathcal{A}_i$ holds.
(IV) there exists $\lambda \in (0, 1)$, such that for any $s(\mathcal{Q})$,

$$2NN_1 + 2s(\mathcal{Q}) \leq \lambda \, \Upsilon(\mathcal{U}, \mathcal{V}) \tag{19}$$

holds for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}(n)$ with $\|\mathcal{U}\|, \|\mathcal{V}\| \leq M$, $\mathcal{U} \leq \mathcal{V}$ and $\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i \neq \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i$, where

$$\Upsilon(\mathcal{U}, \mathcal{V}) = \max \begin{Bmatrix} |s^+(\mathcal{U})| + |s^+(\mathcal{V})|, |s^+(\mathcal{U})| + |s^+(\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i)| \\ |s^+(\mathcal{V})| + |s^+(\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i)| \end{Bmatrix}. \tag{20}$$

Then the NME (18) has a unique solution $\widehat{\mathcal{U}} \in \mathcal{P}(n)$ with $\|\widehat{\mathcal{U}}\| \leq M$. Further, the solution can be obtained as the limit of the iterative sequence $\{\mathcal{U}_n\}$, where for $j \geq 0$,

$$\mathcal{U}_{j+1} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}_j) \mathcal{A}_i \tag{21}$$

and $\mathcal{U}_0$ is an arbitrary element of $\mathcal{P}(n)$ satisfying $\|\mathcal{U}_0\| \leq M$.

Proof. Denote $\Lambda := \{\mathcal{U} \in \mathcal{P}(n) : \|\mathcal{U}\| \leq M\}$, being a closed subset of $\mathcal{P}(n)$. According to (II), any solution of (18) in Λ has to be positive definite. We have, for any $\mathcal{U} \in \Lambda$,

$$\begin{aligned} \|\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| &\leq \|\mathcal{Q}\| + \|\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| \\ &\leq \|\mathcal{Q}\| + \sum_{i=1}^{k} \|\mathcal{A}_i^*\| \, \|\mathcal{A}_i\| \, \|\hbar(\mathcal{U})\| = \|\mathcal{Q}\| + N\|\hbar(\mathcal{U})\|. \end{aligned} \tag{22}$$

Since all singular values of $\mathcal{U}$ satisfy $s(\hbar(\mathcal{U})) \leq N_1$, it follows that $\|\hbar(\mathcal{U})\| \leq N_1 n$. Thus, (22) implies

$$\|\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| \leq \|\mathcal{Q}\| + NN_1 n \leq M - NN_1 n + NN_1 n = M.$$

Define now an operator $\mathfrak{I} : \Lambda \to \Lambda$ by

$$\mathfrak{I}(\mathcal{U}) = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i, \text{ for all } \mathcal{U} \in \Lambda.$$

Also, define a binary relation

$$\mathcal{R} = \{(\mathcal{U}, \mathcal{V}) \in \Lambda \times \Lambda : \mathcal{U} \preceq \mathcal{V}\}.$$

It is clear that finding positive definite solution(s) of the equation (18) is equivalent to finding fixed point(s) of $\mathfrak{I}$. Notice that $\mathfrak{I}$ is well defined, $\mathcal{R}$-continuous and $\mathcal{R}$ is $\mathfrak{I}$-closed. Since

$$\sum_{i=1}^{k} \mathcal{A}_i^* \hbar_j(\mathcal{K}) \mathcal{A}_i \succ 0,$$

for some $\mathcal{K} \in \Lambda$, we have $(\mathcal{K}, \mathfrak{I}(\mathcal{K})) \in \mathcal{R}$ and hence $\Lambda(\mathfrak{I}; \mathcal{R}) \neq \emptyset$.

Now, let $(\mathcal{U}, \mathcal{V}) \in \mathcal{R}^* = \{(\mathcal{U}, \mathcal{V}) \in \mathcal{R} : \mathfrak{I}(\mathcal{U}) \neq \mathfrak{I}(\mathcal{V})\}$. Then we have

$$\begin{aligned}
\|\mathfrak{I}(\mathcal{U})\| + \|\mathfrak{I}(\mathcal{V})\| &= \|Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| + \|Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\| \\
&\leq 2\|Q\| + \|\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| + \|\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\| \\
&\leq 2\|Q\| + \sum_{i=1}^{k} \left[\|\mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| + \|\mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\|\right] \\
&\leq 2\|Q\| + \sum_{i=1}^{k} \|\mathcal{A}_i^*\| \, \|\mathcal{A}_i\| \left[\|\hbar(\mathcal{U})\| + \|\hbar(\mathcal{V})\|\right] \\
&\leq 2\|Q\| + N(\|\hbar(\mathcal{U})\| + \|\hbar(\mathcal{V})\|) \\
&\leq 2\|Q\| + N(N_1 n + N_1 n) \\
&= 2\|Q\| + 2NN_1 n.
\end{aligned}$$

Thus, for any $\mathcal{U}, \mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$, we have

$$\|\mathfrak{I}(\mathcal{U})\| + \|\mathfrak{I}(\mathcal{V})\| \leq 2\|Q\| + 2NN_1 n. \tag{23}$$

For some fixed $\mathcal{U}, \mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$, from (19) and (20), we have

$$\begin{aligned}
2NN_1 + 2s(Q) &\leq \lambda \max \begin{Bmatrix} |s^+(\mathcal{U})| + |s^+(\mathcal{V})|, \\ |s^+(\mathcal{U})| + |s^+(Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i)| \\ |s^+(\mathcal{V})| + |s^+(Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i)| \end{Bmatrix} \\
&= \lambda \max \begin{Bmatrix} \|\mathcal{U}\| + \|\mathcal{V}\|, \|\mathcal{U}\| + \|Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| \\ \|\mathcal{V}\| + \|Q + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\| \end{Bmatrix}.
\end{aligned}$$

The above relation holds for every singular value of Q, so adding up, we obtain

$$2NN_1 n + 2\|Q\| \leq \lambda \max\{\|\mathcal{U}\| + \|\mathcal{V}\|, \|\mathcal{U}\| + \|\mathfrak{I}(\mathcal{U})\|, \|\mathcal{V}\| + \|\mathfrak{I}(\mathcal{V})\|\}.$$

Therefore, from (23) we get

$$\|\mathfrak{I}(\mathcal{U})\| + \|\mathfrak{I}(\mathcal{V})\| \leq \lambda \max\{\|\mathcal{U}\| + \|\mathcal{V}\|, \|\mathcal{U}\| + \|\mathfrak{I}(\mathcal{U})\|, \|\mathcal{V}\| + \|\mathfrak{I}(\mathcal{V})\|\}. \tag{24}$$

Let $w \colon \Lambda \times \Lambda \to \mathbb{R}_+$ be defined by

$$w(\mathcal{U}, \mathcal{V}) = \|\mathcal{U}\| + \|\mathcal{V}\| \text{ for all } \mathcal{U}, \mathcal{V} \in \Lambda.$$

Then $(\Lambda, \|.\|, w)$ is a complete relational metric space with the above w-distance. It follows from (24) that

$$w(\mathfrak{I}(\mathcal{U}), \mathfrak{I}(\mathcal{V})) \leq \lambda \max\{w(\mathcal{U}, \mathcal{V}), w(\mathcal{U}, \mathfrak{I}(\mathcal{U})), w(\mathcal{V}, \mathfrak{I}(\mathcal{V}))\}. \tag{25}$$

Taking $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as $\theta(\xi, \zeta) = \lambda\,\zeta - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$, (25) can be rewritten as

$$\theta(w(\Im(\mathcal{U}), \Im(\mathcal{V})), \max\{w(\mathcal{U}, \mathcal{V}), w(\mathcal{U}, \Im(\mathcal{U})), w(\mathcal{V}, \Im(\mathcal{V}))\}) \geq 0.$$

Now, all the hypotheses of Theorem 2.5 are satisfied, and therefore there exists $\hat{\mathcal{X}} \in \mathcal{P}(n)$ such that $\Im(\hat{\mathcal{X}}) = \hat{\mathcal{X}}$. Hence, the matrix equation (18) has a solution in $\mathcal{P}(n)$. Furthermore, due to the existence of the least upper bound and the greatest lower bound for each pair $\mathcal{U}, \mathcal{V} \in \Im(\mathcal{P}(n))$, we have $\mathcal{Y}(\mathcal{U}, \mathcal{V}; \mathcal{R}|_{\Im(\mathcal{P}(n))}) \neq \emptyset$ for all $\mathcal{U}, \mathcal{V} \in \Im(\mathcal{P}(n))$. Hence, using Theorem 2.7, $\Im$ has a unique fixed point, and hence we conclude that the matrix equation (18) has a unique solution in $\mathcal{P}(n)$. □

5. Numerical experiments

In this section, we consider some numerical results. All experiments were run on a macOS Mojave version 10.14.6 CPU @1.6 GHz intel core i5 8GB with MATLAB R2020b as the programming language (Online). The number of necessary iterations is denoted by Iter. No., initial matrix is denoted by Int. Mat., Dimension is denoted by Dim., Minimum eigenvalue of a matrix is denoted by Min(Eng) and the trace norm of the residual is denoted by Res ($\mathrm{Res}(\mathcal{X}) = \|\mathcal{X}_{n+1} - \mathcal{X}_n\|_{tr}$). We have assigned $tol = 10^{-10}$ in all studies.

Three examples of key variables, as well as tables and graphs displaying various input-data, such as solutions, iteration number, error, CPU time, computing time, are shown here. We use line graphs, bar graphs, and surface plotting to obtain a clearer understanding.

Example 5.1. *Consider matrices $\mathcal{A}_1, \mathcal{A}_2, Q \in C^{3\times 3}$ given as*

$$\mathcal{A}_1 = \begin{bmatrix} 0.036664411116369 & 0.089269870544203 & 0.066952402908152 \\ 0.057387773921273 & 0.135498910647450 & 0.071734717401591 \\ 0.153034063790062 & 0.133904805816304 & 0.103616814024521 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0.105210918855667 & 0.017535153142611 & 0.086081660881910 \\ 0.074922927063884 & 0.133904805816304 & 0.151439958958915 \\ 0.103616814024521 & 0.065358298077005 & 0.036664411116369 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.0609 & -0.0011 & 0.0017 \\ -0.0011 & 1.0572 & -0.0014 \\ 0.0017 & -0.0014 & 1.1389 \end{bmatrix}.$$

To see the convergence of the sequence $\{\mathcal{U}_n\}$ defined in (21), we start with three different initializations:

$$\mathcal{U}_0 = \begin{bmatrix} 0.0014 & 0.0019 & 0.0025 \\ 0.0019 & 0.0027 & 0.0034 \\ 0.0025 & 0.0034 & 0.0052 \end{bmatrix}, \qquad \mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{W}_0 = \begin{bmatrix} 1.1255 & -0.0023 & 0.0037 \\ -0.0023 & 1.1177 & -0.0030 \\ 0.0037 & -0.0030 & 1.2970 \end{bmatrix},$$

where $\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0 \in \mathcal{P}(3)$.

Table 1

Int. Mat.	$\hbar(\mathcal{U})$	λ	*Dim.*	*Iter No.*	*CPU*	*Error*	*Min(Eig)*
$\mathcal{U}_0$	$\mathcal{U}^2$	0.83	3	26	0.012246	0.7923×10^{-10}	1.0745
$\mathcal{V}_0$	$\mathcal{U}^2$	0.83	3	25	0.012068	0.9888×10^{-10}	1.0745
$\mathcal{W}_0$	$\mathcal{U}^2$	0.83	3	25	0.012444	0.5412×10^{-10}	1.0745

We get the positive definite solution

$$\hat{\mathcal{X}} = \begin{bmatrix} 1.1634 & 0.0968 & 0.0911 \\ 0.0968 & 1.1799 & 0.0983 \\ 0.0911 & 0.0983 & 1.2328 \end{bmatrix}.$$

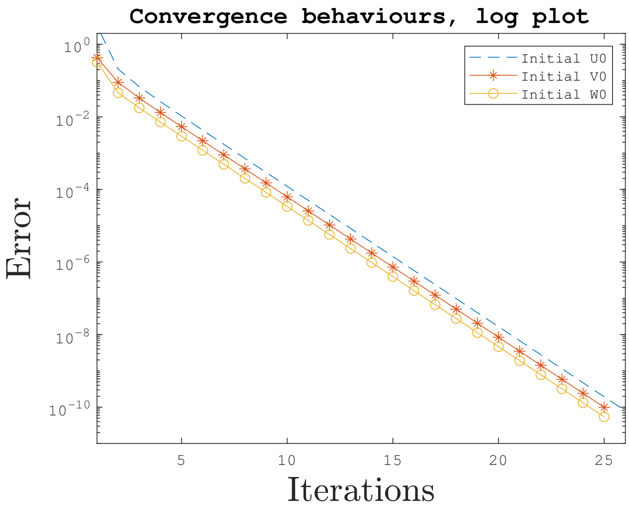

Fig.1 : Iteration vs Error graph

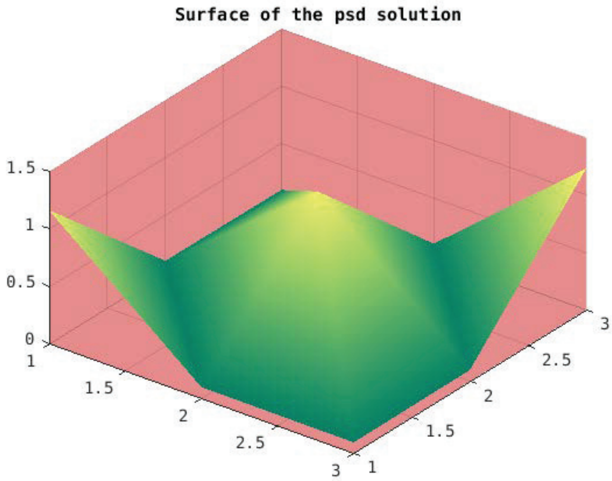

Fig.2 : Solution's surface plot

Example 5.2. *Consider the following matrices $\mathcal{A}_1, \mathcal{A}_2, Q \in C^{4\times4}$:*

$$\mathcal{A}_1 = \begin{bmatrix} 0.0024+0.0194i & 0.0022-0.0067i & 0.0046-0.0037i & 0.0058+0.0157i \\ 0.0084-0.0062i & 0.0023+0.0022i & 0.0064+0.0032i & 0.0043-0.0087i \\ 0.0086-0.0030i & 0.0054+0.0165i & 0.0092+0.0187i & 0.0088+0.0108i \\ 0.0096-0.0093i & 0.0076+0.0064i & 0.0016-0.0063i & 0.0039+0.0194i \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0.0194+0.0020i & -0.0067+0.0019i & -0.0037+0.0013i & 0.0157+0.0087i \\ -0.0062+0.0035i & 0.0022+0.0045i & 0.0032+0.0080i & -0.0087+0.0069i \\ -0.0030+0.0022i & 0.0165+0.0068i & 0.0187+0.0008i & 0.0108+0.0051i \\ -0.0093+0.0000i & 0.0064+0.0047i & -0.0063+0.0012i & 0.0194+0.0062i \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.0008+0.0000i & -0.0002-0.0002i & -0.0000+0.0001i & 0.0002+0.0001i \\ -0.0002+0.0002i & 1.0003+0.0000i & 0.0001+0.0003i & -0.0000+0.0001i \\ -0.0000-0.0001i & 0.0001-0.0003i & 1.0008+0.0000i & 0.0003-0.0000i \\ 0.0002-0.0001i & -0.0000-0.0001i & 0.0003+0.0000i & 1.0006+0.0000i \end{bmatrix}.$$

To see the convergence of the sequence $\{\mathcal{U}_n\}$ defined in (21)*, we start with the following initializations:*

$$\mathcal{U}_0 = \begin{bmatrix} 0.0748 + 0.0000i & -0.0204 + 0.0238i & 0.0055 + 0.0058i & 0.0175 + 0.0117i \\ -0.0204 - 0.0238i & 0.0265 - 0.0000i & 0.0203 - 0.0266i & 0.0009 - 0.0052i \\ 0.0055 - 0.0058i & 0.0203 + 0.0266i & 0.1011 + 0.0000i & 0.0399 + 0.0101i \\ 0.0175 - 0.0117i & 0.0009 + 0.0052i & 0.0399 - 0.0101i & 0.0711 - 0.0000i \end{bmatrix},$$

$$\mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{W}_0 = \begin{bmatrix} 1.0015 + 0.0000i & -0.0004 - 0.0004i & -0.0000 + 0.0001i & 0.0003 + 0.0002i \\ -0.0004 + 0.0004i & 1.0005 + 0.0000i & 0.0002 + 0.0007i & -0.0001 + 0.0002i \\ -0.0000 - 0.0001i & 0.0002 - 0.0007i & 1.0017 + 0.0000i & 0.0006 - 0.0001i \\ 0.0003 - 0.0002i & -0.0001 - 0.0002i & 0.0006 + 0.0001i & 1.0012 + 0.0000i \end{bmatrix},$$

where $\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0 \in \mathcal{P}(n)$. We obtain

Table 2

Int. Mat.	$\hbar(\mathcal{U})$	λ	*Dim.*	*Iter No.*	*CPU*	*Error*	*Min(Eig)*
$\mathcal{U}_0$	$\mathcal{U}^3$	0.9	4	6	0.008690	0.2387×10^{-11}	1.0004
$\mathcal{V}_0$	$\mathcal{U}^3$	0.9	4	5	0.008575	0.3721×10^{-11}	1.0004
$\mathcal{W}_0$	$\mathcal{U}^3$	0.9	4	5	0.008223	0.1050×10^{-11}	1.0004

The PDS is given by

$$\hat{\mathcal{X}} = \begin{bmatrix} 1.0021 + 0.0000i & -0.0005 - 0.0000i & 0.0000 + 0.0001i & 0.0007 + 0.0003i \\ -0.0005 + 0.0000i & 1.0012 + 0.0000i & 0.0008 + 0.0002i & 0.0005 + 0.0001i \\ 0.0000 - 0.0001i & 0.0008 - 0.0002i & 1.0019 + 0.0000i & 0.0005 + 0.0000i \\ 0.0007 - 0.0003i & 0.0005 - 0.0001i & 0.0005 - 0.0000i & 1.0026 - 0.0000i \end{bmatrix}.$$

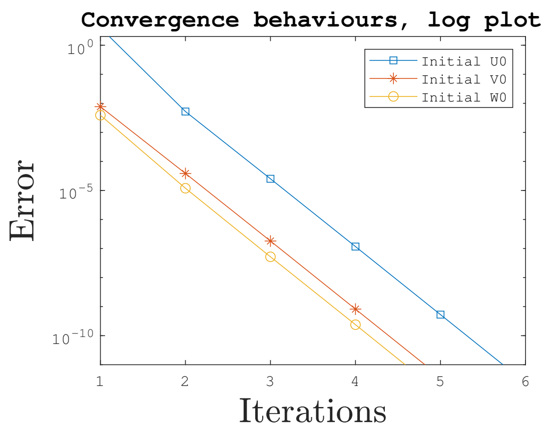

Fig.3 : Iteration vs Error graph

Example 5.3. *In this example, we consider matrices with randomly generated coefficients by*

$$\mathcal{A}_1 = (1/2^n) \times rand(n); \quad \mathcal{A}_2 = (1/2^n) \times rand(n);$$

where $\mathcal{A}_1, \mathcal{A}_2 \in C^{n\times n}$. *For* $n = 4$, *we obtain*

$$\mathcal{A}_1 = \begin{bmatrix} 0.0108 & 0.0038 & 0.0411 & 0.0010 \\ 0.0244 & 0.0250 & 0.0392 & 0.0615 \\ 0.0520 & 0.0329 & 0.0182 & 0.0104 \\ 0.0502 & 0.0260 & 0.0270 & 0.0066 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0.0233 & 0.0595 & 0.0168 & 0.0261 \\ 0.0124 & 0.0575 & 0.0264 & 0.0614 \\ 0.0306 & 0.0033 & 0.0342 & 0.0188 \\ 0.0212 & 0.0461 & 0.0589 & 0.0438 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.0073 & 0.0048 & 0.0066 & 0.0035 \\ 0.0048 & 1.0045 & 0.0040 & 0.0019 \\ 0.0066 & 0.0040 & 1.0068 & 0.0043 \\ 0.0035 & 0.0019 & 0.0043 & 1.0036 \end{bmatrix}.$$

We use the initial values

$$\mathcal{U}_0 = 10^-5 \times \begin{bmatrix} 0.1818 & 0.2030 & 0.1447 & 0.1757 \\ 0.2030 & 0.6543 & 0.3450 & 0.3344 \\ 0.1447 & 0.3450 & 0.4227 & 0.4028 \\ 0.1757 & 0.3344 & 0.4028 & 0.3972 \end{bmatrix}, \quad \mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{W}_0 = \begin{bmatrix} 1.0147 & 0.0096 & 0.0133 & 0.0070 \\ 0.0096 & 1.0090 & 0.0081 & 0.0039 \\ 0.0133 & 0.0081 & 1.0137 & 0.0086 \\ 0.0070 & 0.0039 & 0.0086 & 1.0073 \end{bmatrix},$$

where $\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0 \in \mathcal{P}(n)$.

Table 3

Int. Mat.	$\hbar(\mathcal{U})$	λ	*Dim.*	*Iter No.*	*CPU*	*Error*	*Min(Eig)*
$\mathcal{U}_0$	$\mathcal{U}^2$	*0.95*	*4*	*10*	*0.008158*	0.0933×10^{-10}	*1.0013*
$\mathcal{V}_0$	$\mathcal{U}^2$	*0.95*	*4*	*9*	*0.0080555*	0.1341×10^{-10}	*1.0013*
$\mathcal{W}_0$	$\mathcal{U}^2$	*0.95*	*4*	*8*	*0.008241*	0.8007×10^{-10}	*1.0013*

The PDS is

$$\hat{X} = \begin{bmatrix} 1.0161 & 0.0123 & 0.0141 & 0.0094 \\ 0.0123 & 1.0168 & 0.0127 & 0.0120 \\ 0.0141 & 0.0127 & 1.0176 & 0.0132 \\ 0.0094 & 0.0120 & 0.0132 & 1.0151 \end{bmatrix}$$

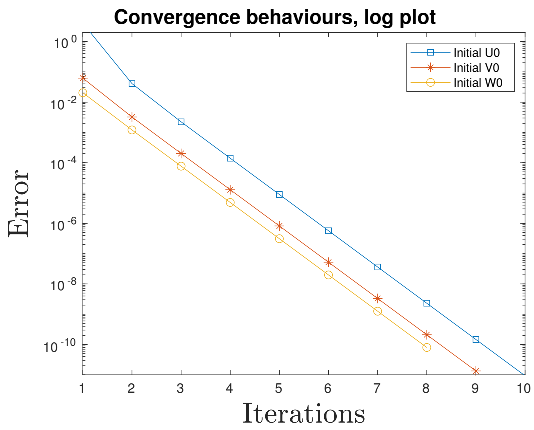

Fig.4 : Iteration vs Error graph

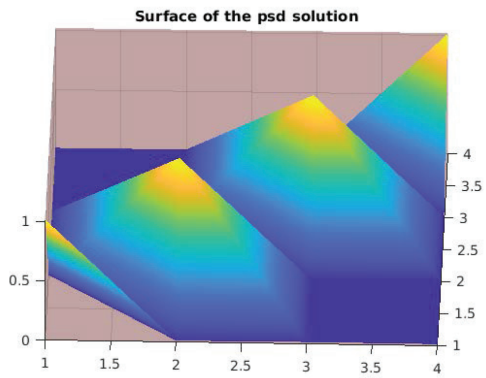

Fig.5 : Solution's surface plot

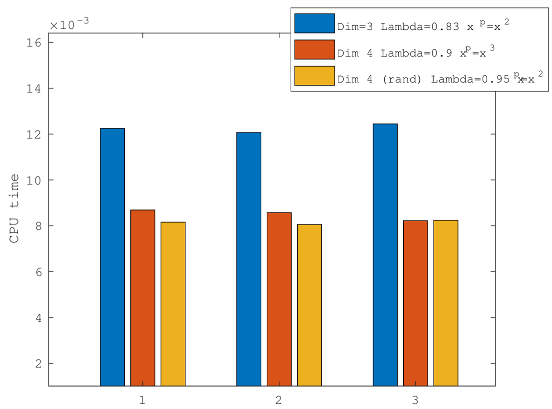

Fig.6 : Dimension vs CPU time

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Conflict of interest

The authors declare that there is no conflict of interest.

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