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Generalized Resolvents of Linear Relations Generated by Integral Equations with Operator Measures

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Abstract. We consider a symmetric minimal relation L_0 generated by an integral equation with operators measures. We obtain a form of generalized resolvents of L_0 and give a description of boundary value problems associated to generalized resolvents.

1. Introduction

Generalized resolvents of symmetric operators were introduced by M.A. Naimark in 1940 (see, for example, [1]). In [27], A.V. Straus described the generalized resolvents of a symmetric operator generated by a formally self-adjoint differential expression of even order in the scalar case. In [5], these results were spread to the operator case, and in [9] to the case of a differential-operator expression with a non-negative weight operator function. Further, the generalized resolvents of differential operators were studied in many works (a detailed bibliography is available, for example, in [25], [21]).

In this paper, we consider the integral equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \tag{1}$$

where y is an unknown function, $a \leqslant t \leqslant b$; J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is the identical operator); $\mathbf{p}$, $\mathbf{m}$ are operator-valued measures defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in H; $x_0 \in H$, $f \in L_2(H, d\mathbf{m}; a, b)$. We assume that the measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations and $\mathbf{p}$ is self-adjoint, $\mathbf{m}$ is non-negative.

We consider a symmetric minimal relation L_0 generated by equation (1). We obtain a form of generalized resolvents of L_0 and give a description of boundary value problems associated to generalized resolvents. We give a detailed example of constructing a generalized resolvent.

If the measures $\mathbf{p}$, $\mathbf{m}$ are absolutely continuous (i.e., $\mathbf{p}(\Delta) = \int_\Delta p(t)dt$, $\mathbf{m}(\Delta) = \int_\Delta m(t)dt$ for all Borel sets $\Delta \subset [a, b]$, where $p(t)$, $m(t)$ are bounded operators for fixed t and the functions $\|p(t)\|$, $\|m(t)\|$ belong to $L_1(a, b)$), then integral equation (1) is transformed to a differential equation with a non-negative weight operator function. Linear relations and operators generated by such differential equations were considered in many works (see [23], [6], [9], further detailed bibliography can be found, for example, in [21], [3]).

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The study of integral equation (1) differs essentially from the study of differential equations by the presence of the following features: i) a representation of a solution of equation (1) using an evolutional family of operators is possible if the measures $\mathbf{p}$, $\mathbf{m}$ have not common single-point atoms (see [12]); ii) the Lagrange formula contains summands relating to single-point atoms of the measures $\mathbf{p}$, $\mathbf{m}$ (see [13]). This article substantially uses the results of [17]. Also note that this article partially corrects the errors made in the work [11]. Moreover, equation (1) was considered in [14] under the assumption that $\mathbf{m}$ is the usual Lebesque measure on $[a,b]$ and the set of single-point atoms of the measure $\mathbf{p}$ can be arranged as an increasing sequence converging to b. In [14], a formula for generalized resolvents of L_0 is obtained and a description of boundary value problems related to generalized resolvents is given. In [14], L_0, L_0^* are operators.

2. Preliminary assertions

Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. We consider a function $\Delta\to\mathbf{P}(\Delta)$ defined on Borel sets $\Delta\subset[a,b]$ and taking values in the set of linear bounded operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, for example, [4, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\sum_{n=1}^{\infty}\mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n, where the series converges weakly. Further, we extend any measure $\mathbf{P}$ on $[a,b]$ to a segment $[a,b_0]$ $(b_0>b)$ letting $\mathbf{P}(\Delta)=0$ for each Borel set $\Delta\subset(b,b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$ we denote $\mathbf{V}_\Delta(\mathbf{P})=\rho_{\mathbf{P}}(\Delta)=\sup\sum_n\|\mathbf{P}(\Delta_n)\|$, where the supremum is taken over all finite sums of disjoint Borel sets $\Delta_n\subset\Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called the variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for $\rho_{\mathbf{P}}$-almost all $s\in[a,b]$ there exists an operator function $s\to\Psi_{\mathbf{P}}(s)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of linear bounded operators acting in H, $\|\Psi_{\mathbf{P}}(s)\|=1$, and the equality

$$\mathbf{P}(\Delta)=\int_\Delta\Psi_{\mathbf{P}}(s)d\rho_{\mathbf{P}} \tag{2}$$

holds for each Borel set $\Delta\subset[a,b]$. The function $\Psi_{\mathbf{P}}$ is uniquely determined up to values on a set of zero $\rho_{\mathbf{P}}$-measure. Integral (2) converges with respect to the usual operator norm ([4, ch. 5]).

Further, $\int_{t_0}^{t}$ stands for $\int_{[t_0t)}$ if $t_0<t$, for $-\int_{[t,t_0)}$ if $t_0>t$, and for 0 if $t_0=t$. This implies that $y(a)=x_0$ in equation (1). A function h is integrable with respect to the measure $\mathbf{P}$ on a set Δ if there exists the Bochner integral $\int_\Delta\Psi_{\mathbf{P}}(t)h(t)d\rho_{\mathbf{P}}=\int_\Delta(d\mathbf{P})h(t)$. Then the function $y(t)=\int_{t_0}^{t}(d\mathbf{P})h(s)$ is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$ denote a set of single-point atoms of the measure $\mathbf{P}$ (i.e., a set $t\in[a,b]$ such that $\mathbf{P}(\{t\})\neq0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure $\mathbf{P}$ is continuous if $\mathcal{S}_{\mathbf{P}}=\varnothing$, it is self-adjoint if $(\mathbf{P}(\Delta))^*=\mathbf{P}(\Delta)$ for each Borel set $\Delta\subset[a,b]$, it is non-negative if $(\mathbf{P}(\Delta)x,x)\geqslant0$ for all Borel sets $\Delta\subset[a,b]$ and for all elements $x\in H$.

In following Lemma 2.1, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations on $[a,b]$ and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint. We assume that these measures are extended on the segment $[a,b_0]\supset[a,b_0)\supset[a,b]$ in the manner described above.

Lemma 2.1. [13] *Let f, g be functions integrable on $[a,b_0]$ with respect to the measure $\mathbf{q}$ and $y_0,z_0\in H$. Then any functions*

$$y(t)=y_0-iJ\int_{t_0}^{t}d\mathbf{p}_1(s)y(s)-iJ\int_{t_0}^{t}d\mathbf{q}(s)f(s),\quad z(t)=z_0-iJ\int_{t_0}^{t}d\mathbf{p}_2(s)z(s)-iJ\int_{t_0}^{t}d\mathbf{q}(s)g(s)\quad(a\leqslant t_0<b_0,\ t_0\leqslant t\leqslant b_0)$$

satisfy the following formula (*analogous to the Lagrange one*):

$$\int_{c_1}^{c_2}(d\mathbf{q}(t)f(t),z(t)) - \int_{c_1}^{c_2}(y(t),d\mathbf{q}(t)g(t)) = (iJy(c_2),z(c_2)) - (iJy(c_1),z(c_1)) + \int_{c_1}^{c_2}(y(t),d\mathbf{p}_2(t)z(t)) -$$
$$- \int_{c_1}^{c_2}(d\mathbf{p}_1(t)y(t),z(t)) - \sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{p}_2(\{t\})z(t)) - \sum_{t\in\mathcal{S}_{\mathbf{q}}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{p}_2(\{t\})z(t)) -$$
$$- \sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{q}(\{t\})g(t)) - \sum_{t\in\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{q}(\{t\})g(t)), \quad t_0 \leqslant c_1 < c_2 \leqslant b_0. \quad (3)$$

Further we assume that measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations and $\mathbf{p}$ is self-adjoint, $\mathbf{m}$ is non-negative. We consider equation (1), where $x_0 \in H$, f is integrable with respect to the measure $\mathbf{m}$ on $[a,b]$, $a \leqslant t \leqslant b_0$. We construct a continuous measure $\mathbf{p}_0$ from the measure $\mathbf{p}$ in the following way. We set $\mathbf{p}_0(\{t_k\}) = 0$ for $t_k \in \mathcal{S}_{\mathbf{p}}$ and we set $\mathbf{p}_0(\Delta) = \mathbf{p}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. Similarly, we construct a continuous measure $\mathbf{m}_0$ from the measure $\mathbf{m}$. We denote $\widehat{\mathbf{p}} = \mathbf{p} - \mathbf{p}_0$, $\widehat{\mathbf{m}} = \mathbf{m} - \mathbf{m}_0$. Then $\widehat{\mathbf{p}}(\{t_k\}) = \mathbf{p}(\{t_k\})$ for all $t_k \in \mathcal{S}_{\mathbf{p}}$ and $\widehat{\mathbf{p}}(\Delta) = 0$ for all Borel sets Δ such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. The similar equalities hold for the measure $\widehat{\mathbf{m}}$. The measures $\mathbf{p}_0$, $\widehat{\mathbf{p}}$, $\mathbf{m}_0$, $\widehat{\mathbf{m}}$ are self-adjoint and the measures $\mathbf{m}_0$, $\widehat{\mathbf{m}}$ are non-negative.

We replace $\mathbf{p}$ by $\mathbf{p}_0$ and $\mathbf{m}$ by $\mathbf{m}_0$ in (1). Then we obtain the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t d\mathbf{m}_0(s)f(s). \quad (4)$$

Equations (1), (4) have unique solutions (see [12]).

By $W(t,\lambda)$ denote an operator solution of the equation

$$W(t,\lambda)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s,\lambda)x_0 - iJ\lambda\int_a^t d\mathbf{m}_0(s)W(s,\lambda)x_0, \quad (5)$$

where $x_0 \in H$, $\lambda \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers). It follows from Lemma 2.1 that $W^*(t,\overline{\lambda})JW(t,\lambda) = J$. The functions $t \to W(t,\lambda)$ and $t \to W^{-1}(t,\lambda) = JW^*(t,\overline{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently there exist constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the inequality $\varepsilon_1\|x\|^2 \leqslant \|W(t,\lambda)x\|^2 \leqslant \varepsilon_2\|x\|^2$ holds for all $x \in H$, $t \in [a,b_0]$, $\lambda \in C \subset \mathbb{C}$ (C is a compact set).

Lemma 2.2. [17]. *Suppose that a function f is integrable with respect to the measure* $\mathbf{m}$. *A function y is a solution of the equation*

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda\int_a^t d\mathbf{m}_0(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \quad x_0 \in H, \quad a \leqslant t \leqslant b_0, \quad (6)$$

if and only if y has the form

$$y(t) = W(t,\lambda)x_0 - W(t,\lambda)iJ\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi).$$

3. Linear relations generated by the integral equation

This article is a continuation of the work [17]. In this section, we provide definitions and statements from [17] that are used in this article.

Let $\mathbf{B}$ be a Hilbert space. A linear relation T is understood as any linear manifold $T \subset \mathbf{B} \times \mathbf{B}$. The terminology on the linear relations can be found, for example, in [19], [25], [2]. In what follows we make use of the following notations: $\{\cdot,\cdot\}$ is an ordered pair; $\mathcal{D}(T)$ is the domain of T; $\mathcal{R}(T)$ is the range of T; $\ker T$ is a set of elements $x \in \mathbf{B}$ such that $\{x,0\} \in T$; T^{-1} is the relation inverse for T, i.e., the relation formed by the pairs $\{x', x\}$, where $\{x, x'\} \in T$. A relation T is called surjective if $\mathcal{R}(T) = \mathbf{B}$. A relation T is called invertible or injective if $\ker T = \{0\}$ (i.e., the relation T^{-1} is an operator); it is called continuously invertible if it is closed, invertible, and surjective (i.e., T^{-1} is a bounded everywhere defined operator). A relation T^* is called adjoint for T if T^* consists of all pairs $\{y_1, y_2\}$ such that equality $(x_2, y_1) = (x_1, y_2)$ holds for all pairs $\{x_1, x_2\} \in T$. A relation T is called symmetric if $T \subset T^*$ and self-adjoint if $T = T^*$.

It is known (see, for example, [20, ch.3], [19, ch.1]) that the graph of an operator $T : \mathcal{D}(T) \to \mathbf{B}$ is the set of pairs $\{x, Tx\} \in \mathbf{B} \times \mathbf{B}$, where $x \in \mathcal{D}(T) \subset \mathbf{B}$. Consequently, the linear operators can be treated as linear relations; this is why the notation $\{x_1, x_2\} \in T$ is used also for the operator T. Since all considered relations are linear, we shall often omit the word "linear".

Let $\mathbf{m}$ is a non-negative operator measure defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in the space H. The measure $\mathbf{m}$ is assumed to have a bounded variation on $[a, b]$. We introduce the quasi-scalar product $(x, y)_{\mathbf{m}} = \int_a^{b_0} ((d\mathbf{m})x(t), y(t))$ on a set of step-like functions with values in H defined on the segment $[a, b_0]$. Identifying with zero functions y obeying $(y, y)_{\mathbf{m}} = 0$ and making the completion, we arrive at the Hilbert space denoted by $L_2(H, d\mathbf{m}; a, b) = \mathfrak{H}$. The elements of $\mathfrak{H}$ are the classes of functions identified with respect to the norm $\|y\|_{\mathbf{m}} = (y, y)_{\mathbf{m}}^{1/2}$. In order not to complicate the terminology, the class of functions with a representative y is indicated by the same symbol and we write $y \in \mathfrak{H}$. The equality of functions in $\mathfrak{H}$ is understood as the equality for associated equivalence classes.

Let us define a *minimal relation* L_0 in the following way. The relation L_0 consists of all pairs $\{\widetilde{y}, \widetilde{f_0}\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the condition: for each pair $\{\widetilde{y}, \widetilde{f_0}\}$ there exists a pair $\{y, f_0\}$ such that the pairs $\{\widetilde{y}, \widetilde{f_0}\}$, $\{y, f_0\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and $\{y, f_0\}$ satisfies equation (1) and the equalities

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \quad \mathbf{m}(\{\beta\}) f_0(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{7}$$

Further, without loss of generality it can be assumed that if $\{y, f_0\} \in L_0$, then equalities (1), (7) hold for this pair. In general, the relation L_0 is not an operator since a function y can happen to be identified with zero in $\mathfrak{H}$, while f is non-zero. The relation L_0 is symmetric and closed. We note that if $y \in \mathcal{D}(L_0)$, then y is continuous and $y(b) = 0$ (see[16], [17]).

By $\mathfrak{X}_A = \mathfrak{X}_A(t)$ denote an operator characteristic function of a set A, i.e., $\mathfrak{X}_A(t) = E$ if $t \in A$ and $\mathfrak{X}_A(t) = 0$ if $t \notin A$. We shall often omit the argument t in the notation $\mathfrak{X}_A$. By $\overline{\mathcal{S}}_{\mathbf{p}}$ denote the closure of the set $\mathcal{S}_{\mathbf{p}}$. Let $\mathcal{S}_0$ be the set $t \in [a, b]$ such that $y(t) = 0$ for all $y \in \mathcal{D}(L_0)$. The set $\mathcal{S}_0$ is closed and $\overline{\mathcal{S}}_{\mathbf{p}} \cup \{a\} \cup \{b\} \subset \mathcal{S}_0$ (see[17]).

Lemma 3.1. [17]. *Suppose $\{y, f\} \in L_0$. Then $f(t) = 0$ for $\mathbf{m}$-almost all $t \in \mathcal{S}_0$.*

By $\mathfrak{H}_0$ (by $\mathfrak{H}_1$) denote a subspace of functions that vanish on $[a, b] \setminus \mathcal{S}_0$ (on $\mathcal{S}_0$, respectively) with respect to the norm in $\mathfrak{H}$. The subspaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ are orthogonal and $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. We note that $\mathfrak{H}_0 = \{0\}$ if and only if $\mathbf{m}(\mathcal{S}_0) = 0$. We denote $L_{10} = L_0 \cap (\mathfrak{H}_1 \times \mathfrak{H}_1)$. Then $\mathcal{D}(L_{10}) \subset \mathfrak{H}_1$, $\mathcal{R}(L_{10}) \subset \mathfrak{H}_1$. It follows from Lemma 3.1 that

$$L_0^* = (\mathfrak{H}_0 \times \mathfrak{H}_0) \oplus L_{10}^*, \tag{8}$$

i.e., the relation L_0^* consists of all pairs $\{y, f\} \in \mathfrak{H}$ of the form $\{y, f\} = \{u, v\} + \{z, g\} = \{u + z, v + g\}$, where $u, v \in \mathfrak{H}_0$, $\{z, g\} \in L_{10}^*$.

The set $\mathcal{T}_{\mathbf{p}} = (a, b) \setminus \mathcal{S}_0$ is open and it is the union of at most a countable number of disjoint open intervals $\mathcal{J}_k$, i.e., $\mathcal{T}_{\mathbf{p}} = \bigcup_{k=1}^{\Bbbk_1} \mathcal{J}_k$ and $\mathcal{J}_k \cap \mathcal{J}_j = \varnothing$ for $k \neq j$, where $\Bbbk_1$ is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). By $\mathbb{J}$ denote the set of these intervals $\mathcal{J}_k$. We note that the boundaries α_k, β_k of any interval $\mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$ belong to $\mathcal{S}_0$.

We denote

$$w_k(t, \lambda) = \mathfrak{X}_{[\alpha_k, \beta_k)} W(t, \lambda) W^{-1}(\alpha_k, \lambda), \tag{9}$$

where $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then (see[17])

$$w_k^*(t, \overline{\lambda}) J w_k(t, \lambda) = J, \quad \alpha_k \leqslant t < \beta_k. \tag{10}$$

By $\mathfrak{H}_{10}$ (by $\mathfrak{H}_{11}$) denote a subspace of functions that belong to $\mathfrak{H}_1$ and vanish on $\mathcal{S}_{\mathbf{m}}$ (on $[a,b] \setminus \mathcal{S}_{\mathbf{m}}$, respectively) with respect to the norm in $\mathfrak{H}$. So, $\mathfrak{H}_{10}$ ($\mathfrak{H}_{11}$) consists of functions of the form $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup\mathcal{S}_{\mathbf{m}})}h$ (of the form $\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\setminus\mathcal{S}_0}h$, respectively), where $h \in \mathfrak{H}$ is an arbitrary function. Therefore,

$$\mathfrak{H}_1 = \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}, \quad \mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}.$$

Obviously, the space $\mathfrak{H}_{11}$ is the closure in $\mathfrak{H}$ of the linear span of functions that have the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$, where $x \in H$, $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. By (7), it follows that $\mathfrak{H}_{11} \subset \ker L^*_{10}$.

Let $u_k(t,\lambda,\tau): H \to \mathfrak{H}_1$ be an operator acting by the formula

$$u_k(t,\lambda,\tau)x = -\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s) \lambda \mathfrak{X}_{\{\tau\}}(s)x, \tag{11}$$

where $x \in H$, $\tau \in (\alpha_k,\beta_k) \cap \mathcal{S}_{\mathbf{m}}$, $(\alpha_k,\beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then (see[17]) for any $x \in H$ the function

$$u_k(\cdot,\lambda,\tau)x + \mathfrak{X}_{\{\tau\}}(\cdot)x \in \ker(L^*_{10} - \lambda E).$$

Lemma 3.2. [17]. *The linear span of functions of the form* $\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(\cdot,\lambda)x_0$ *and* $u_k(\cdot,\lambda,\tau)B_k x_j + \mathfrak{X}_{\{\tau\}}(\cdot)B_k x_j$ *is dense in* $\ker(L^*_{10} - \lambda E)$. *Here* $x_j, x_0 \in H$; $\tau \in (\alpha_k,\beta_k) \cap \mathcal{S}_{\mathbf{m}}$; $B_k : H \to H$ *is a bounded continuously invertible operator;* $k = 1,\dots,\mathbb{k}_1$ *if* $\mathbb{k}_1$ *is finite and* k *is any natural number if* $\mathbb{k}_1$ *is infinite.*

Let $\mathbb{M}$ be a set consisting of intervals $\mathcal{J} \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. The set $\mathbb{M}$ is at most countable. Let $\mathbb{k}$ be the number of elements in $\mathbb{M}$. We arrange the elements of $\mathbb{M}$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_k$, where k is any natural number if the number of elements in $\mathbb{M}$ is infinite, and $1 \leqslant k \leqslant \mathbb{k}$ if the number of elements in $\mathbb{M}$ is finite.

To each element $\mathcal{E}_k \in \mathbb{M}$ assign an operator function v_k in the following way. If $\mathcal{E}_k$ is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k,\beta_k) \in \mathbb{J}$, then

$$v_k(t,\lambda) = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda). \tag{12}$$

If $\mathcal{E}_k$ is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, $\tau_k \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$, and $\tau_k \in \mathcal{J}_n = (\alpha_n,\beta_n) \in \mathbb{J}$, then

$$v_k(t,\lambda) = u_n(t,\lambda,\tau_k) w_n(\tau_k,\lambda) + \mathfrak{X}_{\{\tau_k\}}(t) w_n(\tau_k,\lambda). \tag{13}$$

Further, we denote $v_k(t,0) = v_k(t)$. We note that $u_k(t,0,\tau) = 0$ (see equality (11)).

Let $Q_{k,0}$ be a set $x \in H$ such that the functions $t \to v_k(t)x$ are identical with zero in $\mathfrak{H}$. We put $Q_k = H \ominus Q_{k,0}$. On the linear space Q_k we introduce a norm $\|\cdot\|_-$ by the equality

$$\|\xi_k\|_- = \|v_k(\cdot)\xi_k\|_{\mathfrak{H}}, \quad \xi_k \in Q_k. \tag{14}$$

By Q_k^- denote the completion of Q_k with respect to norm (14). The space Q_k^- can be treated as a space with a negative norm with respect to Q_k ([4, ch. 1], [19, ch.2]). By Q_k^+ denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that $Q_k^+ \subset Q_k \subset Q_k^-$. By $(\cdot,\cdot)_+$ and $\|\cdot\|_+$ we denote the scalar product and the norm in Q_k^+, respectively.

Remark 3.3. *The set* $Q_{k,0}$ *will not change if the function* $v_k(\cdot) = v_k(\cdot,0)$ *is replaced by* $v_k(\cdot,\lambda)$ *in the definition of* $Q_{k,0}$. *Moreover, with such replacement, the space* Q_k^- *will not change in the following sense: the set* Q_k^- *will not change, and the norm in it will be replaced by the equivalent one. The similar statement holds for the space* Q_k^+ (*see* [17]).

Suppose that a sequence $\{x_{kn}\}$, $x_{kn} \in Q_k$, converges in the space Q_k^- to $x_0 \in Q_k^-$ as $n \to \infty$. Then the sequence $\{v_k(\cdot,\lambda)x_{kn}\}$ is fundamental in $\mathfrak{H}$. Therefore this sequence converges to some element in $\mathfrak{H}$. By $v_k(\cdot,\lambda)x_0$ we denote this element.

Let $\widetilde{Q_N^-} = Q_1^- \times \dots \times Q_N^-$ ($\widetilde{Q_N^+} = Q_1^+ \times \dots \times Q_N^+$) be the Cartesian product of the first N sets Q_k^- (Q_k^+, respectively) and let $V_N(t,\lambda) = (v_1(t,\lambda),\dots,v_N(t,\lambda))$ be the operator one-row matrix. It is convenient to treat elements from $\widetilde{Q_N^-}$ as one-column matrices, and to assume that $V_N(t,\lambda)\widetilde{\xi_N} = \sum_{k=1}^N v_k(t,\lambda)\xi_k$, where we denote $\widetilde{\xi_N} =$

$\operatorname{col}(\xi_1, ..., \xi_N) \in \widetilde{Q}_N^-$, $\xi_k \in Q_k^-$. Let $\ker_k(\lambda)$ be a linear space of functions $t \to v_k(t,\lambda)\xi_k$, $\xi_k \in Q_k^-$. The space $\ker_k(\lambda)$ is closed in $\mathfrak{H}$. We denote $\mathcal{K}_N(\lambda) = \ker_1(\lambda) \dotplus ... \dotplus \ker_N(\lambda)$. Obviously, $\mathcal{K}_{N_1}(\lambda) \subset \mathcal{K}_{N_2}(\lambda)$ for $N_1 < N_2$. By $\mathcal{V}_N(\lambda)$ denote the operator $\widetilde{\xi}_N \to V_N(\cdot,\lambda)\widetilde{\xi}_N$, where $\widetilde{\xi}_N \in \widetilde{Q}_N^-$. The operator $\mathcal{V}_N(\lambda)$ maps continuously and one-to-one $\widetilde{Q}_N^-$ onto $\mathcal{K}_N(\lambda) \subset \mathfrak{H}_1 \subset \mathfrak{H}$.

Let $\mathcal{Q}_-, \mathcal{Q}_+, \mathcal{Q}$ be linear spaces of sequences, respectively, $\widetilde{\eta} = \{\eta_k\}$, $\widetilde{\varphi} = \{\varphi_k\}$, $\widetilde{\xi} = \{\xi_k\}$, where $\eta_k \in Q_k^-$, $\varphi_k \in Q_k^+$, $\xi_k \in Q_k$; $k \in \mathbb{N}$ if $\Bbbk = \infty$, and $1 \leqslant k \leqslant \Bbbk$ if $\Bbbk$ is finite; $\Bbbk$ is the number of elements in $\mathbb{M}$. We assume that the series $\sum_{k=1}^{\infty} \|\eta_k\|_-^2$, $\sum_{k=1}^{\infty} \|\varphi_k\|_+^2$, $\sum_{k=1}^{\infty} \|\xi_k\|^2$ converge if $\Bbbk = \infty$. These spaces become Hilbert spaces if we introduce scalar products by the formulas

$$(\widetilde{\eta}, \widetilde{\zeta})_- = \sum_{k=1}^{\Bbbk} (\eta_k, \zeta_k)_-, \quad \widetilde{\eta}, \widetilde{\zeta} \in \mathcal{Q}_-; \quad (\widetilde{\varphi}, \widetilde{\psi})_+ = \sum_{k=1}^{\Bbbk} (\varphi_k, \psi_k)_+, \quad \widetilde{\varphi}, \widetilde{\psi} \in \mathcal{Q}_+; \quad (\widetilde{\xi}, \widetilde{\sigma}) = \sum_{k=1}^{\Bbbk} (\xi_k, \sigma_k), \quad \widetilde{\xi}, \widetilde{\sigma} \in \mathcal{Q}.$$

The spaces $\mathcal{Q}_+$,$\mathcal{Q}_-$ can be treated as spaces with positive and negative norms with respect to $\mathcal{Q}$ ([4, ch. 1], [19, ch.2]). So $\mathcal{Q}_+ \subset \mathcal{Q} \subset \mathcal{Q}_-$ and $\gamma_1 \|\widetilde{\varphi}\|_- \leqslant \|\widetilde{\varphi}\| \leqslant \gamma_2 \|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi} \in \mathcal{Q}_+$, $\gamma_1, \gamma_2 > 0$. The "scalar product" $(\widetilde{\eta}, \widetilde{\varphi})$ is defined for all $\widetilde{\varphi} \in \mathcal{Q}_+$, $\widetilde{\eta} \in \mathcal{Q}_-$. If $\widetilde{\eta} \in \mathcal{Q}$, then $(\widetilde{\eta}, \widetilde{\varphi})$ coincides with the scalar product in $\mathcal{Q}$.

Let $\mathcal{M} \subset \mathcal{Q}_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set $\mathcal{M}$ is dense in the space $\mathcal{Q}_-$. The operator $\mathcal{V}_N(\lambda)$ is the restriction of $\mathcal{V}_{N+1}(\lambda)$ to $\widetilde{Q}_N^-$. By $\mathcal{V}'(\lambda)$ denote an operator in $\mathcal{M}$ such that $\mathcal{V}'(\lambda)\widetilde{\eta} = \mathcal{V}_N(\lambda)\widetilde{\eta}_N$ for all $N \in \mathbb{N}$, where $\widetilde{\eta} = (\widetilde{\eta}_N, 0, ...)$, $\widetilde{\eta}_N \in \widetilde{Q}_N^-$. The operator $\mathcal{V}'(\lambda)$ admits an extension by continuity to the space $\mathcal{Q}_-$. By $\mathcal{V}(\lambda)$ denote the extended operator. This operator maps continuously and one-to-one $\mathcal{Q}_-$ onto $\ker(L_{10}^* - \lambda E) \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Moreover, we denote $\widetilde{V}(t,\lambda)\widetilde{\eta} = (\mathcal{V}(\lambda)\widetilde{\eta})(t)$, where $\widetilde{\eta} = \{\eta_k\} \in \mathcal{Q}_-$.

The adjoint operator $\mathcal{V}^*(\lambda)$ maps continuously $\mathfrak{H}$ onto $\mathcal{Q}_+$ and

$$\mathcal{V}^*(\lambda) f = \int_a^{b_0} \widetilde{V}^*(t,\lambda) d\mathbf{m}(t) f(t). \tag{15}$$

Lemma 3.4. [17]. *The operator $\mathcal{V}(\lambda)$ maps $\mathcal{Q}_-$ onto $\ker(L_{10}^* - \lambda E)$ continuously and one to one. A function z belongs to $\ker(L_{10}^* - \lambda E)$ if and only if there exists an element $\widetilde{\eta} = \{\eta_k\} \in \mathcal{Q}_-$ such that $z(t) = (\mathcal{V}(\lambda)\widetilde{\eta})(t) = \widetilde{V}(t,\lambda)\widetilde{\eta}$. The operator $\mathcal{V}^*(\lambda)$ maps $\mathfrak{H}$ onto $\mathcal{Q}_+$ continuously, and acts by formula* (15), *and* $\ker \mathcal{V}^*(\lambda) = \mathfrak{H}_0 \oplus \mathcal{R}(L_{10} - \overline{\lambda} E)$. *Moreover, $\mathcal{V}^*(\lambda)$ maps $\ker(L_{10}^* - \lambda E)$ onto $\mathcal{Q}_+$ one to one.*

The following theorem is proved in [17]. We have changed some designations from [17] to shorten the record.

Theorem 3.5. *A pair $\{\widetilde{y}, \widetilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to $L_0^* - \lambda E$ if and only if there exist a pair $\{\widehat{y}, \widehat{f}\} \in \mathfrak{H} \times \mathfrak{H}$, functions $y_0, y_0' \in \mathfrak{H}_0$, $y, f \in \mathfrak{H}_1$, and an element $\widetilde{\eta} \in \mathcal{Q}_-$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{\widehat{y}, \widehat{f}\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and the equalities*

$$\widehat{y} = y_0 + y, \ \widehat{f} = y_0' + f,$$

$$y(t) = \widetilde{V}(t,\lambda)\widetilde{\eta} - \sum_{k=1}^{\Bbbk_1} \mathfrak{X}_{[a,b]\setminus \mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) iJ \int_a^t w_k^*(s,\overline{\lambda}) d\mathbf{m}(s) f(s) \tag{16}$$

hold, where the series in (16) *converges in $\mathfrak{H}$, $\Bbbk_1$ is the number of intervals $\mathcal{J}_k \in \mathbb{J}$.*

4. The description of generalized resolvents

Let T be a symmetric relation, $T \subset \mathbf{B} \times \mathbf{B}$ ($\mathbf{B}$ is a Hilbert space), and let $\widetilde{T}$ be a self-adjoint extension of T to $\widetilde{\mathbf{B}}$, where $\widetilde{\mathbf{B}}$ is a Hilbert space, $\widetilde{\mathbf{B}} \supset \mathbf{B}$, and scalar products coincide in $\mathbf{B}$ and $\widetilde{\mathbf{B}}$. By P denote an orthogonal projection of $\widetilde{\mathbf{B}}$ onto $\mathbf{B}$. The function $\lambda \to R_\lambda$ defined by the formula $R_\lambda = P(\widetilde{T} - \lambda E)^{-1}|_{\mathbf{B}}$, $\operatorname{Im}\lambda \neq 0$, is called the generalized resolvent of the relation T (see, for example, [1, ch.9]).

A.V. Straus (see [26]) obtained a formula for all generalized resolvents of a symmetric operator. It is shown in [18] that this formula remains true for symmetric relations also. By $\mathfrak{N}_\lambda$ denote a defect subspace of the symmetric relation T, i.e., the orthogonal complement in $\mathbf{B}$ of the range of the relation $T-\lambda E$. We fix some number λ_0 ($\mathrm{Im}\lambda_0 \neq 0$). Let $\lambda \to \mathcal{F}(\lambda)$ be a holomorphic operator function, where $\mathcal{F}(\lambda):\mathfrak{N}_{\lambda_0}\to\mathfrak{N}_{\overline{\lambda}_0}$ is a bounded operator, $\|\mathcal{F}(\lambda)\| \leqslant 1$, $\mathrm{Im}\lambda\cdot\mathrm{Im}\lambda_0 > 0$. Let $T_{\mathcal{F}(\lambda)}$ be the relation consisting of all pairs of the form $\{y_0+\mathcal{F}(\lambda)z-z, y_1+\lambda_0\mathcal{F}(\lambda)z-\overline{\lambda}_0 z\}$, where $\{y_0,y_1\}\in T$, $z\in\mathfrak{N}_{\lambda_0}$. Then (see [26], [18]) the family of operators R_λ is a generalized resolvent of T if and only if R_λ can be represented in the form

$$R_\lambda = (T_{\mathcal{F}(\lambda)}-\lambda E)^{-1}, \quad \mathrm{Im}\lambda\cdot\mathrm{Im}\lambda_0 > 0. \tag{17}$$

Theorem 4.1. *Let R_λ ($\mathrm{Im}\lambda \neq 0$) be a generalized resolvent of the relation L_{10} and $y = R_\lambda f$. Then*

$$y(t) = \int_a^b \widetilde{V}(t,\lambda)M(\lambda)\widetilde{V}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+$$
$$+2^{-1}\sum_{n=1}^{\Bbbk_1}\int_a^b \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}(t)w_n(t,\lambda)\mathrm{sgn}(s-t)iJw_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(s)f(s)-\lambda^{-1}\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_n,\beta_n)}(t)f(t), \tag{18}$$

where $M(\lambda):\mathcal{Q}_+\to\mathcal{Q}_-$ is the bounded operator such that $M(\overline{\lambda}) = M^*(\lambda)$, $\mathrm{Im}\lambda\neq 0$. The function $\lambda\to M(\lambda)\widetilde{x}$ is holomorphic for every $\widetilde{x}\in\mathcal{Q}_+$ in the half-planes $\mathrm{Im}\lambda\neq 0$. If $\mathcal{S}_\mathbf{m}=\varnothing$, then

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x}) \geqslant 0 \tag{19}$$

for every λ ($\mathrm{Im}\lambda\neq 0$) and for every $\widetilde{x}\in\mathcal{Q}_+$.

Proof. Suppose $y = R_\lambda f$. By (17), it follows that the pair $\{y,f\}\in L_{10}^*-\lambda E$. Equality (18) follows from (17) and [17, Theorem 4.3]. Using (18), we get

$$y(t)=\widetilde{V}(t,\lambda)M(\lambda)\int_a^b\widetilde{V}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+\sum_{n=1}^{\Bbbk_1}\Big(-2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(s)f(s)+$$
$$+2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_\mathbf{m}}(t)w_n(t,\lambda)iJ\int_t^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(s)f(s)\Big)-\lambda^{-1}\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{\mathcal{S}_\mathbf{m}\cap(\alpha_n,\beta_n)}(t)f(t). \tag{20}$$

Let us prove that the function $\lambda\to M(\lambda)\widetilde{x}$ is holomorphic for every $\widetilde{x}\in\mathcal{Q}_+$ ($\mathrm{Im}\lambda\neq 0$). We denote $S(\lambda) = M(\lambda)\mathcal{V}^*(\overline{\lambda})$. It follows from (18) and the holomorphicity of the function $\lambda\to R_\lambda$ that the function $\lambda\to\mathcal{V}(\lambda)S(\lambda)f$ is holomorphic. Using (10), we obtain that the function $\lambda\to S(\lambda)f$ is holomorphic. Now the holomorphicity of the function $\lambda\to M(\lambda)$ follows from Lemma 4.2. This Lemma is formulated after the proof of the Theorem. In Lemma 4.2 it should be taken that $\mathcal{B}_1=\mathfrak{H}_1$, $\mathcal{B}_2=\mathcal{Q}_+$, $\mathcal{B}_3=\mathcal{Q}_-$, $T_1(\lambda)=\mathcal{V}^*(\overline{\lambda})$, $T_2(\lambda)=M(\lambda)$, $T_3(\lambda)=S(\lambda)$.

We note that the equality $R_\lambda^* = R_{\overline{\lambda}}$ implies $M(\overline{\lambda}) = M^*(\lambda)$.

Let us prove that (19) holds under the condition $\mathcal{S}_\mathbf{m}=\varnothing$. Then $\mathbf{m}=\mathbf{m}_0$. It follows from Lemma 3.4 that there exists a function $f\in\mathfrak{H}$ such that $\widetilde{x}=\mathcal{V}^*(\overline{\lambda})f$. Let $p_n:\mathcal{Q}_-\to Q_n^-$ be the operator defined by the formula $p_n\widetilde{\xi}=\xi_n$, where $\widetilde{\xi}=\{\xi_n\}\in\mathcal{Q}_-$. We denote $M_n(\lambda)=p_nM(\lambda)$, $x_n=p_n\widetilde{x}$. Since $\mathcal{S}_\mathbf{m}=\varnothing$, we obtain from (20)

$$y(t)=\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{[\alpha_n,\beta_n)}(t)w_n(t,\lambda)M_n(\lambda)\widetilde{x}+2^{-1}\sum_{n=1}^{\Bbbk_1}\Big(-\mathfrak{X}_{[\alpha_n,\beta_n)}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+$$
$$+\mathfrak{X}_{[\alpha_n,\beta_n)}(t)w_n(t,\lambda)iJ\int_t^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)\Big). \tag{21}$$

We denote

$$z(t) = \widetilde{V}(t,\lambda)(M(\lambda)\widetilde{x} - 2^{-1}i\widetilde{J}\widetilde{x}) = \sum_{n=1}^{\Bbbk_1} z_n(t), \quad z_n(t) = \mathfrak{X}_{[\alpha_n,\beta_n)} z = w_n(t,\lambda)(M_n(\lambda)\widetilde{x} - 2^{-1}iJx_n),$$

where $\widetilde{J}$ is the operator in Q acting as $\widetilde{J}\widetilde{\xi} = \{J\xi_k\}$, $\widetilde{\xi} = \{\xi_k\} \in Q$. Using (9), (10), (21), we get

$$y(\alpha_n) = M_n(\lambda)\widetilde{x} + 2^{-1}iJx_n, \quad z_n(\alpha_n) = M_n(\lambda)\widetilde{x} - 2^{-1}iJx_n, \tag{22}$$

$$\lim_{t\to\beta_n-0} y(t) = \lim_{t\to\beta_n-0} z_n(t) = W(\beta_n,\lambda)W^{-1}(\alpha_n,\lambda)(M_n(\lambda)\widetilde{x} - 2^{-1}iJx_n). \tag{23}$$

It follows from Lemmas 3.2, 3.4 that $z \in \ker L_{10}^* - \lambda E$. Consequently, $\{y-z, f\} \in L_{10}^* - \lambda E$, $\{y+z, f\} \in L_{10}^* - \lambda E$. Then the pairs $\{y - z, g_1\} \in L_{10}^*$, $\{y + z, g_2\} \in L_{10}^*$, where

$$g_1 = f + \lambda(y - z), \quad g_2 = f + \lambda(y + z). \tag{24}$$

We denote $y_n = \mathfrak{X}_{[\alpha_n,\beta_n)} y$, $g_{1n} = \mathfrak{X}_{[\alpha_n,\beta_n)} g_1$, $g_{2n} = \mathfrak{X}_{[\alpha_n,\beta_n)} g_2$, $f_n = \mathfrak{X}_{[\alpha_n,\beta_n)} f$. Then $\{y_n - z_n, g_{1n}\} \in L_{10}^*$, $\{y_n + z_n, g_{2n}\} \in L_{10}^*$. Taking into account Theorem 3.5 (for $\lambda = 0$) and (22), we obtain

$$y_n(t) - z_n(t) = w_n(t,0)iJx_n - w_n(t,0)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}(s)g_{1n}(s),$$

$$y_n(t) + z_n(t) = 2w_n(t,0)M_n(\lambda)\widetilde{x} - w_n(t,0)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}(s)g_{2n}(s).$$

It follows from Lemma 2.2 that formula (3) can be applied to the functions $y_n - z_n$, $y_n + z_n$ on the interval $[\alpha_n, \beta]$ ($\alpha_n < \beta < \beta_n$). Using (3), we get

$$\int_{\alpha_n}^{\beta}(d\mathbf{m}(t)g_{1n}(t), y_n(t) + z_n(t)) - \int_{\alpha_n}^{\beta}(y_n(t) - z_n(t), d\mathbf{m}(t)g_{2n}) =$$
$$= (iJ(y_n(\beta) - z_n(\beta)), y_n(\beta) + z_n(\beta)) - (iJ(y_n(\alpha_n) - z_n(\alpha_n)), y_n(\alpha_n) + z_n(\alpha_n)). \tag{25}$$

Passing to the limit as $\beta \to \beta_n - 0$ in (25) and taking into account (22), (23), we obtain

$$\int_{\alpha_n}^{\beta_n}(d\mathbf{m}(t)g_{1n}(t), y_n(t) + z_n(t)) - \int_{\alpha_n}^{\beta_n}(y_n(t) - z_n(t), d\mathbf{m}(t)g_{2n}) = 2(x_n, M_n(\lambda)\widetilde{x}). \tag{26}$$

On the other hand, using (24), we get

$$(f_n, y_n + z_n)_{\mathfrak{H}} - (y_n - z_n, f_n)_{\mathfrak{H}} = (g_{1n} - \lambda(y_n - z_n), y_n + z_n)_{\mathfrak{H}} - (y_n - z_n, g_{2n} - \lambda(y_n + z_n))_{\mathfrak{H}} =$$
$$= (g_{1n}, y_n + z_n)_{\mathfrak{H}} - (y_n - z_n, g_{2n})_{\mathfrak{H}} - (\lambda - \overline{\lambda})(y_n - z_n, y_n + z_n)_{\mathfrak{H}}. \tag{27}$$

Combining (26) and (27), we obtain

$$(f_n, y_n + z_n)_{\mathfrak{H}} - (y_n - z_n, f_n)_{\mathfrak{H}} = 2(x_n, M_n(\lambda)\widetilde{x}) - (\lambda - \overline{\lambda})(y_n - z_n, y_n + z_n)_{\mathfrak{H}}.$$

Therefore,

$$(f, y + z)_{\mathfrak{H}} - (y - z, f)_{\mathfrak{H}} = 2(\widetilde{x}, M(\lambda)\widetilde{x}) - (\lambda - \overline{\lambda})(y - z, y + z)_{\mathfrak{H}}. \tag{28}$$

Equation (28) implies that

$$\mathrm{Im}[(f, y + z)_{\mathfrak{H}} - (y - z, f)_{\mathfrak{H}}] = 2\mathrm{Im}(\widetilde{x}, M(\lambda)\widetilde{x}) - \mathrm{Im}[(\lambda - \overline{\lambda})((y, y)_{\mathfrak{H}} - (z, y)_{\mathfrak{H}} + (y, z)_{\mathfrak{H}} - (z, z)_{\mathfrak{H}})].$$

Therefore,

$$\mathrm{Im}[(f,y)_{\mathfrak{H}} - (y,f)_{\mathfrak{H}}] = 2\mathrm{Im}(\widetilde{x}, M(\lambda)\widetilde{x}) - \mathrm{Im}[(\lambda - \overline{\lambda})((y,y)_{\mathfrak{H}} - (z,z)_{\mathfrak{H}})].$$

Consequently,

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x}, \widetilde{x}) = \|z\|^2_{\mathfrak{H}} + (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R_\lambda f, f)_{\mathfrak{H}} - (R_\lambda f, R_\lambda f)_{\mathfrak{H}}.$$

Since $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R_\lambda f, f)_{\mathfrak{H}} - (R_\lambda f, R_\lambda f)_{\mathfrak{H}} \geqslant 0$, we see that (19) holds. The theorem is proved. □

Lemma 4.2. [10]. *Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ be Banach spaces. Suppose bounded operators $T_3(\lambda):\mathcal{B}_1 \to \mathcal{B}_3$, $T_1(\lambda):\mathcal{B}_1 \to \mathcal{B}_2$, $T_2(\lambda):\mathcal{B}_2 \to \mathcal{B}_3$ satisfy the equality $T_3(\lambda) = T_2(\lambda)T_1(\lambda)$ for every fixed λ belonging to some neighborhood of a point λ_1 and suppose the range of operator $T_1(\lambda_1)$ coincides with $\mathcal{B}_2$. If functions $T_1(\lambda)$, $T_3(\lambda)$ are strongly differentiable at the point λ_1, then function $T_2(\lambda)$ is strongly differentiable at λ_1.*

Remark 4.3. *It follows from Lemma 3.1 and* (8) *that $L_0 \cap \mathfrak{H}_0 \times \mathfrak{H}_0 = \{0,0\}$. Therefore any generalized resolvent $\widetilde{R}_\lambda$ of the relation L_0 has the form $\widetilde{R}_\lambda = R_{0\lambda} \oplus R_\lambda$, where R_λ is some generalized resolvent of L_{10} and $R_{0\lambda}$ is a generalized resolvent of the relation $\{0,0\}$, i.e., $R_{0\lambda} = (T_{\mathcal{F}(\lambda)} - \lambda E)^{-1}$ (see* (17)*), $T_{\mathcal{F}(\lambda)}$ is the relation consisting of pairs of the form $\{\mathcal{F}(\lambda)z - z, \lambda_0\mathcal{F}(\lambda)z - \overline{\lambda}_0 z\}$ (here $\mathcal{F}(\lambda):\mathfrak{H}_0 \to \mathfrak{H}_0$ is a bounded operator, $\|\mathcal{F}(\lambda)\| \leqslant 1$, $z \in \mathfrak{H}_0$, the operator function $\lambda \to \mathcal{F}(\lambda)$ is holomorphic,* $\mathrm{Im}\lambda \cdot \mathrm{Im}\lambda_0 > 0$*).*

Remark 4.4. *In general, if $\mathcal{S}_{\mathbf{m}} \neq \varnothing$, then the inequality* $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x}, \widetilde{x}) < 0$ *is possible (see Remark 6.1).*

5. Boundary value problems connected with generalized resolvents

To shorten the notation, we denote $w_k(t,0) = w_k(t)$, $\widetilde{V}(t,0) = \widetilde{V}(t)$, $\mathcal{V}(0) = \mathcal{V}$. It follows from Lemma 3.4 (for $\lambda = 0$) that $\mathcal{V}^*f$ ($f \in \mathfrak{H}$) is an element of the space $\mathcal{Q}_+ \subset \mathcal{Q}$, i.e., a sequence with elements of the form

$$\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} \int_{\alpha_n}^{\beta_n} w_n^*(t)d\mathbf{m}(t)f(t), \tag{29}$$

$$w_n^*(\tau_{nk})\mathbf{m}(\{\tau_{nk}\})f(\tau_{nk}) \tag{30}$$

(and possibly with zeros), where $\tau_{nk} \in (\mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0) \cap \mathcal{J}_n$; $(\alpha_n, \beta_n) = \mathcal{J}_n$; $\mathcal{J}_n \in \mathbb{J}$; $1 \leqslant n \leqslant \Bbbk_1$ if the number $\Bbbk_1$ of intervals $\mathcal{J}_n \in \mathbb{J}$ is finite, and n is any natural number if $\Bbbk_1 = \infty$. We replace elements (29) by zeros in $\mathcal{V}^*f$. By $\mathcal{V}_0^*f$ denote the resulting sequence. So, $\mathcal{V}_0^*f$ is a sequence with elements of form (30) (and possibly with zeros). Further, we replace each element (29) and (30) in $\mathcal{V}^*f$ by the element

$$\sigma_n = \int_{\alpha_n}^{\beta_n} w_n^*(t)d\mathbf{m}(t)f(t) = \int_{\alpha_n}^{\beta_n} \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} w_n^*(t)d\mathbf{m}(t)f(t) + \sum_{\tau_{nk}\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)} w_n^*(\tau_{nk})\mathbf{m}(\{\tau_{nk}\})f(\tau_{nk}). \tag{31}$$

By $\mathcal{V}_*f$ denote the resulting sequence. We claim that $\mathcal{V}_*f \in \mathcal{Q}_-$. Indeed, let $\mathcal{V}_*f = \widetilde{\sigma} = \{\sigma_n\}$. It follows from (9), (10), (31) that $\|\sigma_n\| < \varepsilon_1 \left\|f\right\|_{\mathfrak{H}} = \varepsilon_2$, where $\varepsilon_1 > 0$, ε_1 is independent of n. Then

$$\mathcal{V}\widetilde{\sigma} = \mathcal{V}(0)\widetilde{\sigma} = \sum_{n=1}^{\Bbbk_1}\left(\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} w_n(t)\sigma_n + \sum_{\tau_{nk}\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)} \mathfrak{X}_{\{\tau_{nk}\}} w_n(\tau_{nk})\sigma_n\right),$$

and

$$\left\|\mathcal{V}\widetilde{\sigma}\right\|^2_{\mathfrak{H}} = \sum_{n=1}^{\Bbbk_1}\left(\left\|\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} w_n(t)\sigma_n\right\|^2_{\mathfrak{H}} + \sum_{\tau_{nk}\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)} \left\|\mathfrak{X}_{\{\tau_{nk}\}} w_n(\tau_{nk})\sigma_n\right\|^2_{\mathfrak{H}}\right) = \sum_{n=1}^{\Bbbk_1} \|w_n(t)\sigma_n\|^2_{\mathfrak{H}} \leqslant \varepsilon_3, \quad \varepsilon_3 > 0. \tag{32}$$

By (14), (15), (32), and the definition of Q_-, it follows that $\widetilde{\sigma} \in Q_-$. We note that this proof uses only the boundedness of the sequence $\{\sigma_n\}$ in H.

Further, we replace each element (30) in $\mathcal{V}_0^* f$ by the element $\int_{\alpha_n}^{\tau_{nk}} w^*(s)d\mathbf{m}(s)f(s)$. By $\mathcal{V}_{*\tau} f$ denote the resulting sequence. Then $\mathcal{V}_{*\tau} f \in Q_-$ (the proof is the same as for $\mathcal{V}_* f$). It follows from the definition $\mathcal{V}^* f$, $\mathcal{V}_0^* f$, $\mathcal{V}_* f$ $\mathcal{V}_{*\tau} f$ that the equalities

$$(\mathcal{V}^* f, \mathcal{V}_0^* g) = (\mathcal{V}_0^* f, \mathcal{V} g) = (\mathcal{V}_0^* f, \mathcal{V}_0^* g), \quad (\mathcal{V}_{*\tau} f, \mathcal{V}^* g) = (\mathcal{V}_{*\tau} f, \mathcal{V}_0^* g), \quad f, g \in \mathfrak{H}, \tag{33}$$

$$\sum_{n=1}^{\Bbbk_1} \left(iJ \int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)f(s), \int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)g(s) \right) = (i\widetilde{J}\mathcal{V}_* f, \mathcal{V}^* g) = (i\widetilde{J}\mathcal{V}^* f, \mathcal{V}_* g), \quad f, g \in \mathfrak{H} \tag{34}$$

hold. Using (10), we obtain

$$(iJw_n^*(\tau_{nk})\mathbf{m}(\{\tau_{nk}\})f(\tau_{nk}), w_n^*(\tau_{nk})\mathbf{m}(\{\tau_{nk}\})g(\tau_{nk})) = (iJ\mathbf{m}(\{\tau_{nk}\})f(\tau_{nk}), \mathbf{m}(\{\tau_{nk}\})g(\tau_{nk})), \quad f, g \in \mathfrak{H}.$$

Therefore,

$$\sum_{n=1}^{\Bbbk_1} (iJw_n^*(\tau_{nk})\mathbf{m}(\{\tau_{nk}\})f(\tau_{nk}), w_n^*(\tau_{nk})\mathbf{m}(\{\tau_{nk}\})g(\tau_{nk})) = \sum_{n=1}^{\Bbbk_1} (iJ\mathbf{m}(\{\tau_{nk}\})f(\tau_{nk}), \mathbf{m}(\{\tau_{nk}\})g(\tau_{nk})) = (i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}_0^* g). \tag{35}$$

We denote $\mathbf{H}_- = \mathfrak{H}_0 \times Q_-$, $\mathbf{H}_+ = \mathfrak{H}_0 \times Q_+$. Suppose a pair $\{\widetilde{y}, \widetilde{f}\} \in L_0^*$. By Theorem 3.5 (for $\lambda = 0$), there exists a pair $\{\widehat{y}, \widehat{f}\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{\widehat{y}, \widehat{f}\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and equalities

$$\widehat{y} = y_0 + y, \ \widehat{f} = y_0' + f, \ y(t) = \widetilde{V}(t)\widetilde{\eta} - \sum_{n=1}^{\Bbbk_1} \mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(t) w_n(t) iJ \int_a^t w_n^*(s)d\mathbf{m}(s)f(s) \tag{36}$$

hold, where $y_0, y_0' \in \mathfrak{H}_0$, $\{y, f\} \in L_{10}^*$, $\widetilde{\eta} \in Q_-$, the series in (36) converges in $\mathfrak{H}$, $\Bbbk_1$ is the number of intervals $\mathcal{J}_n \in \mathbb{J}$. With each such pair $\{\widehat{y}, \widehat{f}\}$ we associate a pair of boundary values $\{Y, Y'\} \in \mathbf{H}_- \times \mathbf{H}_+$ by formulas

$$Y = \{y_0, Y_{10}\} \in \mathbf{H}_- = \mathfrak{H}_0 \times Q_-, \quad Y' = \{y_0', Y_{10}'\} \in \mathbf{H}_+ = \mathfrak{H}_0 \times Q_+, \tag{37}$$

where

$$Y_{10} = \widetilde{\eta} - 2^{-1} i\widetilde{J}\mathcal{V}_* f + 2^{-1} i\widetilde{J}\mathcal{V}_0^* f + i\widetilde{J}\mathcal{V}_{*\tau} f, \quad Y_{10}' = \mathcal{V}^* f. \tag{38}$$

Let Γ denote the operator that takes each pair $\{\widehat{y}, \widehat{f}\} \in L_0^*$ to the ordered pair $\{Y, Y'\}$ of boundary values Y, Y', i.e., $\Gamma\{\widehat{y}, \widehat{f}\} = \{Y, Y'\}$. We put $\Gamma_1\{\widehat{y}, \widehat{f}\} = Y$, $\Gamma_2\{\widehat{y}, \widehat{f}\} = Y'$. It follows from Lemma 3.4 that if pairs $\{\widehat{y_1}, \widehat{f_1}\}$, $\{\widehat{y}, \widehat{f}\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$, then their boundary values coincide.

Theorem 5.1. *The range $\mathcal{R}(\Gamma)$ of the operator Γ coincides with $\mathbf{H}_- \times \mathbf{H}_+$ and "the Green formula"*

$$(\widehat{f}, \widehat{z})_{\mathfrak{H}} - (\widehat{y}, \widehat{g})_{\mathfrak{H}} = (Y', Z) - (Y, Z') \tag{39}$$

holds, where $\{\widehat{y}, \widehat{f}\}, \{\widehat{z}, \widehat{g}\} \in L_0^*$, $\Gamma\{\widehat{y}, \widehat{f}\} = \{Y, Y'\}$, $\Gamma\{\widehat{z}, \widehat{g}\} = \{Z, Z'\}$.

Proof. The equality $\mathcal{R}(\Gamma) = \mathbf{H}_- \times \mathbf{H}_+$ follows from Lemma 3.4 and formulas (8), (36)-(38). Let us prove (39). Suppose that a pair $\{y, f\}$ has form (36) and a pair $\{\widehat{z}, \widehat{g}\}$ has the form $\widehat{z} = z_0 + z$, $\widehat{g} = z_0' + g$, where $\{z, g\} \in L_{10}^*$, $z_0, z_0' \in \mathfrak{H}_0$, and

$$z(t) = \widetilde{V}(t)\widetilde{\zeta} - \sum_{n=1}^{\Bbbk_1} \mathfrak{X}_{[a,b]\setminus\mathcal{S}_\mathbf{m}}(t) w_n(t) iJ \int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)g(s), \quad \widetilde{\zeta} \in Q_-. \tag{40}$$

Then

$$(\widehat{f},\widehat{z})_{\mathfrak{H}} - (\widehat{y},\widehat{g})_{\mathfrak{H}} = (y_0', z_0)_{\mathfrak{H}} - (y_0, z_0')_{\mathfrak{H}} + (f,z)_{\mathfrak{H}} - (y,g)_{\mathfrak{H}}.$$

Thus, it is enough to prove the equality

$$(f,z)_{\mathfrak{H}} - (y,g)_{\mathfrak{H}} = (Y_{10}', Z_{10}) - (Y_{10}, Z_{10}').$$

We define the functions F_n, G_n, $\widetilde{F}$, $\widetilde{G}$ by the equalities

$$F_n(t) = -w_n(t)iJ\int_{\alpha_n}^{t} w_n^*(s)d\mathbf{m}(s)f(s),\quad G_n(t) = -w_n(t)iJ\int_{\alpha_n}^{t} w_n^*(s)d\mathbf{m}(s)g(s),\quad \widetilde{F}(t) = \sum_{n=1}^{\Bbbk_1} F_n(t),\quad \widetilde{G}(t) = \sum_{n=1}^{\Bbbk_1} G_n(t). \quad (41)$$

It follows from Lemma 2.2 that the functions F_n, G_n are solutions of equation (6) on $[\alpha_n, \beta_n)$ for $x_0 = 0$ (G_n is the solution if f, y are replaced by g, z, respectively, in (6)). Using (10) and Lemma 2.1 for $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_0$, $\mathbf{q} = \mathbf{m}$, $c_1 = \alpha_n$, $c_2 = \beta < \beta_n$, we obtain

$$\int_{\alpha_n}^{\beta} (f(s), d\mathbf{m}(s)G_n(s)) - \int_{\alpha_n}^{\beta} (F_n(s), d\mathbf{m}(s)g(s)) = \left(iJw_n(\beta)iJ\int_{\alpha_n}^{\beta} w_n^*(s)d\mathbf{m}(s)f(s), w_n(\beta)iJ\int_{\alpha_n}^{\beta} w_n^*(s)d\mathbf{m}(s)g(s)\right) -$$
$$- \sum_{\tau\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_n,\beta)} (iJ\mathbf{m}(\{\tau\})f(\tau), \mathbf{m}(\{\tau\})g(\tau)) = \left(iJ\int_{\alpha_n}^{\beta} w_n^*(s)d\mathbf{m}(s)f(s), \int_{\alpha_n}^{\beta} w_n^*(s)d\mathbf{m}(s)g(s)\right) -$$
$$- \sum_{\tau\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_n,\beta)} (iJ\mathbf{m}(\{\tau\})f(\tau), \mathbf{m}(\{\tau\})g(\tau)). \quad (42)$$

Passing to the limit as $\beta \to \beta_n - 0$ in (42), we obtain that (42) will remain true if β is replaced by β_n. Therefore,

$$\int_{\alpha_n}^{\beta_n} (f(s), d\mathbf{m}(s)G_n(s)) - \int_{\alpha_n}^{\beta_n} (F_n(s), d\mathbf{m}(s)g(s)) = \left(iJ\int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)f(s), \int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)g(s)\right) -$$
$$- \sum_{\tau\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_n,\beta_n)} (iJ\mathbf{m}(\{\tau\})f(\tau), \mathbf{m}(\{\tau\})g(\tau)). \quad (43)$$

Taking into account (41), (43), and (35), we obtain

$$(f,G)_{\mathfrak{H}} - (F,g)_{\mathfrak{H}} = \sum_{n=1}^{\Bbbk_1} \left(iJ\int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)f(s), \int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)g(s)\right) - (i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}_0^* g). \quad (44)$$

Further, we define the functions F_{n0}, G_{n0}, $\widetilde{F}_0$, $\widetilde{G}_0$ by the equalities

$$F_{n0}(t) = \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} F_n(t),\quad G_{n0}(t) = \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} G_n(t),\quad \widetilde{F}_0 = \sum_{n=1}^{\Bbbk_1} F_{n0},\quad \widetilde{G}_0 = \sum_{n=1}^{\Bbbk_1} G_{n0}. \quad (45)$$

Using (43), we get

$$(f, G_{n0})_{\mathfrak{H}} - (F_{n0}, g)_{\mathfrak{H}} = (f, G_n)_{\mathfrak{H}} - (F_n, g)_{\mathfrak{H}} +$$
$$+ (f, \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}} w_n(t)iJ\int_{\alpha_n}^{t} w_n^*(s)d\mathbf{m}(s)g(s))_{\mathfrak{H}} - (\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}} w_n(t)iJ\int_{\alpha_n}^{t} w_n^*(s)d\mathbf{m}(s)f(s), g(s))_{\mathfrak{H}} =$$
$$= \left(iJ\int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)f(s), \int_{\alpha_n}^{\beta_n} w_n^*(s)d\mathbf{m}(s)g(s)\right) - \sum_{\tau\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_n,\beta_n)} (iJ\mathbf{m}(\{\tau\})f(\tau), \mathbf{m}(\{\tau\})g(\tau)) -$$
$$- \sum_{\tau\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_n,\beta_n)} (iJw_n^*(\tau)f(\tau), \int_{\alpha_n}^{\tau} w_n^*(s)d\mathbf{m}(s)g(s)) - \sum_{\tau\in\mathcal{S}_{\mathbf{m}}\cap[\alpha_n,\beta_n)} (iJ\int_{\alpha_n}^{\tau} w_n^*(s)d\mathbf{m}(s)f(s), w_n^*(\tau)g(\tau)). \quad (46)$$

By (35), (45), (46), we obtain

$$(f, G_0)_{\mathfrak{H}} - (F_0, g)_{\mathfrak{H}} = \sum_{n=1}^{\Bbbk_1} \left(iJ \int_{\alpha_n}^{\beta_n} w_n^*(s) d\mathbf{m}(s) f(s), \int_{\alpha_n}^{\beta_n} w_n^*(s) d\mathbf{m}(s) g(s) \right) - \\ - (i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}_0^* g) - (i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}_{*\tau} g) - (i\widetilde{J}\mathcal{V}_{*\tau} f, \mathcal{V}_0^* g). \quad (47)$$

It follows from (38), (40) that the equalities

$$(f, \mathcal{V}\widetilde{\zeta})_{\mathfrak{H}} = (\mathcal{V}^* f, \widetilde{\zeta}) = (\mathcal{V}^* f, Z_{10} + 2^{-1} i\widetilde{J}\mathcal{V}_* g - 2^{-1} i\widetilde{J}\mathcal{V}_0^* g - i\widetilde{J}\mathcal{V}_{*\tau} g), \quad (48)$$

$$(\mathcal{V}\widetilde{\eta}, g)_{\mathfrak{H}} = (\widetilde{\eta}, \mathcal{V}^* g) = (Y_{10} + 2^{-1} i\widetilde{J}\mathcal{V}_* f - 2^{-1} i\widetilde{J}\mathcal{V}_0^* f - i\widetilde{J}\mathcal{V}_{*\tau} f, \mathcal{V}^* g) \quad (49)$$

hold. Using (33), (34), (47), (48), (49), we get

$$(f, z)_{\mathfrak{H}} - (y, g)_{\mathfrak{H}} = (f, \mathcal{V}\zeta + G_0)_{\mathfrak{H}} - (\mathcal{V}\eta + F_0, g)_{\mathfrak{H}} = (\mathcal{V}^* f, \zeta) - (\eta, \mathcal{V}^* g) + (f, G_0)_{\mathfrak{H}} - (F_0, g)_{\mathfrak{H}} = \\ = (\mathcal{V}^* f, Z_{10} + 2^{-1} i\widetilde{J}\mathcal{V}_* g - 2^{-1} i\widetilde{J}\mathcal{V}_0^* g - i\widetilde{J}\mathcal{V}_{*\tau} g) - (Y_{10} + 2^{-1} i\widetilde{J}\mathcal{V}_* f - 2^{-1} i\widetilde{J}\mathcal{V}_0^* f - i\widetilde{J}\mathcal{V}_{*\tau} g, \mathcal{V}^* g) + (f, G_0)_{\mathfrak{H}} - (F_0, g)_{\mathfrak{H}} = \\ = (Y'_{10}, Z_{10}) - 2^{-1}(i\widetilde{J}\mathcal{V}^* f, \mathcal{V}_* g) + 2^{-1}(i\widetilde{J}\mathcal{V}^* f, \mathcal{V}_0^* g) + (i\widetilde{J}\mathcal{V}^* f, \mathcal{V}_{*\tau} g) - \\ - (Y_{10}, Z'_{10}) - 2^{-1}(i\widetilde{J}\mathcal{V}_* f, \mathcal{V}^* g) + 2^{-1}(i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}^* g) + (i\widetilde{J}\mathcal{V}_{*\tau} f, \mathcal{V}^* g) + \\ + (i\widetilde{J}\mathcal{V}_*, \mathcal{V}^* g) - (i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}_0^* g) - (i\widetilde{J}\mathcal{V}_0^* f, \mathcal{V}_{*\tau} g) - (i\widetilde{J}\mathcal{V}_{*\tau} f, \mathcal{V}_0^* g) = (Y'_{10}, Z_{10}) - (Y_{10}, Z'_{10}).$$

The theorem is proved. □

By Lemma 3.2 (for $\lambda = 0$), it follows that functions $\mathfrak{X}_{\{\tau\}}(\cdot)x$ ($x \in H$, $\tau \in \mathcal{S}_{\mathbf{m}}$) belong to $\ker L_{10}^*$. Consequently equality (36) is reduced to the form

$$\widehat{y} = y_0 + y, \quad \widehat{f} = y'_0 + f, \quad y(t) = \widetilde{V}(t)\widetilde{\xi} - \sum_{n=1}^{\Bbbk_1} w_n(t) iJ \int_a^t w_n^*(s) d\mathbf{m}(s) f(s), \quad (50)$$

where $\widetilde{\xi} = \{\xi_k\} \in \mathcal{Q}_-$, $\xi_k = \eta_k$ (see (36)) if v_k has form (12) and

$$\xi_k = \eta_k + iJ \int_{\alpha_n}^{\tau_k} w_n^*(s) d\mathbf{m}(s) f(s) \quad (51)$$

if v_k has form (13) for $\lambda = 0$.

Corollary 5.2. *If the pair* $\{y, f\}$ *has form* (50), *then*

$$Y_{10} = \widetilde{\xi} - 2^{-1} i\widetilde{J}\mathcal{V}_* f + 2^{-1} i\widetilde{J}\mathcal{V}_0^* f, \quad Y'_{10} = \mathcal{V}^* f. \quad (52)$$

Proof. Equality (52) follows from (38) and (51). □

We note that the case where functions y, f have form (50) was considered in [16]. Equality (44) is proved in [16]; however, in [16], there is a mistake in formula (52): $\mathcal{V}^*$ is written in the first equality instead of $\mathcal{V}_*$.

From the theory of spaces with positive and negative norms (see [4, ch. 1], [19, ch.2]), it follows that there exist isometric operators $\delta_- : \mathcal{Q}_- \to \mathcal{Q}$, $\delta_+ : \mathcal{Q}_+ \to \mathcal{Q}$ such that the equality $(\widetilde{\eta}, \widetilde{\varphi}) = (\delta_- \widetilde{\eta}, \delta_+ \widetilde{\varphi})$ holds for all $\widetilde{\eta} \in \mathcal{Q}_-$, $\widetilde{\varphi} \in \mathcal{Q}_+$. We denote $\mathcal{H} = \mathfrak{H}_0 \times \mathcal{Q}$. Suppose $\{\widetilde{y}, \widetilde{f}\} \in L_0^*$. According to Theorem 3.5 (for $\lambda = 0$), there exists a pair $\{\widehat{y}, \widehat{f}\}$ such that the pairs $\{\widetilde{y}, \widetilde{f}\}$, $\{\widehat{y}, \widehat{f}\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and equalities (36) hold. To each such pair $\{\widehat{y}, \widehat{f}\}$ assign a pair of boundary values $\gamma\{\widehat{y}, \widehat{f}\} = \{\mathcal{Y}, \mathcal{Y}'\} \in \mathcal{H} \times \mathcal{H}$ by the formulas

$$\mathcal{Y} = \gamma_1\{\widehat{y}, \widehat{f}\} = \{y_0, \delta_- Y_{10}\}, \quad \mathcal{Y}' = \gamma_2\{\widehat{y}, \widehat{f}\} = \{y'_0, \delta_+ Y'_{10}\}.$$

By Theorem 5.1, it follows that the operator γ maps L_0^* onto $\mathcal{H} \times \mathcal{H}$ and equality

$$(\widehat{f}, \widehat{z})_{\mathfrak{H}} - (\widehat{y}, \widehat{g})_{\mathfrak{H}} = (\mathcal{Y}', \mathcal{Z}) - (\mathcal{Y}, \mathcal{Z}') \tag{53}$$

holds, where $\{\widehat{y}, \widehat{f}\}, \{\widehat{z}, \widehat{g}\} \in L_0^*$, $\gamma\{\widehat{y}, \widehat{f}\} = \{\mathcal{Y}, \mathcal{Y}'\}$, $\gamma\{\widehat{z}, \widehat{g}\} = \{\mathcal{Z}, \mathcal{Z}'\}$. This implies that the ordered triple $(\mathcal{H}, \gamma_1, \gamma_2)$ is the space of boundary values (a boundary triplet in another terminology) for L_0 in the sense of papers [22], [7], [8] (see also [19, ch. 3]). It was established in the articles [22], [7], [8] that for the space of boundary values, formula (53) implies the following statement.

Corollary 5.3. *If U is a unitary operator on $\mathcal{H}$, then the restriction of the relation L_0^* to the set of pairs $\{\widehat{y}, \widehat{f}\} \in L_0^*$ satisfying the condition*

$$(U - E)\mathcal{Y}' - (U + E)\mathcal{Y} = 0 \tag{54}$$

is a self-adjoint extension of L_0. Conversely, any self-adjoint extension of L_0 is the restriction of L_0^* to the set of pairs $\{\widehat{y}, \widehat{f}\} \in L_0^*$ satisfying (54)*, where a unitary operator U is uniquely determined by an extension.*

This statement is proved in [16] for the boundary values (52). It is established in [15] provided that $\mathbf{m}$ is the usual Lebesque measure on $[a, b]$ (i.e., $\mathbf{m}([\alpha, \beta)) = \beta - \alpha$, where $a \leqslant \alpha < \beta \leqslant b$). We note that F.S. Rofe-Beketov [24] first applied linear relations to describe self-adjoint extensions of differential operators.

We consider boundary value problem

$$\widehat{f} = \lambda\widehat{y} + h, \quad (K(\lambda) - E)\mathcal{Y}' - i(K(\lambda) + E)\mathcal{Y} = 0, \tag{55}$$

where $\{\mathcal{Y}, \mathcal{Y}'\} = \gamma\{\widehat{y}, \widehat{f}\}$; $h \in \mathfrak{H}$; $\lambda \to K(\lambda)$ is a holomorphic operator function in $\mathcal{H}$ such that $\|K(\lambda)\| \leqslant 1$; $\mathrm{Im}\lambda > 0$.

From (53) and [7], [8] we obtain the following statement.

Theorem 5.4. *There exists a one-to-one mapping between boundary problems* (55) *and generalized resolvents of the operator L_0. Every solution y of problem* (55) *determines a generalized resolvent $\widetilde{R}_\lambda$ by the formula $y = \widetilde{R}_\lambda h$ and, conversely, for any generalized resolvent $\widetilde{R}_\lambda$ there exists a function $K(\lambda)$ such that the function $y = \widetilde{R}_\lambda h$ is the solution of* (55)*.*

6. The example

We consider equation (1) on a segment $[0, b]$ and assume that $H = \mathbb{C}$, $J = E = 1$, $\mathbf{p} = 0$, $\mathbf{m} = \mathbf{m}_0 + \widehat{\mathbf{m}}$, where $\mathbf{m}_0$ is the usual Lebesque measure (we write ds instead of $d\mathbf{m}_0(s)$), $0 < \tau < b$, $\widehat{\mathbf{m}}(\{\tau\}) = 1$ and $\widehat{\mathbf{m}}(\Delta) = 0$ for all Borel sets such that $\tau \notin \Delta$. So, $\mathcal{S}_{\mathbf{m}} = \{\tau\}$. Thus, equation (1) has the form

$$y(t) = x_0 - i\int_0^t d\mathbf{m}(s)f(s). \tag{56}$$

It follows from the definition of L_0 and (7), (56) that L_0 is an operator and if $y = L_0 f$, then

$$y(t) = -i\int_0^t f(s)ds, \quad y(b) = 0, \quad f(\tau) = 0 \quad \Leftrightarrow \quad y'(t) = -if(t), \quad y(0) = y(b) = 0, \quad f(\tau) = 0.$$

Since $\mathcal{S}_0 = \{0, b\}$ and $\mathbf{m}(\mathcal{S}_0) = 0$, we have $\mathfrak{H}_0 = \{0\}$ and $L_{10}^* = L_0^*$ in equality (8).

Equation (5) (for $x_0 = 1$) takes the form

$$W(t, \lambda) = 1 - i\lambda\int_0^t W(s, \lambda)ds, \quad \lambda \in \mathbb{C}.$$

Therefore, $W(t,\lambda) = e^{-i\lambda t}$. Obviously, if $\lambda = 0$, then $W(t,0) = 1$. The number of intervals $\mathcal{J}_k \in \mathbb{J}$ is $\Bbbk_1 = 1$. We write $w(t,\lambda)$ instead of $w_1(t,\lambda)$. Then $w(t,\lambda) = \mathfrak{X}_{[0,b)}W(t,\lambda)$. Without loss of generality it can be assumed that $w(t,\lambda) = W(t,\lambda) = e^{-i\lambda t}$.

The set $\mathbb{M}$ consists of the interval $(0,b)$ and the single-point set $\{\tau\}$. Hence the number of elements of $\mathbb{M}$ is $\Bbbk = 2$. Using (12), (13), and the equality $\mathbf{m}(\{\tau\}) = 1$, we get

$$v_1(t,\lambda) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}w(t,\lambda) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t} = \begin{cases} e^{-i\lambda t} & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau \end{cases} ; \tag{57}$$

$$v_2(t,\lambda) = u_1(t,\lambda,\tau)e^{-i\lambda\tau} + \mathfrak{X}_{\{\tau\}}(t)e^{-i\lambda\tau} = -\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t} i\int_0^t e^{i\lambda s}d\mathbf{m}(s)\lambda\mathfrak{X}_{\{\tau\}}(s)e^{-i\lambda\tau} + \mathfrak{X}_{\{\tau\}}(t)e^{-i\lambda\tau} = \begin{cases} 0 & \text{for } t < \tau, \\ e^{-i\lambda\tau} & \text{for } t = \tau, \\ -\lambda i e^{-i\lambda t} & \text{for } t > \tau. \end{cases} \tag{58}$$

Therefore,

$$v_1^*(t,\overline{\lambda}) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{i\lambda t} = \begin{cases} e^{i\lambda t} & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau \end{cases} ; \quad v_2^*(t,\overline{\lambda}) = \begin{cases} 0 & \text{for } t < \tau, \\ e^{i\lambda\tau} & \text{for } t = \tau, \\ \lambda i e^{i\lambda t} & \text{for } t > \tau. \end{cases} \tag{59}$$

If $\lambda = 0$, then equalities (58), (59) imply that

$$v_1(t) = v_1(t,0) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t) = \begin{cases} 1 & \text{for } t \neq \tau, \\ 0 & \text{for } t = \tau \end{cases} ; \ v_2(t) = v_2(t,0) = \mathfrak{X}_{\{\tau\}}(t) = \begin{cases} 0 & \text{for } t \neq \tau, \\ 1 & \text{for } t = \tau. \end{cases} \tag{60}$$

By (14), (60), it follows that $Q_{10} = Q_{20} = \{0\}$ and $Q_1 = Q_1^- = Q_1^+ = Q_2 = Q_2^- = Q_2^+ = H = \mathbb{C}$. Therefore, $\mathcal{Q} = \mathcal{Q}_- = \mathcal{Q}_+ = \mathbb{C}^2$.

The domain $\mathcal{D}(L_0)$ of L_0 is dense in $\mathfrak{H} = L_2(\mathbb{C}, d\mathbf{m}; 0, b)$. This yields that L_0^* is an operator. Using Theorem 3.5, we obtain

$$y(t) = v_1(t,\lambda)\eta_1 + v_2(t,\lambda)\eta_2 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)e^{-i\lambda t} i\int_0^t e^{i\lambda s}d\mathbf{m}(s)f(s) \tag{61}$$

for all $y \in \mathcal{D}(L_0^* - \lambda E)$, where $\eta_1, \eta_2 \in \mathbb{C}$, $f = (L_0^* - \lambda E)y$. For $\lambda = 0$, it follows from (61) that

$$y(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)\eta_1 + \mathfrak{X}_{\{\tau\}}(t)\eta_2 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t) i\int_0^t d\mathbf{m}(s)u(s), \tag{62}$$

where $u = L_0^*y$. Since $\mathfrak{X}_{\{\tau\}}\xi \in \ker L_0^*$ for all $\xi \in \mathbb{C}$, we obtain

$$y(t) = \xi_1 + \mathfrak{X}_{\{\tau\}}(t)\xi_2 - i\int_0^t d\mathbf{m}(s)u(s), \quad \xi_1, \xi_2 \in \mathbb{C}.$$

Taking into account (37), (38), (62), and the equality $\mathbf{m}(\{\tau\}) = 1$, we see that the boundary values $Y = \mathcal{Y}$, $Y' = \mathcal{Y}'$ are calculated by the formulas

$$Y = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - 2^{-1}i \begin{pmatrix} \int_0^b d\mathbf{m}(s)u(s) \\ \int_0^b d\mathbf{m}(s)u(s) \end{pmatrix} + 2^{-1}i \begin{pmatrix} 0 \\ u(\tau) \end{pmatrix} + i \begin{pmatrix} 0 \\ \int_0^\tau d\mathbf{m}(s)u(s) \end{pmatrix}, \quad Y' = \begin{pmatrix} \int_0^b u(s)ds \\ u(\tau) \end{pmatrix}, \tag{63}$$

where y has form (62), $u = L_0^*y$.

Let $\mathcal{L}$ be an operator such that $L_0 \subset \mathcal{L} \subset L_0^*$. Suppose that $\mathcal{L}$ is the restriction of L_0^* to a set of functions satisfying the condition $Y = 0$. It follows from Corollary 5.3 that $\mathcal{L}$ is the self-adjoint operator. Let us find the function

$$M(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2 \tag{64}$$

that corresponds to the resolvent $R_\lambda = (\mathcal{L} - \lambda E)^{-1}$. We denote

$$x_1 = x_1(f,\lambda) = \int_0^b v_1^*(s,\overline{\lambda})d\mathbf{m}(s)f(s) = \int_0^b e^{i\lambda s} f(s)ds, \tag{65}$$

$$x_2 = x_2(f,\lambda) = \int_0^b v_2^*(s,\overline{\lambda})d\mathbf{m}(s)f(s) = e^{i\lambda\tau}f(\tau) + \int_\tau^b i\lambda e^{i\lambda s} f(s)ds. \tag{66}$$

Then equality (20) takes the form

$$y(t) = v_1(t,\lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) + v_2(t,\lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) - \\ - 2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^t e^{i\lambda s} f(s)ds + 2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_t^b e^{i\lambda s} f(s)ds - \lambda^{-1}\mathfrak{X}_{\{\tau\}}(t)f(\tau). \tag{67}$$

By elementary transformations, equality (67) is converted to the following form

$$y(t) = v_1(t,\lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) + v_2(t,\lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) - \\ - \mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^t e^{i\lambda s} f(s)ds + 2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t} i \int_0^b e^{i\lambda s} f(s)ds - \lambda^{-1}\mathfrak{X}_{\{\tau\}}(t)f(\tau). \tag{68}$$

Using (57), (58), we obtain that in equality (68)

$$v_1(t,\lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) = \mathfrak{X}_{[0,b]\setminus\{\tau\}} e^{-i\lambda t}(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2),$$

$$v_2(t,\lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) = \begin{cases} 0 \ \text{ for } \ t < \tau, \\ e^{-i\lambda\tau}(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) \ \text{ for } \ t = \tau, \\ -\lambda i e^{-i\lambda t}(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) \ \text{ for } t > \tau. \end{cases}$$

To find $M_{12}(\lambda)$, $M_{22}(\lambda)$, we take the function $f_1(t) = \mathfrak{X}_{\{\tau\}}(t)$, i.e., $f_1(t) = 1$ if $t = \tau$ and $f_1(t) = 0$ if $t \neq \tau$ (by $\{Y_1, Y_1'\}$ denote the corresponding pair of boundary values). It follows from (59), (65), (66) that $x_1 = 0$, $x_2 = e^{i\lambda\tau}$. We denote $y_1 = R_\lambda f_1$. Using (68), we obtain

$$y_1(t) = v_1(t,\lambda)M_{12}(\lambda)e^{i\lambda\tau} + v_2(t,\lambda)M_{22}(\lambda)e^{i\lambda\tau} - \lambda^{-1}\mathfrak{X}_{\{\tau\}}(t). \tag{69}$$

We denote $u_1 = L_0^* y_1 = \lambda y_1 + f_1$. Then using (69), we get

$$u_1(t) = \lambda v_1(t,\lambda)M_{12}(\lambda)e^{i\lambda\tau} + \lambda v_2(t,\lambda)M_{22}(\lambda)e^{i\lambda\tau}. \tag{70}$$

By (60), (62), (69), it follows that

$$y_1(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)M_{12}(\lambda)e^{i\lambda\tau} + \mathfrak{X}_{\{\tau\}}(t)(M_{22}(\lambda) - \lambda^{-1}) - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t) i \int_0^t d\mathbf{m}(s)u_1(s). \tag{71}$$

Using (70) by direct calculations, we obtain

$$\int_0^b d\mathbf{m}(s)u_1(s) = i(e^{-i\lambda b} - 1)M_{12}(\lambda)e^{i\lambda\tau} + \lambda e^{-i\lambda b}M_{22}(\lambda)e^{i\lambda\tau}; \quad \int_0^\tau d\mathbf{m}(s)u_1(s) = i(1 - e^{i\lambda\tau})M_{12}(\lambda). \tag{72}$$

By (63), (71), (72), so that

$$Y_1 = \begin{pmatrix} M_{12}(\lambda)e^{i\lambda\tau} \\ M_{22}(\lambda) - \lambda^{-1} \end{pmatrix} - 2^{-1}i \begin{pmatrix} i(e^{-i\lambda b} - 1)M_{12}(\lambda)e^{i\lambda\tau} + \lambda e^{-i\lambda b}M_{22}(\lambda)e^{i\lambda\tau} \\ i(e^{-i\lambda b} - 1)M_{12}(\lambda)e^{i\lambda\tau} + \lambda e^{-i\lambda b}M_{22}(\lambda)e^{i\lambda\tau} \end{pmatrix} + 2^{-1}i \begin{pmatrix} 0 \\ \lambda M_{22}(\lambda) \end{pmatrix} + i \begin{pmatrix} 0 \\ i(1 - e^{i\lambda\tau})M_{12}(\lambda) \end{pmatrix}.$$

The equality $Y_1 = 0$ is equivalent to two equalities

$$\begin{cases} M_{12}(\lambda)e^{i\lambda\tau} + 2^{-1}(e^{-i\lambda b} - 1)M_{12}(\lambda)e^{i\lambda\tau} - 2^{-1}\lambda i e^{-i\lambda b}M_{22}(\lambda)e^{i\lambda\tau} = 0, \\ M_{22}(\lambda) - \lambda^{-1} + 2^{-1}(e^{-i\lambda b} - 1)M_{12}(\lambda)e^{i\lambda\tau} - 2^{-1}\lambda i e^{-i\lambda b}M_{22}(\lambda)e^{i\lambda\tau} + 2^{-1}\lambda i M_{22}(\lambda) - (1 - e^{i\lambda\tau})M_{12}(\lambda) = 0. \end{cases} \tag{73}$$

Solving the system of equations (73), we get

$$M_{12}(\lambda) = \frac{2ie^{-i\lambda b}}{2(e^{-i\lambda b} + 1) - i\lambda(e^{-i\lambda b} - 1)}\ ; \quad M_{22}(\lambda) = \frac{2(e^{-i\lambda b} + 1)}{\lambda(2(e^{-i\lambda b} + 1) - i\lambda(e^{-i\lambda b} - 1))}\ . \tag{74}$$

To find $M_{11}(\lambda)$, $M_{21}(\lambda)$, we take the function $f_2(t) = \mathfrak{X}_{[0,\tau)}(t)$, i.e., $f_2(t) = 1$ if $t < \tau$ and $f_2(t) = 0$ if $t \geqslant \tau$ (by $\{Y_2, Y_2'\}$ denote the corresponding pair of boundary values). It follows from (65) that $x_1 = i\lambda^{-1}(1 - e^{i\lambda\tau})$, $x_2 = 0$. We denote $y_2 = R_\lambda f_2$. Using (68), we obtain

$$y_2(t) = v_1(t,\lambda)M_{11}(\lambda)x_1 + v_2(t,\lambda)M_{21}(\lambda)x_1 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^t e^{i\lambda s}f_2(s)ds + 2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^b e^{i\lambda s}f_2(s)ds. \tag{75}$$

The equality $f_2(t) = \mathfrak{X}_{[0,\tau)}(t)$ implies

$$-\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^t e^{i\lambda s}f_2(s)ds = \begin{cases} \lambda^{-1}(e^{-i\lambda t} - 1) \ \text{ for } \ t < \tau, \\ 0 \ \text{ for } \ t = \tau, \\ \lambda^{-1}e^{-i\lambda t}(1 - e^{i\lambda\tau}) \ \text{ for } t > \tau; \end{cases} \tag{76}$$

$$2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^b e^{i\lambda s}f_2(s)ds = \begin{cases} 2^{-1}ie^{-i\lambda t}x_1 \ \text{ for } \ t \neq \tau, \\ 0 \ \text{ for } \ t = \tau. \end{cases} \tag{77}$$

By (62), (75)-(77), it follows that

$$y_2(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(M_{11}(\lambda)x_1 + 2^{-1}ix_1) + \mathfrak{X}_{\{\tau\}}(t)e^{-i\lambda\tau}M_{21}(\lambda)x_1 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)i\int_0^t d\mathbf{m}(s)u_2(s), \tag{78}$$

where $u_2 = L_0^* y_2 = \lambda y_2 + f_2$. Equalities (57), (58), (75)-(77) imply that $u_2(t) = u_{21}(t) + u_{22}(t) + u_{23}(t) + u_{24}(t)$, where

$$u_{21}(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}\lambda e^{-i\lambda t}M_{11}(\lambda)x_1; \quad u_{22}(t) = \begin{cases} 0 \ \text{ for } \ t < \tau, \\ \lambda e^{-i\lambda\tau}M_{21}(\lambda)x_1 \ \text{ for } \ t = \tau, \\ -\lambda^2 ie^{-i\lambda t}M_{21}(\lambda)x_1 \ \text{ for } t > \tau; \end{cases} \tag{79}$$

$$u_{23}(t) = \begin{cases} e^{-i\lambda t} \ \text{ for } \ t < \tau, \\ 0 \ \text{ for } \ t = \tau, \\ e^{-i\lambda t}(1 - e^{i\lambda\tau}) \ \text{ for } t > \tau; \end{cases} \qquad u_{24}(t) = \begin{cases} 2^{-1}\lambda ie^{-i\lambda t}x_1 \ \text{ for } \ t \neq \tau, \\ 0 \ \text{ for } \ t = \tau. \end{cases} \ . \tag{80}$$

Using (79), (80), and equality $x_1 = i\lambda^{-1}(1 - e^{i\lambda\tau})$, by direct calculations, we obtain

$$\int_0^b d\mathbf{m}(s)u_2(s) = i(e^{-i\lambda b} - 1)M_{11}(\lambda)x_1 + \lambda e^{-i\lambda b}M_{21}(\lambda)x_1 + i\lambda^{-1}(e^{-i\lambda\tau} - 1) + (e^{-i\lambda b} - e^{-i\lambda\tau})x_1 - 2^{-1}(e^{-i\lambda b} - 1)x_1, \tag{81}$$

$$\int_0^\tau d\mathbf{m}(s)u_2(s) = i(e^{-i\lambda\tau} - 1)M_{11}x_1 + i\lambda^{-1}(e^{-i\lambda\tau} - 1) - 2^{-1}(e^{-i\lambda\tau} - 1)x_1. \quad (82)$$

By (63), (78), so that

$$Y_2 = \begin{pmatrix} M_{11}(\lambda)x_1 + 2^{-1}ix_1 \\ e^{-i\lambda\tau}M_{21}(\lambda)x_1 \end{pmatrix} - 2^{-1}i\begin{pmatrix} \int_0^b d\mathbf{m}(s)u_2(s) \\ \int_0^b d\mathbf{m}(s)u_2(s) \end{pmatrix} + 2^{-1}i\begin{pmatrix} 0 \\ \lambda e^{-i\lambda\tau}M_{21}(\lambda)x_1 \end{pmatrix} + i\begin{pmatrix} 0 \\ \int_0^\tau d\mathbf{m}(s)u_2(s) \end{pmatrix},$$

where the integrals $\int_0^b d\mathbf{m}(s)u_2(s)$, $\int_0^\tau d\mathbf{m}(s)u_2(s)$ are calculated by formulas (81), (82), respectively.

The equality $Y_2 = 0$ is equivalent to two equalities

$$\begin{cases} M_{11}(\lambda)x_1 + 2^{-1}ix_1 - 2^{-1}i\int_0^b d\mathbf{m}(s)u_2(s) = 0, \\ e^{-i\lambda\tau}M_{21}(\lambda)x_1 - 2^{-1}i\int_0^b d\mathbf{m}(s)u_2(s) + 2^{-1}\lambda i e^{-i\lambda\tau}M_{21}(\lambda)x_1 + i\int_0^\tau d\mathbf{m}(s)u_2(s) = 0. \end{cases} \quad (83)$$

Solving the system of equations (83), we obtain

$$M_{11}(\lambda) = i\frac{(2 - i\lambda)e^{-i\lambda b} - (2 + i\lambda)}{2((2 - i\lambda)e^{-i\lambda b} + (2 + i\lambda))}\ ; \quad M_{21}(\lambda) = i\frac{-2}{(2 - i\lambda)e^{-i\lambda b} + (2 + i\lambda)}\ . \quad (84)$$

Thus the matrix $M(\lambda)$ (64) is calculated by equalities (84), (74).

Remark 6.1. *It follows from* (84), (74) *that*

$$M_{11}(i) = \frac{3e^b - 1}{2(3e^b + 1)}i; \quad M_{21}(i) = \frac{-2}{3e^b + 1}i; \quad M_{12}(i) = \frac{2e^b}{3e^b + 1}i; \quad M_{22}(i) = \frac{-2(e^b + 1)}{3e^b + 1}i.$$

Suppose that $f_1(t) = \mathfrak{X}_{\{\tau\}}(t)$. *Then* $x_1 = x_1(f_1, i) = 0$, $x_2 = x_2(f_1, i) = e^{-\tau}$ (*see* (59), (65), (66)). *We denote* $\widetilde{x} = \mathrm{col}(x_1, x_2)$. *Therefore,* $(M(i)\widetilde{x}, \widetilde{x}) = M_{22}(i)e^{-2\tau}$. *Thus,* $\mathrm{Im}(M(i)\widetilde{x}, \widetilde{x}) = \mathrm{Im}M_{22}(i)e^{-2\tau} = -2(e^b + 1)e^{-2\tau}/(3e^b + 1) < 0$.

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