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Bounded and Compact Hankel Operators on the Fock-Sobolev Spaces

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Abstract. This paper focuses on the operator-theoretic properties (boundedness and compactness) of Hankel operators on the Fock-Sobolev spaces $\mathscr{F}^{p,m}$ in terms of symbols in $\mathcal{BMO}_r^p$ and $\mathcal{VMO}_r^p$ spaces, respectively, for a non-negative integers m, $1 \leq p < \infty$ and $r > 0$. Along the way, we also study Berezin transform of Hankel operators on $\mathscr{F}^{p,m}$.

1. Introduction

The investigation of Hankel operators on several spaces like Hardy spaces, Bergman spaces, Bergman spaces on a certain domains, Fock spaces, Fock-type spaces etc., has a long history in mathematics. We refer to [2, 6, 7, 9–11] for the detailed study of Hankel operators on these spaces. Zhu [9] obtained a characterization of bounded and compact Hankel operators on the Bergman space by defining the spaces of bounded mean oscillation and vanishing mean oscillation with respect to the Bergman metric and analogous results were obtained by Perälä, Schuster and Virtanen [5] on the weighted Fock spaces. Motivated by these developments, the properties of Hankel operators on the Fock-Sobolev spaces are discussed in this paper. In particular, we examine the boundedness and compactness of these operators in terms of $\mathcal{BMO}_r^p$ and $\mathcal{VMO}_r^p$ spaces for the generating symbols.

Let dA be the Lebesgue area measure on $\mathbb{C}$. For $1 \leq p \leq \infty$, let $\mathscr{F}^p$ be the space of all entire functions g on the complex plane $\mathbb{C}$ such that $g(v)e^{-\frac{1}{2}|v|^2} \in L^p(\mathbb{C}, dA(v))$ with norm

$$||g||_p = \left\{ \frac{p}{2\pi} \int_{\mathbb{C}} |g(v)|^p e^{-\frac{p}{2}|v|^2} dA(v) \right\}^{\frac{1}{p}},$$

for $1 \leq p < \infty$ and

$$||g||_\infty = \operatorname*{ess\,sup}_{v \in \mathbb{C}} \{|g(v)|e^{-\frac{1}{2}|v|^2}\},$$

for $p = \infty$.

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Throughout the paper, m is a fixed non-negative integer. The Fock-Sobolev space $\mathscr{F}^{p,m}$ is the space of all entire functions g on $\mathbb{C}$ such that

$$\|g\|_{p,m} = \sum_{0\le k\le m} \|g^{(k)}\|_p < \infty,$$

where g^k denotes the k^{th} derivative of g. Cho and Zhu [3, 4] gave a very useful Fourier characterization of Fock-Sobolev spaces on $\mathbb{C}^n$ ($n \geq 1$). They proved that $g \in \mathscr{F}^{p,m}$ if and only if $z^m g \in \mathscr{F}^p$ and $\|g\|_{p,m}$ can be taken as

$$\|g\|_{p,m} = \left\{\omega_{p,m} \int_{\mathbb{C}} |g(v)|^p |v|^{mp} e^{-\frac{p}{2}|v|^2} dA(v)\right\}^{\frac{1}{p}} ; 1 \le p < \infty$$

and

$$\|g\|_\infty = \sup_{v\in\mathbb{C}} |g(v)v^m e^{-\frac{1}{2}|v|^2}|; p = \infty,$$

where $\omega_{p,m} = (\frac{p}{2})^{\frac{mp}{2}+1} \frac{1}{\pi\Gamma(\frac{mp}{2}+1)}$.

Let $L^{p,m}$ be the space of all Lebesgue measurable functions g such that $g(v)|v|^m e^{-\frac{1}{2}|v|^2} \in L^p(\mathbb{C}, dA(v))$ and $\mathscr{F}^{p,m}$ is a closed subspace of $L^{p,m}$.

The space $\mathscr{F}^{2,m}$ is a closed subspace of the Hilbert space $L^{2,m}$ with inner product

$$\langle f, g\rangle_{p,m} = \frac{1}{\pi} \int_{\mathbb{C}} f(v)\overline{g(v)}|v|^{2m} e^{-|v|^2} dA(v) \text{ for all } f, g \in \mathscr{F}^{2,m},$$

and having reproducing kernel

$$K^m(v, z) = K^m_z(v) = \sum_{k=0}^{\infty} \frac{m!}{(k+m)!}(\bar{z}v)^k = m!\frac{(e^{\bar{z}v} - Q_m(\bar{z}v))}{(\bar{z}v)^m},$$

where $Q_m(w)$ is the Taylor polynomial of e^w of order $(m-1)$ that is, $Q_m(w) = \sum_{k=0}^{m-1} \frac{w^k}{k!}$. Let

$$k^m_z(v) = \frac{K^m_z(v)}{\sqrt{K^m_z(z)}} = \frac{(e^{\bar{z}v} - Q_m(\bar{z}v))}{(\bar{z}v)^m} \left\{\frac{m!|z|^{2m}}{(e^{|z|^2} - Q_m(|z|^2))}\right\}^{\frac{1}{2}}$$

denote the normalized reproducing kernel of $\mathscr{F}^{2,m}$. Also, the sequence $b_k(v)_{k=0}^{\infty}$ forms an orthonormal basis of $\mathscr{F}^{2,m}$, where

$$b_k(v) = \sqrt{\frac{m!}{(k+m)!}} v^k.$$

Cho and Zhu [4] showed that the orthogonal projection $P^m : L^{2,m} \to \mathscr{F}^{2,m}$ given by

$$P^m g(z) = \langle g, K^m_z\rangle_{2,m} = \frac{1}{\pi} \int_{\mathbb{C}} g(v)\overline{K^m_z(v)}|v|^{2m} e^{-|v|^2} dA(v)$$

is a bounded projection from $L^{p,m}$ onto $\mathscr{F}^{p,m}$ for $1 \le p \le \infty$.

2. $\mathcal{BMO}_r^p$ spaces and boundedness of Hankel operators on $\mathscr{F}^{p,m}$

For $1 \leq p \leq \infty$, let Ω_m^p denote the space of all Lebesgue measurable functions g on $\mathbb{C}$ such that $gk_v^m \in L^{p,m}$ for each $v \in \mathbb{C}$. Let I denotes the identity operator on $L^{p,m}$.

The following result can be found in [3], from which it is clear that for each $v \in \mathbb{C}$, the reproducing kernel $\|K_v^m\|_{p',m}$ is finite for all possible $p' \geq 1$.

Lemma 2.1. *Suppose m is a fixed non-negative integers and $Q_m(z)$ is the Taylor polynomial of e^z of order $m-1$(with the convention that $Q_0 = 0$). For any parameter $p' > 0$, $\sigma > 0$, $c > 0$ and $d > -mp' - 2$, we can find a positive constant C_0 such that*

$$\int_{\mathbb{C}} |e^{\bar{z}w} - Q_m(\bar{z}w)|^{p'} e^{-c|w|^2} |w|^d dA(w) \leq C_0 |z|^d e^{\frac{p'^2}{4c}|z|^2},$$

for all $|z| \geq \sigma$. Furthermore, this holds for all z if $d \leq p'm$ as well.

Therefore, it follows that if $g \in \Omega_m^p$, then the Hankel operator $H_g^p : \mathscr{F}^{p,m} \to L^{p,m}$ with symbol g, defined by $H_g^p f = (I - P^m)gf$ for all $f \in \mathscr{F}^{p,m}$, is densely defined on $\mathscr{F}^{p,m}$, since the set of linear span of all kernel functions $\{k_v^m : v \in \mathbb{C}\}$ is dense in the space $\mathscr{F}^{p,m}$. By using the definition of P^m, we write

$$H_g^p f(z) = \frac{1}{\pi} \int_{\mathbb{C}} (g(z) - g(v)) f(v) \overline{K_z^m(v)} |v|^{2m} e^{-|v|^2} dA(v). \tag{1}$$

Henceforth, for the convergence of integral in (1), we will assume that the symbol g is in Ω_m^p.

For some $z \in \mathbb{C}$, $1 \leq p < \infty$ and $0 < r < \infty$, let $\mathscr{B}(z;r) = \{v \in \mathbb{C} : |v - z| \leq r\}$ be the Euclidean disk centred at z and of radius r. Let L_{Loc}^p denote the space of all Lebesgue measurable functions g on $\mathbb{C}$ such that $g(v) \in L^p(K, dA(v))$ for each compact subset K of $\mathbb{C}$. Let $\mathcal{BA}_r$ be the set of all L_{Loc}^1 integrable functions g on $\mathbb{C}$ such that $\tilde{g}_r$ defined by

$$\tilde{g}_r(z) = \frac{1}{\pi r^2} \int_{\mathscr{B}(z;r)} g(v) dA(v)$$

is bounded on $\mathbb{C}$. For finite $p \geq 1$ and $g \in L_{Loc}^p$, denote

$$\tilde{g}_r^p(z) = \frac{1}{\pi r^2} \int_{\mathscr{B}(z;r)} |g(v)|^p dA(v).$$

Let $\mathcal{BA}_r^p$ be the set of all L_{Loc}^p integrable functions g on $\mathbb{C}$ such that $\tilde{g}_r^p$ is bounded on $\mathbb{C}$. Let $\mathcal{BMO}_r^p$ denote the set of all L_{Loc}^p integrable functions g such that

$$\|g\|_{\mathcal{BMO}_r^p} = \underset{z \in \mathbb{C}}{Sup} \left\{ \frac{1}{\pi r^2} \int_{\mathscr{B}(z;r)} |g(v) - \tilde{g}_r(z)|^p dA(v) \right\}^{\frac{1}{p}}$$

is finite. Let $\mathcal{BO}_r$ be the set of all continuous functions g on $\mathbb{C}$ such that

$$\|g\|_{\mathcal{BO}_r} = \underset{z \in \mathbb{C}}{Sup} \left\{ \underset{v \in \mathscr{B}(z;r)}{Sup} |g(v) - g(z)| \right\} < \infty.$$

The following results will be instrumental in the study of Hankel operators on $\mathscr{F}^{p,m}$.

Lemma 2.2. [5] *Let $p \geq 1$. Then the following conditions hold:*

1. *Let $g \in L^p_{Loc}$ then $g \in \mathcal{BMO}^p_r$ if and only if there is a constant $C > 0$ such that for every $z \in \mathbb{C}$ there exists a constant μ_z such that*

$$\int_{\mathscr{B}(z;r)} |g(v) - \mu_z|^p dA(v) \leq C.$$

2. *For $0 < r < R$, $\mathcal{BMO}^p_R \subset \mathcal{BMO}^p_r$.*

3. *$\mathcal{BO}_r$ is independent of r. Moreover, for any continous function g on $\mathbb{C}$, $g \in \mathcal{BO}$ if and only if there exists a constant $C_0 > 0$ such that*

$$|g(z) - g(v)| \leq C_0(|z - v| + 1)$$

for all $z, v \in \mathbb{C}$.

4. *If $g \in \mathcal{BMO}^p_{2r}$, then $\tilde{g}_r \in \mathcal{BO}_r$.*

5. *If $g \in \mathcal{BMO}^p_{2r}$, then $g - \tilde{g}_r \in \mathcal{BA}^p_r$.*

6. *$\mathcal{BMO}^p_r \subset \mathcal{BO}_r + \mathcal{BA}^p_r$ for $0 < r < \infty$.*

Lemma 2.3. *[3] Suppose $t \in \mathbb{R}$ and $M > 0$ be a fixed real number.*

1. *Then there exists a constant $C_0 > 0$ such that*

$$\sum_{k=0}^{\infty} \left(\frac{y}{k+1}\right)^t \frac{y^k}{k!} \leq C_0 e^y$$

for all real $y \geq M$. Futhermore, this holds for all $y \geq 0$ if $t \geq 0$.

2. *Then there exists a constant $C_0 > 0$ such that*

$$\sum_{k=0}^{\infty} \left(\frac{y}{k+1}\right)^t \frac{y^k}{k!} \geq C_0 e^y$$

for all real $y \geq M$. Futhermore, this holds for all $y \geq 0$ if $t \leq 0$.

For any two points u and v such that u and v do not lie on the same ray emanating from the origin, the lattice generated by u and v is the set $\{au + bv | a, b \in \mathbb{Z}\}$.

Lemma 2.4. *[8] Suppose λ is a locally integrable positive measure, $p > 0$, $r > 0$, m is a non- negative integer and $\{b_n\}$ is the lattice in $\mathbb{C}$ generated by r and ri. Then the following conditions are equivalent.*

1. *There exists a constant C_0 such that*

$$\int_{\mathbb{C}} |g(v) v^m e^{-\frac{|v|^2}{2}}|^p d\lambda \leq C_0 ||g||^p_{p,m}$$

for all entire functions g.

2. *There exists a constant $C_0 > 0$ such that $\lambda(\mathfrak{B}(z; r)) < C_0$ for all $z \in \mathbb{C}$.*

3. *There exists a constant $C_0 > 0$ such that $\lambda(\mathfrak{B}(b_n; r)) < C_0$ for all positive integers n.*

The Berezin transform of a function g is given by

$$\begin{aligned}\mathfrak{B}_{\mathrm{m}}(g)(z) &= \langle g k_z^m, k_z^m \rangle_{2,m} = \frac{1}{\pi m!}\int_{\mathbb{C}} g(v)|k_z^m(v)|^2|v|^{2m}e^{-|v|^2}dA(v)\\ &= \frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathbb{C}} g(v)|e^{\bar{z}v}-Q_m(\bar{z}v)|^2e^{-|v|^2}dA(v),\end{aligned}$$

where k_z^m denotes the normalized reproducing kernel of $\mathscr{F}^{2,m}$.

Proposition 2.5. *Let $g \in \Omega_m^p$. For $1 \le p \le \infty$, the following conditions are equivalent:*

1. *$g \in \mathcal{BA}_r^p$;*
2. *There exists a positive constant C such that*
$$\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathbb{C}} |g(v)|^p|e^{\bar{z}v}-Q_m(\bar{z}v)|^2e^{-|v|^2}dA(v) \le C$$
for all $z \in \mathbb{C}$;
3. *The multiplication operator $L_g^p : \mathscr{F}^{p,m} \to L^{p,m}$ is bounded.*

Proof. (1) ⇔ (2) Let $g \in \mathcal{BA}_r^p$ then $\int_{\mathscr{B}(z;r)} |g(v)|^p dA(v)$ is bounded on $\mathbb{C}$. Then Lemma 2.4 gives

$$\int_{\mathbb{C}} |h(v)v^m e^{-\frac{|v|^2}{2}}|^p d\lambda \le C_0\|h\|_{p,m}^p$$

for all entire functions h where $d\lambda(v) = |g(v)|^p dA(v)$ if and only if $g \in \mathcal{BA}_r^p$ and hence, it follows that $g \in \mathcal{BA}_r^p$ if and only if $\mathfrak{B}_{\mathrm{m}}|g|^p$ is bounded on $\mathbb{C}$ where

$$\mathfrak{B}_{\mathrm{m}}|g(z)|^p = \frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathbb{C}} |g(v)|^p|e^{\bar{z}v}-Q_m(\bar{z}v)|^2e^{-|v|^2}dA(v).$$

(1) ⇔ (3) Let $g \in \mathcal{BA}_r^p$ then by definition $\tilde{g}_r^p$ is bounded. Define a non-negative measure $d\lambda(z) = |g(z)|^p dA(z)$ on $\mathbb{C}$ then $\lambda(\mathscr{B}(z;r)) = \int_{\mathscr{B}(z;r)} d\lambda(v) = \int_{\mathscr{B}(z;r)} |g(v)|^p dA(v)$. Therefore, from Lemma 2.4, it follows that

$$\int_{\mathbb{C}} |h(v)v^m e^{-\frac{|v|^2}{2}}|^p d\lambda \le C_0\|h\|_{p,m}^p$$

for all entire functions h if and only if $g \in \mathcal{BA}_r^p$. Thus, for all $h \in \mathscr{F}^{p,m}$, we have

$$\begin{aligned}\|L_g^p(h)\|_{p,m}^p = \|hg\|_{p,m}^p &= \omega_{p,m}\int_{\mathbb{C}} |h(v)|^p|g(v)|^p|v|^{mp}e^{-\frac{p}{2}|v|^2}dA(v)\\ &= \omega_{p,m}\int_{\mathbb{C}} |h(v)v^m e^{-\frac{1}{2}|v|^2}|^p d\lambda(v) \le C\|h\|_{p,m}^p\end{aligned}$$

for some constant $C > 0$. □

Thus, from Lemma 2.2 and Proposition 2.5, it is obtained that $\mathcal{BO}_r$ and $\mathcal{BA}_r^p$ are independent of r and hence, we will denote them by $\mathcal{BO}$ and $\mathcal{BA}^p$, respectively.

Lemma 2.6. *Let $g \in \mathcal{BMO}_r^p$. Then*

$$\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathbb{C}} |g(v)-\mathfrak{B}_{\mathrm{m}}g(z)|^p|e^{\bar{z}v}-Q_m(\bar{z}v)|^2e^{-|v|^2}dA(v)$$

is bounded for $|z| > M$, for some positive constant M.

Proof. Let $g \in \mathcal{BMO}_r^p \subset \mathcal{BO} + \mathcal{BA}_r^p$, therefore, there exist two functions g_+, g_- on $\mathbb{C}$ such that $g_+ \in \mathcal{BO}_r$ and $g_- \in \mathcal{BA}_r^p$. Since $g_+ \in \mathcal{BO}_r$, therefore, by using Lemma 2.2 and Lemma 2.3 and the fact that $\mathcal{BO}_r$ is independent of r and

$$\lim_{\substack{v\in\mathscr{B}(z;r)\\|z|\to\infty}}(1-e^{-\bar{z}v}Q_m(\bar{z}v))=1,$$

it follows that

$$\begin{aligned}
&\left\{\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathscr{B}(z;r)}|g_+(v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|^p|e^{\bar{z}v}-Q_m(\bar{z}v)|^2e^{-|v|^2}dA(v)\right\}\\
&=\left\{\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathscr{B}(z;r)}|g_+(v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|^p|e^{\bar{z}v}|^2|1-e^{-\bar{z}v}Q_m(\bar{z}v)|^2\right.\\
&\qquad\left.e^{-|v|^2}dA(v)\right\}\\
&\le C\left\{\int_{\mathscr{B}(z;r)}|g_+(v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|^pe^{-|z-v|^2}dA(v)\right\}\\
&\le C\left\{\int_{\mathbb{C}}|g_+(v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|^pe^{-|z-v|^2}dA(v)\right\}\\
&= C\left\{\int_{\mathbb{C}}|g_+(z-v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|^pe^{-|v|^2}dA(v)\right\},
\end{aligned}$$

for all $|z| > M$ for some positive constants C and M, where

$$\begin{aligned}
&|g_+(z-v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|\\
&=|g_+(z-v)-\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2)}\int_{\mathbb{C}}g_+(u)|e^{\bar{z}u}-Q_m(\bar{z}u)|^2e^{-|v|^2}dA(u)|\\
&=|\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2)}\int_{\mathbb{C}}(g_+(z-v)-g_+(u))|e^{\bar{z}u}-Q_m(\bar{z}u)|^2e^{-|v|^2}dA(u)|\\
&=|\lim_{r\to\infty}\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2)}\int_{\mathscr{B}(z;r)}(g_+(z-v)-g_+(u))|e^{\bar{z}u}-Q_m(\bar{z}u)|^2\\
&\qquad e^{-|u|^2}dA(u)|\\
&\le\lim_{r\to\infty}|\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2)}\int_{\mathscr{B}(z;r)}(g_+(z-v)-g_+(u))|e^{\bar{z}u}|^2|1-e^{-\bar{z}u}Q_m(\bar{z}u)|^2\\
&\qquad e^{-|u|^2}dA(u)|\\
&\le C\lim_{r\to\infty}|\int_{\mathscr{B}(z;r)}(g_+(z-v)-g_+(u))e^{-|z-u|^2}dA(u)|\\
&\le C|\int_{\mathbb{C}}(g_+(z-v)-g_+(u))e^{-|z-u|^2}dA(u)|\\
&= C\int_{\mathbb{C}}|g_+(z-v)-g_+(z-u)|e^{-|u|^2}dA(u)|
\end{aligned}$$

for all $|z| > M$. Therefore,

$$\begin{aligned}
&\left\{\frac{1}{\pi(e^{|z|^2}-Q_m(|z|^2))}\int_{\mathscr{B}(z;r)}|g_+(v)-\mathfrak{B}_{\mathrm{m}}g_+(z)|^p|e^{\bar{z}v}-Q_m(\bar{z}v)|^2e^{-|v|^2}dA(v)\right\}\\
&\le C^2\iint_{\mathbb{C}}|g_+(z-v)-g_+(z-u)|^pe^{-|u|^2}dA(u)e^{-|v|^2}dA(v)
\end{aligned}$$

$$\leq \iint_{\mathbb{C}} (|u-v|+1)^p e^{-|u|^2} dA(u) e^{-|v|^2} dA(v)$$

for all $|z| > M$ which is a constant term. Now, since $g_- \in \mathcal{BA}_r^p$, therefore, by Proposition 2.5, there exists a positive constant C such that

$$\frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \leq C$$

for all $z \in \mathbb{C}$. Consider

$$\begin{aligned}
&\left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v) - \mathfrak{B}_m g_-(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\}^{\frac{1}{p}} \\
&\leq \left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\}^{\frac{1}{p}} + |\mathfrak{B}_m g_-(z)| \\
&\quad ||k_z||_{2,m}^{\frac{2}{p}} \\
&\leq \left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\}^{\frac{1}{p}} + |\mathfrak{B}_m g_-(z)| \\
&\leq C + |\mathfrak{B}_m g_-(z)|,
\end{aligned}$$

where

$$\begin{aligned}
|\mathfrak{B}_m g_-(z)| &\leq \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v)| |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \\
&\leq \left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\}^{\frac{1}{p}} \\
&\quad ||k_z||_{2,m}^{\frac{2}{q}} \\
&= \left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\}^{\frac{1}{p}} \\
&\leq C.
\end{aligned}$$

Therefore,

$$\left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g_-(v) - \mathfrak{B}_m g_-(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\}^{\frac{1}{p}} \leq 2C$$

and hence, we get the result. □

Lemma 2.7. *Suppose there exists a positive constant M such that*

$$\underset{|z|>M}{Sup} \left\{ \frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))} \int_{\mathbb{C}} |g(v) - \mathfrak{B}_m g(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v) \right\},$$

is bounded. Then, there exists a constant $M' > 0$ such that for each $z \in \mathbb{C}$, there exists a constant μ_z such that

$$\underset{|z|>M'}{Sup} \left\{ \frac{1}{\pi r^2} \int_{\mathscr{B}(z;r)} |g(v) - \mu_z|^p dA(v) \right\}$$

is bounded.

Proof. By using the fact that $e^{-|z-v|^2} \geq a$ for $v \in \mathscr{B}(z;r)$ and for some constant $a > 0$, it follows that

$$
\begin{aligned}
& aC\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)} |g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|^p dA(v)\right\} \\
& \leq C\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)} |g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|^p e^{-|z-v|^2} dA(v)\right\} \\
& = C\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)} |g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|^p e^{-|z|^2}|e^{\bar{z}v}|^2 e^{-|v|^2} dA(v)\right\} \\
& \leq \left\{\frac{1}{\pi^2 r^2((e^{|z|^2} - Q_m(|z|^2))}\int_{\mathscr{B}(z;r)} |g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v)\right\} \qquad (2) \\
& \leq \left\{\frac{1}{\pi^2 r^2((e^{|z|^2} - Q_m(|z|^2))}\int_{\mathbb{C}} |g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v)\right\}
\end{aligned}
$$

for all $|z| > M'$, where the Eq. (2) follows from Lemma 2.3 and

$$\lim_{\substack{v\in\mathscr{B}(z;r) \\ |z|\to\infty}} (1 - e^{-\bar{z}v} Q_m(\bar{z}v)) = 1.$$

□

Lemma 2.6 and Lemma 2.7 jointly give the following result:

Theorem 2.8. *Let $g \in \mathcal{BMO}^p_r$. Then the following conditions are equivalent:*

1. *There exists a constant $M > 0$ such that*

$$\mathop{Sup}_{|z|>M}\left\{\frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2)}\int_{\mathbb{C}} |g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v)\right\} < \infty;$$

2. *There exists a constant $M > 0$ such that for each $z \in \mathbb{C}$, there exists a constant μ_z such that*

$$\mathop{Sup}_{|z|>M}\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)} |g(v) - \mu_z|^p dA(v)\right\} < \infty;$$

3. *There exists a constant $M > 0$ such that for each $z \in \mathbb{C}$, there exists a constant μ_z such that*

$$\mathop{Sup}_{|z|>M}\left\{\frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2)}\int_{\mathbb{C}} |g(v) - \mu_z|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v)\right\} < \infty.$$

Proof. (1) implies (2) follows from Lemma 2.7 and (1) implies (3) follows from Lemma 3.1 [5]. □

Proposition 2.9. *Let $g \in \mathcal{BMO}^p_{2r}$. Then there exists a positive constant M such that the following hold:*

(1) $\mathop{Sup}_{|z|>M+r}\left\{\mathop{Sup}_{v\in\mathscr{B}(z;r)} |\mathfrak{B}_{\mathfrak{m}} g(v) - \mathfrak{B}_{\mathfrak{m}} g(z)|\right\} < \infty$

(2) $\mathop{Sup}_{|z|>M+r}\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)} |(g - \mathfrak{B}_{\mathfrak{m}} g)(v)|^p dA(v)\right\} < \infty.$

Proof. Let $g \in \mathcal{BMO}^p_{2r} \subset \mathcal{BMO}^p_r$. Consider

$$|\mathfrak{B}_{\mathfrak{m}} g(z) - \tilde{g}_r(z)|$$

$$= |\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)}\mathfrak{B}_{\mathfrak{m}}g(z)dA(v) - \frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)}g(v)dA(v)|$$
$$\leq \frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)}|g(v) - \mathfrak{B}_{\mathfrak{m}}g(z)|dA(v)$$
$$\leq \left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)}|g(v) - \mathfrak{B}_{\mathfrak{m}}g(z)|^p dA(v)\right\}^{\frac{1}{p}}$$
$$\leq C\left\{\frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2)}\int_{\mathbb{C}}|g(v) - \mathfrak{B}_{\mathfrak{m}}g(z)|^p|e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2}dA(v)\right\}^{\frac{1}{p}}$$
$$\leq C_0,$$

for all $|z| > M > 0$ and for some constant $C_0 > 0$. Thus,

$$\underset{|z|>M+r}{Sup}\left\{\underset{v\in\mathscr{B}(z;r)}{Sup}|(\mathfrak{B}_{\mathfrak{m}}g - \tilde{g}_r)(v) - (\mathfrak{B}_{\mathfrak{m}}g - \tilde{g}_r)(z)|\right\} < \infty$$

and

$$\underset{|z|>M+r}{Sup}\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)}|(\mathfrak{B}_{\mathfrak{m}}g - \tilde{g}_r)(v)|^p dA(v)\right\} < \infty.$$

Since $g \in \mathcal{BMO}^p_{2r} \subset \mathcal{BMO}^p_r$, therefore by Lemma 2.2, it follows that $\tilde{g}_r \in \mathcal{BO}$ and $g - \tilde{g}_r \in \mathcal{BA}^p$, so $\mathfrak{B}_{\mathfrak{m}}g = \mathfrak{B}_{\mathfrak{m}}g - \tilde{g}_r + \tilde{g}_r$, $g - \mathfrak{B}_{\mathfrak{m}}g = g - \tilde{g}_r + \tilde{g}_r - \mathfrak{B}_{\mathfrak{m}}g$, we get the desired result. □

Lemma 2.10. *If $g \in \mathcal{BA}^p$, then H^p_g is bounded on $\mathscr{F}^{p,m}$ for finite $p \geq 1$.*

Proof. By Proposition 2.5, $g \in \mathcal{BA}^p$ if and only if L^p_g is bounded on $\mathscr{F}^{p,m}$ and hence if $g \in \mathcal{BA}^p$ then H^p_g is bounded on $\mathscr{F}^{p,m}$, since P^m is bounded. □

For two quantities X and Y, the equation $X \lesssim Y$ represents there exists a constant $C > 0$ such that $X \leq CY$ (C is independent of X and Y).

Lemma 2.11. *If $g \in \mathcal{BO}$, then H^p_g is bounded on $\mathscr{F}^{p,m}$ for all $1 \leq p \leq \infty$.*

Proof. Let $h \in \mathscr{F}^{p,m}$ and $1 < p < \infty$. Since $g \in \mathcal{BO}$, therefore, by Lemma 2.2, we obtain that

$$|H^p_g(h)(z)|^p \leq \left\{\omega_{p,m}\int_{\mathbb{C}}|g(z) - g(v))||h(v)||\frac{e^{z\bar{v}} - Q_m(z\bar{v})}{(z\bar{v})^m}|e^{-|v|^2}|v|^{2m}dA(v)\right\}^p$$
$$\leq C\left\{\omega_{p,m}\int_{\mathbb{C}}(|z - v| + 1)|h(v)||\frac{e^{z\bar{v}} - Q_m(z\bar{v})}{(z\bar{v})^m}|e^{-|v|^2}|v|^{2m}dA(v)\right\}^p.$$

Further from [4], we have

$$|\frac{e^{z\bar{v}} - Q_m(z\bar{v})}{(z\bar{v})^m}| \lesssim \frac{e^{\frac{1}{2}|z|^2 + \frac{1}{2}|v|^2 - \frac{1}{8}|z-v|^2}}{(1 + |z||v|)^m} \leq \frac{e^{\frac{1}{2}|z|^2 + \frac{1}{2}|v|^2 - \frac{1}{8}|z-v|^2}}{(|z||v|)^m}.$$

Therefore,

$$|H^p_g(h)(z)|^p e^{-\frac{p}{2}|z|^2}|z|^{pm}$$
$$\leq Ce^{-\frac{p}{2}|z|^2}|z|^{pm}\left\{\omega_{p,m}\int_{\mathbb{C}}(|z - v| + 1)|h(v)||\frac{e^{z\bar{v}} - Q_m(z\bar{v})}{(z\bar{v})^m}|e^{-|v|^2}|v|^{2m}dA(v)\right\}^p$$
$$\leq C\left\{\omega_{p,m}\int_{\mathbb{C}}(|z - v| + 1)|h(v)|e^{-\frac{1}{2}|v|^2}e^{-\frac{1}{8}|z-v|^2}|v|^m dA(v)\right\}^p$$

$$\leq C\left\{\omega_{p,m}\int_{\mathbb{C}}|h(v)|^p e^{-\frac{p}{2}|v|^2}|v|^{pm}dA(v)\right\}\left\{\omega_{p,m}\int_{\mathbb{C}}(|z-v|+1)^q e^{-\frac{q}{8}|z-v|^2}dA(v)\right\}^{\frac{p}{q}}$$
$$= C\|h\|_{p,m}^p\left\{\omega_{p,m}\int_{\mathbb{C}}(|z-v|+1)^q e^{-\frac{q}{8}|z-v|^2}dA(v)\right\}^{\frac{p}{q}},$$

for some constant $C > 0$. Therefore, $\|H_g^p(h)\|_{p,m}^p \leq C_0\|h\|_{p,m}^p$ and hence, H_g^p is bounded on $\mathscr{F}^{p,m}$ for $1 < p < \infty$. For $p = 1$, we can conclude by using Fubini's theorem that

$$\begin{aligned}\|H_g^p(h)\|_{1,m} &= \omega_{1,m}\int_{\mathbb{C}}|H_g^p(h)(z)|e^{-\frac{1}{2}|z|^2}|z|^m dA(z)\\ &\leq C_0\left\{\int_{\mathbb{C}}|h(v)|e^{-\frac{1}{2}|v|^2}|v|^m dA(v)\right\}\left\{\int_{\mathbb{C}}(|z-v|+1)e^{-\frac{1}{8}|z-v|^2}dA(z)\right\}\\ &\leq C_0\|h\|_{1,m},\end{aligned}$$

where C_0 is a constant. For $p = \infty$,

$$\begin{aligned}\|H_g^p(h)\|_{\infty,m} &= |H_g^p(h)(z)|e^{-\frac{1}{2}|z|^2}|z|^m\\ &\leq \|h\|_{\infty,m}\left\{\int_{\mathbb{C}}(|z-v|+1)e^{-\frac{1}{8}|z-v|^2}dA(z)\right\} \leq C_1,\end{aligned}$$

where $C_1 > 0$ is a constant and hence, the result follows for all $1 \leq p \leq \infty$. □

Theorem 2.12. *Let $g \in \mathcal{BMO}_r^p$. Then the operators H_g^p and $H_{\bar{g}}^p$ are bounded for all $1 \leq p < \infty$.*

Proof. The proof of the theorem follows from Lemma 2.2, Lemma 2.10 and Lemma 2.11 and the fact that if $g \in \mathcal{BMO}_r^p$ then so $\bar{g}$. □

3. $\mathcal{VMO}_r^p$ spaces and Compactness of Hankel operators on $\mathscr{F}^{p,m}$

Define $\mathcal{VA}_r$ be the set of all L_{Loc}^1 integrable functions g on $\mathbb{C}$ such that $\lim_{|z|\to\infty}\tilde{g}_r = 0$. For finite $p \geq 1$, let $\mathcal{VMO}_r^p$ denote the set of all L_{Loc}^p integrable functions g such that

$$\lim_{|z|\to\infty}\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)}|g(v)-\tilde{g}_r(z)|^p dA(v)\right\}^{\frac{1}{p}} = 0.$$

Let $\mathcal{VO}_r \subset \mathcal{BO}_r$ be the set of all continuous functions g on $\mathbb{C}$ such that

$$\lim_{|z|\to\infty}\ \underset{v\in\mathscr{B}(z;r)}{Sup}|g(v)-g(z)| = 0.$$

Let $\mathcal{VA}_r^p$ be the set of all L_{Loc}^p integrable functions g on $\mathbb{C}$ such that

$$\lim_{|z|\to\infty}\tilde{g}_r^p = 0.$$

The following Lemma will be useful in the study of compact Hankel operators on $\mathscr{F}^{p,m}$ and the related results.

Lemma 3.1. *[8] Let λ is a positive Borel measure, $0 < p < \infty$, $r > 0$, m is a non- negative integer and $\{b_n\}$ is the lattice in $\mathbb{C}$ generated by r and ri. Then the following conditions are equivalent:*

1. $\lim_{n\to\infty}\int_{\mathbb{C}} |g_n(v)v^m e^{-\frac{|v|^2}{2}}|^p d\lambda = 0$ *for all bounded sequence* $\{g_n\}$ *in* $\mathscr{F}^{p,m}$ *that converges to* 0 *uniformly on compact sets;*
2. $\lim_{|z|\to\infty} \lambda(\mathscr{B}(z;r)) = 0;$
3. $\lim_{n\to\infty} \lambda(\mathscr{B}(b_n;r)) = 0.$

Similar to $\mathcal{BO}_r$ and $\mathcal{BA}_r^p$, it is easy to observe that $\mathcal{VO}_r$ and $\mathcal{VA}_r^p$ are independent of r, so we will denote them by $\mathcal{VO}$ and $\mathcal{VA}^p$, respectively.

The following results are analogues to Lemma 2.2 and Theorem 2.8.

Theorem 3.2. *Let p is any natural number. Then the following conditions are equivalent:*

1. $g \in \mathcal{VMO}^p$;
2. $g \in \mathcal{VO} + \mathcal{VA}^p$;
3. $\lim_{|z|\to\infty}\left\{\frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))}\int_{\mathbb{C}} |g(v) - \mathfrak{B}_m g(z)|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v)\right\} = 0;$
4. *There exists a constant* $M > 0$ *such that for each* $z \in \mathbb{C}$*, there exists a constant* μ_z *such that*

$$\lim_{|z|\to\infty}\left\{\frac{1}{\pi r^2}\int_{\mathscr{B}(z;r)} |g(v) - \mu_z|^p dA(v)\right\} = 0;$$

5. *There exists a constant* $M > 0$ *such that for each* $z \in \mathbb{C}$*, there exists a constant* μ_z *such that*

$$\lim_{|z|\to\infty}\left\{\frac{1}{\pi(e^{|z|^2} - Q_m(|z|^2))}\int_{\mathbb{C}} |g(v) - \mu_z|^p |e^{\bar{z}v} - Q_m(\bar{z}v)|^2 e^{-|v|^2} dA(v)\right\} = 0.$$

From Theorem 3.2, it follows that $\mathcal{VMO}_r^p$ is independent of r, so we will write $\mathcal{VMO}^p$.

Lemma 3.3. 1. *If* $g \in \mathcal{VMO}^p$*, then* $\tilde{g}_r \in \mathcal{VO}$.

2. *If* $g \in \mathcal{VMO}^p$*, then* $g - \tilde{g}_r \in \mathcal{VA}^p$ *for every* $r > 0$.
3. *If* $g \in \mathcal{VMO}^p$*, then* $\mathfrak{B}_m(g) \in \mathcal{VO}$.
4. *If* $g \in \mathcal{VMO}^p$*, then* $g - \mathfrak{B}_m(g) \in \mathcal{VA}^p$.
5. *The function* $g \in \mathcal{VO}$ *if and only if for each constant* $C > 0$*, there exists* $r > 0$ *such that* $|g(z) - g(v)| \le C(1 + |z - v|)$ *for all* $z, v \in \mathbb{C}\backslash\mathscr{B}(0;r)$ *(see [1]).*

Lemma 3.4. *[1] For* $r > 0$*, consider a function* $g : \mathbb{C}\backslash\mathscr{B}(0;r) \to \mathbb{C}$ *with*

$$|g(z) - g(v)| \le C(1 + |z - v|) \text{ for all } z, v \in \mathbb{C}\backslash\mathscr{B}(0;r),$$

where $C > 0$ *is independent of* g*. Then, there exists a function* G *on* $\mathbb{C}$ *such that* $g = G$ *on* $\mathbb{C}\backslash\mathscr{B}(0;r)$ *and* $|G(z) - G(v)| \le 2C(1 + |z - v|)$ *for all* $z, v \in \mathbb{C}$.

Theorem 3.5. *Let* $g \in \mathcal{VMO}^p$ *where* $1 \le p < \infty$*. Then the Hankel operators* H_g^p *and* $H_{\bar{g}}^p$ *are both compact.*

Proof. Let $g \in \mathcal{VA}^p$. This gives the positive measure $d\lambda = |g|^p dA$ satisfying $\lim_{|z|\to\infty} \lambda(\mathscr{B}(z;r)) = 0$. So, by Lemma 3.1, the multiplication operator $L_g^p : \mathscr{F}^{p,m} \to L^{p,m}$ is compact and so is H_g^p.

Let $g \in \mathcal{VO}$. Let $\epsilon > 0$ be arbitrary. Using Lemma 3.3 and Lemma 3.4, it follows that there exists a function G on $\mathbb{C}$ such that $g = G$ on $\mathbb{C}\backslash\mathscr{B}(0;r)$ and $|G(z) - G(v)| \le 2\epsilon(1 + |z - v|)$ for all $z, v \in \mathbb{C}$. Then Lemma 2.2 and Lemma 2.11 give H_G^p is bounded with $\|H_G^p\| \le 2\epsilon C_0$ for some constant $C_0 > 0$. Also, H_{g-G}^p is compact, since $g - G$ has compact support, and $\|H_G^p - H_{g-G}^p\| = \|H_g^p\| \le 2\epsilon C_0$. Since $\epsilon > 0$ is arbitrary, therefore, the Hankel operators H_g^p is compact. Similarly, it can be proved that the Hankel operators $H_{\bar{g}}^p$ is compact. Hence, by using Theorem 3.2, the result follows. □

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