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# Some Characterizations of Pseudo $\mathcal{Z}$ Symmetric Spacetimes

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**Abstract.** The motive of this work is to investigate pseudo $\mathcal{Z}$ symmetric spacetimes. At first we present some basic properties of pseudo $\mathcal{Z}$ symmetric spacetimes showing that the 1-forms $A$ and $B$ and the scalars $a$ and $b$ associated with the spacetime agree with a specific relation. Next we explore conditions under which a pseudo $\mathcal{Z}$ symmetric spacetime to be a GRW spacetime and a quasi-Einstein spacetime respectively. Also we provide some results on pseudo Ricci symmetric spacetimes.

## 1. Introduction

In present day scenario, Lorentzian manifold[7] is a rapid growing sub-section of the semi-Riemannian manifold. It has numerous applications in mathematics and applied physics, especially in the theory of relativity, cosmology and astrophysics. The time-like vector field $\psi$ breaks all causal tangent vectors into two distinct classes, called future and past directed. A causal tangent vector $v \in T_pM$ is future (resp., past) directed if $g(\psi(p), v) < 0$ (resp., $g(\psi(p), v) > 0$). A Lorentzian manifold $(M, g)$ is known as time orientable when $(M, g)$ admits a time orientation by some time-like vector field V. A connected, time oriented Lorentzian manifold $(M, g)$ is termed as a spacetime. For more details we cite ([8], [9], [14], [27], [29]).

During the study of conformally flat space of class one Sen and Chaki [26] found that the Ricci tensor $S$ obeys

$$(\nabla_U S)(V, W) = 2\eta(U)S(V, W) + \eta(V)S(W, U) + \eta(W)S(V, U), \tag{1}$$

where $\eta$ is a non-zero 1-form, $U, V, W \in \chi(M)$ and $\nabla$ is the operator of covariant differentiation with respect to the metric tensor $g$. Later Chaki [3] called an $n$-dimensional non-flat Riemannian manifold whose Ricci tensor satisfies the condition (1), a pseudo Ricci symmetric manifold (in short, $(PRS)_n$). Mantica and Suh [19] defined a new tensor, named $\mathcal{Z}-$tensor and characterized pseudo $\mathcal{Z}$ symmetric Riemannian manifolds (briefly, $(P\mathcal{Z}S)_n$ ). Also in [20] the same authors have studied pseudo $\mathcal{Z}$ symmetric spacetimes in dimension $n = 4$. Among others they have corroborated that a $(P\mathcal{Z}S)_4$ is a quasi Einstein manifold and justified their result by constructing an example.

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A Lorentzian manifold of dimension $n > 3$ is said to be pseudo $\mathcal{Z}$ symmetric [19] if the non-null $\mathcal{Z}-$tensor defined by

$$\mathcal{Z}(U,V) = S(U,V) + \mu g(U,V), \tag{2}$$

$\mu$ being a scalar and agrees with the condition

$$(\nabla_U \mathcal{Z})(V,W) = 2A(U)\mathcal{Z}(V,W) + A(V)\mathcal{Z}(U,W) + A(W)\mathcal{Z}(V,W), \tag{3}$$

$A$ being a non-zero 1-form. It is to be noted that if $\mu = 0$, $(P\mathcal{Z}S)_n$ becomes $(PRS)_n$.

Also if $\mu = -\frac{r}{n}$, $r$ being the manifold's scalar curvature, then $(P\mathcal{Z}S)_n$ reduces to pseudo projective Ricci symmetric manifold investigated by Chaki and Saha [5].

In ([23], [19]), the authors have obtained different properties of the $\mathcal{Z}$ tensor elaborately. According to some results in refs. ([23], [19]), the $\mathcal{Z}$ tensor may be needed for describe the Einstein equations of general relativity([12], [15]). In particular, the equation $\mathcal{Z}(U,V) = \kappa T(U,V)$ being $\kappa = \frac{8\pi G}{c^4}$ the Einstein gravitational constant (see Ref. [12]) and the condition $divT = 0$ advancing from the energy-momentum tensor give $\nabla_U(\frac{r}{2} + \mu) = 0$, that is,

$$\mu = -\frac{r}{2} + \Lambda. \tag{4}$$

This terminology $\Lambda$ is nothing but the cosmological constant and Einstein's field equations accept the shape

$$S(U,V) - \frac{r}{2}g(U,V) + \Lambda g(U,V) = \kappa T(U,V).$$

A perfect fluid spacetime is interpreted by the following equation (see [15]):

$$T(U,V) = (\sigma + p)B(U)B(V) + pg(U,V),$$

$\sigma$ being the energy density, $p$ is the isotropic pressure, $T$ is the energy momentum tensor and the vector field $\tilde{\rho}$ defined by $g(X,\tilde{\rho}) = B(X)$, for all $X$, is the fluid flow velocity such that, $g(\tilde{\rho},\tilde{\rho}) = -1$.

The fluid is called perfect due to absence of the heat conduction term and stress term corresponding to viscosity [15].

An $n$-dimensional Lorentzian manifold $(M,g)$, $(n > 3)$ is often called perfect fluid spacetime if its Ricci tensor is of the form

$$S(U,V) = ag(U,V) + bB(U)B(V), \tag{5}$$

$a$ and $b$ are scalars, the vector field $\tilde{\rho}$ metrically equivalent to the 1-form $B$ is a unit timelike vector field, that is, $g(\tilde{\rho},\tilde{\rho}) = -1$. The vector field $\tilde{\rho}$ is called the fluid flow velocity. The manifold satisfying (5) is called quasi Einstein manifold by the geometers [4]. However in a quasi Einstein manifold the vector field $\rho$ is not necessarily a unit timelike vector field. Semi-Riemannian quasi Einstein manifolds come off during the study of exact solutions of Einstein's field equations.

Form (5) of the Ricci tensor is obtained by Einstein's field equations if the energy momentum tensor of the spacetime is a perfect fluid with velocity vector field $\tilde{\rho}$. The scalars $a$ and $b$ are related to the isotropic pressure $p$ and energy density $\sigma$.

In cosmology, the observation that the space is isotropic and homogeneous on an astronomically immense scale chooses the Robertson-Walker (RW) metric. In 1995, Aliàs, Romero and Sànchez [1] generalized the notion of RW metric and called it a generalized Robertson-Walker (GRW) metric. A Lorentzian manifold $M$ of dimension $n \geq 3$ endowed with the Lorentzian metric $g$ defined by

$$ds^2 = g_{ab}dx^a dx^b = -(dt)^2 + f(t)^2 g^*_{lm}(\overrightarrow{x})dx^l dx^m,$$

where $t$ is the time and $g^*_{lm}(\overrightarrow{x})dx^l dx^m$ is the metric tensor of a Riemannian manifold, is a $GRW$ spacetime. In other words, a $GRW$ spacetime is the warped product $-I \times f^2 M^*$, where $I$ is an open interval of the real line, $f$ is a smooth warping function or scale factor such that $f > 0$ and $M^*$ is an $(n-1)$-dimensional Riemannian manifold. In particular, if $M^*$ is a 3− dimensional Riemannian space of constant curvature, then the warped product $-I \times f^2 M^*$ is said to be a $RW$ spacetime. A Robertson-Walker spacetime complies the cosmological principle, that is, the spacetime is spatially isotropic and spatially homogeneous, although the $GRW$ spacetime is not necessarily spatially homogeneous [6]. In [2] Brozos- Vàzquez, Garcia-Rio and Vàzquez-Lorenzo bridged the gap between $RW$ spacetime and $GRW$ spacetime by providing the following: "*A GRW spacetime is conformally flat if and only if it is a RW spacetime*." For more details of $GRW$ spacetimes, we call ( [10], [16], [17], [18], [21], [22]) and their references.

In 2016, the authors [18] prove that a perfect fluid spacetime of dimension $n \geqslant 4$ with (1) irrotational velocity vector field and (2) null divergence of the Weyl tensor, is a generalized Robertson-Walker spacetime with an Einstein fiber.

Subsequently in 2017, Mantica and Molinari [16] obtained a necessary and sufficient condition for a Lorentzian manifold to be a $GRW$ spacetime which is stated as follows:

**Theorem A.** A Lorentzian manifold of dimension $n \geqslant 3$ is a $GRW$ spacetime if and only if it admits a unit timelike torseforming vector, $\nabla_X V = \alpha(X + \omega(X)V)$, that is also an eigenvector the Ricci tensor where $\omega(X) = g(X, V)$ for all $X$.

It is well known that any Robertson-Walker spacetime is a perfect fluid spacetime [24] and for $n = 4$, a GRW spacetime is a perfect fluid if and only if it is a Robertson-Walker spacetime.

In a paper [25] Ray-Guha prove the following:

**Theorem B.** A perfect fluid pseudo Ricci symmetric spacetime is a quasi Einstein manifold.

Motivated by Theorems **A** and **B**, the intention of this paper is to assert the following remarkable result:

**Theorem 1.** A perfect fluid pseudo $\mathcal{Z}$ symmetric spacetime with constant scalar curvature is a generalized Robertson-Walker spacetime.

**Corollary 1.** A perfect fluid pseudo Ricci symmetric spacetime is a generalized Robertson-Walker spacetime.

The corollary improves the result of Theorem **B**. Generalizing Theorem **B**, Mantica and Suh [20] proved:

**Theorem C.** Let $(P\mathcal{Z}S)_4$ be a perfect fluid spacetime. If the condition $A_k u^k = 0$ is fulfilled then the space is quasi Einstein, where $A_k$ is the associated covector of $(P\mathcal{Z}S)_4$ and $u^k$ is the velocity vector of the perfect fluid spacetime.

On the other hand, Failkow[11] introduced a new notion of *concircular* vector field $V$ defined by $\nabla_X V = \psi X$, where $\psi$ being a scalar. Motivated by this one, Chen[6] characterized a GRW-spacetime by proving the following theorem:

**Theorem D.** A lorentzian manifold $M$ of dimension $n \geq 3$ is a GRW-spacetime if and only if it admits a timelike concircular vector $V$ such that $||V|| < 0$ and $\nabla_X V = \psi X$.

Stimulated by the results of Theorems **C** and **D** we also give another two remarkable results as follows:

**Theorem 2.** A pseudo $\mathcal{Z}$ symmetric generalized Robertson-Walker spacetime is a quasi Einstein spacetime, provided the scalar curvature $r$ is constant.

Since the notion of pseudo $Z$ symmetric manifold is a generalization of Pseudo Ricci symmetric manifold, we can state the following:

**Corollary 2.** A pseudo Ricci symmetric generalized Robertson-Walker spacetime is a quasi Einstein spacetime.

**Theorem 3.** A pseudo $\mathcal{Z}$ symmetric spacetime is a quasi Einstein spacetime, provided the $\mathcal{Z}$ tensor is of Codazzi type.

## 2. Preliminaries

From (3) we obtain

$$(\nabla_U \mathcal{Z})(V,W) - (\nabla_V \mathcal{Z})(U,W) = A(U)\mathcal{Z}(V,W) - A(V)\mathcal{Z}(U,W). \tag{6}$$

Now using (2) in the above equation we infer

$$\begin{aligned}(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = (V\mu)g(U,W) - (U\mu)g(V,W) + A(U)S(V,W) - A(V)S(U,W)\\ +[A(U)g(V,W) - A(V)g(U,W)].\end{aligned} \tag{7}$$

Taking a frame field and contracting $V, W$ it follows that

$$dr(U) = 2(1-n)(U\mu) + 2A(U)r - 2S(U,\rho) + 2\mu(n-1)A(U). \tag{8}$$

Again with the help of (2) we get from (3)

$$\begin{aligned}(\nabla_U S)(V,W) + (U\mu)g(V,W) = 2A(U)[S(V,W) + \mu g(V,W)] + A(V)[S(U,W) + \mu g(U,W)]\\ +A(W)[S(V,U) + \mu g(V,U)],\end{aligned} \tag{9}$$

which implies by contraction

$$dr(U) = -n(U\mu) + 2A(U)r + 2(n+1)\mu A(U) + 2S(U,\rho). \tag{10}$$

Equation (8) and (10) together give

$$S(U,\rho) = (U\mu) - \mu A(U). \tag{11}$$

Using (11) in (8) we get

$$dr(U) = -2n(U\mu) + 2A(U)r + 2\mu n A(U). \tag{12}$$

These formulas will be used in the sequel.

## 3. Relation between the associated 1-forms and the associated vector fields

Let $\rho$ be the vector field metrically equivalent to the 1−form $A$ of the $(P\mathcal{Z}S)_n$ spacetime, that is, $g(X,\rho) = A(X)$ for all $X$.

Putting $V = \rho$ in (5) implies

$$S(U,\rho) = aA(U) + bB(U)B(\rho).$$

Let us assume that the scalar curvature $r$ =constant. Hence from equation (4) it follows that $\mu$ is constant. Now using (11) in the above equation we obtain

$$B(U) = -\frac{(\mu + a)A(U)}{bB(\rho)}, \tag{13}$$

provided the scalar curvature $r$ is constant.

Putting $U = \tilde{\rho}$ in the above equation gives

$$b - a = \mu, since \ \ B(\tilde{\rho}) = g(\tilde{\rho},\tilde{\rho}) = -1, A(\tilde{\rho}) = g(\rho,\tilde{\rho}) = B(\rho). \tag{14}$$

Summing up we can state the following:

**Theorem 3.1.** In a perfect fluid $(P\mathcal{Z}S)_n$ spacetime the associated 1-forms $A$ and $B$ are related by (13) and the scalars $a$ and $b$ are related by (14), provided the scalar curvature $r$ is constant.

## 4. Significance of the associated scalars

From (5) it follows that

$$S(U, \tilde{\rho}) = (a - b)B(U).$$

where $B(U) = g(U, \rho)$ for all $U$.
With the help of (14) we get from the above equation

$$S(U, \tilde{\rho}) = -\mu B(U),$$

where $B(U) = g(U, \rho)$ for all $U$.
Let $\psi$ be any vector orthogonal to $\tilde{\rho}$. Then $g(\psi, \tilde{\rho}) = 0$, that is, $B(\psi) = 0$. Hence from (5) we obtain

$$S(U, \psi) = ag(U, \psi),$$

which implies $a$ is an eigenvalue corresponding to the eigen vector $V$. If $a = -\mu$, then (14) implies $b = 0$ which contradicts the definition of perfect fluid spacetime. Thus the eigen values $-\mu$ and $a$ are distinct. Since the spacetime is of dimension $n$ and $\psi$ is any vector orthogonal to $\tilde{\rho}$, it follows from a well known result of Linear algebra [28] that the eigenvalue $a$ of multiplicity $n-1$. Therefore the multiplicity of the eigenvalue $-\mu$ must be 1. This leads to the following:
**Theorem 4.1.** In a perfect fluid $(P\mathcal{Z}S)_n$ spacetime the Ricci tensor $S$ has only two eigenvalues, namely $a$ and $-\mu$ of which the former is of multiplicity $n-1$ and the latter is simple, provided the scalar curvature $r$ is constant.

## 5. Proof of the main theorems

**Proof of Theorem 1.** We assume that the scalar curvature $r$ of the spacetime is a constant. Consequently, $\mu$ in the definition of $\mathcal{Z}-$tensor is a constant. Then from (5) using (14) we get

$$\begin{aligned}(\nabla_W S)(U, V) &= da(W)g(U, V) + da(W)B(U)B(V)\\ &+(a+\mu)[(\nabla_W B)UB(V) + B(U)(\nabla_W B)V].\end{aligned} \tag{15}$$

From (9) with the help of (13) and (14) and by hypothesis $r = constant$, it follows that

$$\begin{aligned}(\nabla_W S)(U, V) = &-2B(W)A(\rho)(a+\mu)[g(U, V) + B(B(V)]\\ &-B(U)A(\rho)(a+\mu)[g(W, V) + B(U)B(V)]\\ &-B(V)A(\rho)(a+\mu)[g(U, W) + B(U)B(W)].\end{aligned} \tag{16}$$

Therefore equation (15) and (16) yield

$$\begin{aligned}(Ua)\{g(U, V) + B(U)B(V)\} + (a+\mu)\{(\nabla_W B)UB(V) + B(U)(\nabla_W B)V\}\\ = -2B(W)A(\rho)(a+\mu)[g(U, V) + B(U)B(V)]\\ -B(U)A(\rho)(a+\mu)[g(U, V) + B(U)B(V)]\\ -B(V)A(\rho)(a+\mu)[g(U, W) + B(U)B(W)].\end{aligned}$$

Substituting $V = \tilde{\rho}$ in the above equation, where $\tilde{\rho}$ is given by $B(X) = g(X, \tilde{\rho})$ for all $X$, we obtain

$$(\nabla_U B)U = -A(\rho)[g(U, U) + B(W)B(U)], \quad since \;\; B(\tilde{\rho}) = -1,$$

which implies by Theorem **A** that a perfect fluid pseudo $\mathcal{Z}$ symmetric spacetime is a $GRW$ spacetime.

This completes the proof.

If $\mu = 0$, then a $(P\mathcal{Z}S)_n$ reduces to a pseudo Ricci symmetric spacetime $(PRS)_n$. Thus we have the corollary 1.

**Proof of Theorem 2.** We assume that a $(P\mathcal{Z}S)_n$ spacetime is $GRW$ spacetime. Then by Chen's Theorem we assume that the vector field associated to the 1-form $A$ of $(P\mathcal{Z}S)_n$ is a timelike concircular vector field, that is, $\nabla_X \rho = \alpha X$ for some scalar function $\alpha$. In [16] the authors obtain the following results for a $GRW$ spacetime:

1) $S(X, \rho) = \xi A(X)$, $\xi = -(n-1)\frac{\nabla_\rho \alpha}{\rho^2}; \nabla_X \xi = A(X)\theta$, where $\theta$ is a scalar function.

2)

$$(\nabla_U S)(V, \rho) = -\alpha S(U, \rho) + \theta A(U)A(V) + \alpha\xi g(U, V). \tag{17}$$

For a $(P\mathcal{Z}S)_n$ with $r = constant$, from (9) we get by replacing $W = \rho$.

$$(\nabla_U S)(V, \rho) = 2A(U)[S(V, \rho) + \mu g(V, \rho)] + A(V)[S(U, \rho) + \mu g(U, \rho)] + A(\rho)[S(V, U) + \mu g(V, U)]. \tag{18}$$

In a $(P\mathcal{Z}S)_n$ from (11) we have for $r = constant$

$$S(U, \rho) = -\mu A(U). \tag{19}$$

Using (17) and (19) in (18) we have

$$A(\rho)S(U, V) = -\alpha S(U, V) + \theta A(U)A(V) + \alpha\xi g(U, V) - \mu A(\rho)g(U, V),$$

which implies

$$S(U, V) = \frac{\alpha\xi - \mu A(\rho)}{\alpha + A(\rho)} g(U, V) + \frac{\theta}{\alpha + A(\rho)} A(U)A(V).$$

The above equation shows that the spacetime is a quasi-Einstein spacetime, provided $\alpha + A(\rho) \neq 0$. Since the vector field $\rho$ is not a unit timelike vector field, therefore the spacetime under consideration does not represent perfect fluid spacetime.

This completes the proof.

For $\mu = 0$ the above equation turns into

$$S(U, V) = \frac{\alpha\xi}{\alpha + A(\rho)} g(U, V) + \frac{\theta}{\alpha + A(\rho)} A(U)A(V),$$

which implies that the spacetime is a quasi Einstein spacetime. Thus we have the Corollary 2.

**Proof of Theorem 3.** In a semi-Riemannian manifold the Ricci tensor $S$ of type $(0, 2)$ is said to be of Codazzi type if it is non zero and satisfies the condition

$$(\nabla_W S)(U, V) - (\nabla_V S)(U, W) = 0. \tag{20}$$

Corresponding to the definition in (20) we can destine Codazzi type of $\mathcal{Z}$ tensor as follows:

The $\mathcal{Z}$ tensor is said to be of Codazzi type if it is non zero and satisfies the condition

$$(\nabla_W \mathcal{Z})(U, V) - (\nabla_U \mathcal{Z})(U, V) = 0. \tag{21}$$

From (2) and (3) we infer

$$(\nabla_W \mathcal{Z})(U, V) - (\nabla_V \mathcal{Z})(U, W) = A(W)\mathcal{Z}(U, V) - A(V)\mathcal{Z}(U, W). \tag{22}$$

Since by hypothesis $\mathcal{Z}$ is of Codazzi type, hence the above equation entails that

$$A(W)\mathcal{Z}(U,V) = A(V)\mathcal{Z}(U,V). \tag{23}$$

Replacing $W$ by $\rho$ in the foregoing equation gives

$$A(\rho)\mathcal{Z}(U,V) = A(W)\mathcal{Z}(U,\rho). \tag{24}$$

Substituting $U = V = e_i$ in (23) and taking $1 \leq i \leq n$, we obtain

$$A(W)Z = \mathcal{Z}(W,\rho), \tag{25}$$

where $Z = \Sigma_{i=1}^{n}\mathcal{Z}(e_i,e_i)$.
From (24) and (25) we get

$$A(\rho)\mathcal{Z}(U,V) = A(U)A(V)Z,$$

which implies that

$$\mathcal{Z}(U,V) = \frac{A(U)A(V)}{A(\rho)}. \tag{26}$$

Equations (2) and (26) together yield

$$S(U,V) = -\mu g(U,V) + \frac{Z}{A(\rho)}A(U)A(V),$$

where $\mu$ and $\frac{Z}{A(\rho)}$ are scalars.
This finishes the proof.

## 6. Discussions

Spacetime is the stage of current modelling of the physical world: a torsion-less, time-oriented Lorentzian manifold [13]. Developments in theoretical physics require a deeper understanding of the geometrical structure. In the present paper some results on spacetime with pseudo symmetric structure for $\mathcal{Z}$ tensor have been obtained. At first the relation between the associated $1-$forms and associated scalars of a $(P\mathcal{Z}S)_4$ space time with constant scalar curvature are obtained. Next it is proven that a perfect fluid $(P\mathcal{Z}S)_n$ spacetime with constant scalar curvature is a GRW spacetime. Conversely, it is shown that under the same restriction on the scalar curvature a $P\mathcal{Z}S_n$ GRW spacetime is a quasi Einstein spacetime. The importance of this study is to obtain relations between $(P\mathcal{Z}S)_n$ spacetime and GRW spacetime.

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