Filomat 36:14 (2022), 4717–4727
https://doi.org/10.2298/FIL2214717L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence Results for Nonlinear Sequential Caputo and Caputo-Hadamard Fractional Differential Equations with Three-Point Boundary Conditions in Banach Spaces

Adel Lachouri[a], Abdelouaheb Ardjouni[b], Ahcene Djoudi[a]

[a] *Applied Mathematics Lab, Faculty of Sciences, Department of Mathematics, University of Annaba, P.O. Box 12, Annaba 23000, Algeria*
[b] *Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria*

Abstract. In this paper, we study the existence of solutions for nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with three-point boundary conditions by using measure of noncompactness combined with fixed point theorem of Mönch. An example illustrating the effectiveness of the theoretical results is presented.

1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of fractional differential equations have received the attention of many authors, see [1], [4]-[10], [13]-[18], [20]-[21], [23]-[28], [31]-[33], [35]-[40] and the references therein.

Measure of non compactness combined with one of fixed point theorems, as Darbo [19] Sadovski [34], Mönch [30] is an important and efficacy tool in study of differential or integral equations.

Kuratowski [29] introduced the concept of measure of noncompactness, which played an important role in fixed point theory, Gohberg [22] gave an other measure called Hausdorff measure later Darbo [19] used Kuratowski's measure of noncompactness to generalize the Schauder's theorem of fixed point. After, that many authors studied and solved some problems by using measure of noncompactness in study of different kind problems, as differential equations, integral equations and integro-differential equations, see [1, 12, 13, 24, 36].

In [38], the authors studied the existence and uniqueness of solutions for two sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations subject to separated boundary conditions as

$$\begin{cases} {}^{C}D^{\alpha}\left[{}^{H}\mathfrak{D}^{\beta}x(t)\right] = f\left(t, x\left(t\right)\right), \; t \in \left(a, b\right), \\ a_1 x(a) + b_1 {}^{H}\mathfrak{D}^{\beta}x(a) = 0, \\ a_2 x(b) + b_2 {}^{H}\mathfrak{D}^{\beta}x(b) = 0, \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 34A08; Secondary 34B15, 47H08
Keywords. Fractional differential equations, nonlinear equations, Caputo and Caputo-Hadamard fractional derivative, measure of noncompactness, Mönch fixed point theorem, existence.
Received: 29 July 2021; Revised: 22 September 2021; Accepted: 23 September 2021
Communicated by Maria Alessandra Ragusa
Email addresses: lachouri.adel@yahoo.fr (Adel Lachouri), abd_ardjouni@yahoo.fr (Abdelouaheb Ardjouni), adjoudi@yahoo.com (Ahcene Djoudi)

and

$$\begin{cases} {}^{H}\mathfrak{D}^{\beta}\left[{}^{C}D^{\alpha}x(t)\right] = f\left(t, x\left(t\right)\right), \ t \in (a,b), \\ a_1 x(a) + b_1 {}^{C}D^{\alpha}x(a) = 0, \\ a_2 x(b) + b_2 {}^{C}D^{\alpha}x(b) = 0, \end{cases}$$

where ${}^{C}D^{\alpha}$ and ${}^{H}\mathfrak{D}^{\beta}$ are the Caputo and Hadamard fractional derivatives of orders α and β, respectively, $0 < \alpha, \beta \leq 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$ and $a_i, b_i \in \mathbb{R}$, $i = 1, 2$.

In [18] Boutiara et al, discussed the existence of solutions of the following fractional-order differential equations with three-point boundary conditions

$$\begin{cases} {}^{C}_{H}\mathfrak{D}^{\alpha}x\left(t\right) = f\left(t, x\left(t\right)\right), \ t \in (1,T), \\ ax(1) + bx\left(T\right) = \lambda \mathfrak{I}^{q} x\left(\eta\right) + \delta, \end{cases}$$

where ${}^{C}_{H}\mathfrak{D}^{\alpha}$ and $\mathfrak{I}^{q}$ are the Caputo-Hadamard fractional derivative and Hadamard fractional integral of order α and q, respectively, $0 < \alpha, q \leq 1$, $f : [1,T] \times E \to E$ is a given continuous function, E is a Banach space, $a, b, \lambda \in \mathbb{R}$ and $\eta \in (1,T)$.

In [20], Derbazi studied the existence of solutions for nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with Dirichlet boundary conditions as

$$\begin{cases} {}^{C}D^{\beta}\left[{}^{C}_{H}\mathfrak{D}^{\alpha}x\left(t\right)\right] = f\left(t, x\left(t\right)\right), \ t \in (a,b), \ a \geq 1, \\ x\left(a\right) = x\left(b\right) = 0, \end{cases}$$

where ${}^{C}_{H}\mathfrak{D}^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo-Hadamard and Caputo fractional derivatives of orders α and β respectively, $0 < \alpha, \beta \leq 1$, $f : [a,b] \times E \to E$ is a given continuous function, E is a Banach space with the norm $\|.\|$.

Motivated by the above works, we study the existence of solutions for nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with three-point boundary conditions as

$$\begin{cases} {}^{C}D^{\beta}\left[{}^{C}_{H}\mathfrak{D}^{\alpha}x\left(t\right)\right] = f\left(t, x\left(t\right)\right), \ t \in (a,b), \ a \geq 1, \\ x\left(a\right) = 0, \ x\left(b\right) = \lambda x\left(\eta\right), \ a < \eta < b, \end{cases} \tag{1}$$

where ${}^{C}_{H}\mathfrak{D}^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo-Hadamard and Caputo fractional derivatives of orders α and β respectively, $0 < \alpha, \beta \leq 1$, $f : [a,b] \times E \to E$ is a given continuous function satisfying some assumptions that will be specified later, and E be a Banach space with the norm $\|.\|$.

2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus which are used throughout this paper.

Let $J = [a,b]$. By $C(J,E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|x\|_{\infty} = \sup\left\{\|x\left(t\right)\| : t \in J\right\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $x : J \to E$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_J \|x\left(t\right)\| \, dt.$$

And $AC(J,E)$ is the space of absolutely continuous valued functions on J, and set

$$AC^n\left(J\right) = \left\{x : J \to \mathbb{R} : x, x', x'', \quad , x^{n-1} \in C(J,E) \text{ and } x^{n-1} \in AC(J,E)\right\}.$$

Now, we give some results and properties of fractional calculus.

Definition 2.1 ([28]). *The fractional integral of order $\alpha > 0$ of a function $x : J \to E$ is given by*

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided the right side is pointwise defined on J, where Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt.$$

Definition 2.2 ([28]). *The Caputo fractional derivative of order $\alpha > 0$ of a function $x : J \to E$ is given by*

$$^{C}D^{\alpha}x(t) = D^{\alpha}\left[x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k\right],$$

where

$$n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}_0,\ n = \alpha \text{ for } \alpha \in \mathbb{N}_0, \tag{2}$$

and $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α defined by

$$D^{\alpha}x(t) = D^{n}I^{n-\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds.$$

The Caputo fractional derivative $^{C}D^{\alpha}$ exists for x belonging to $AC^{n}(J, \mathbb{R})$. In this case, it is defined by

$$^{C}D^{\alpha}x(t) = I^{n-\alpha}x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds.$$

Remark that when $\alpha = n$, we have $^{C}D^{\alpha}x(t) = x^{(n)}(t)$.

Lemma 2.3 ([28]). *Let $\alpha > 0$ and let n be given by (2). If $x \in AC^{n}(J, E)$, then*

$$\left(I^{\alpha\,C}D^{\alpha}x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k,$$

where $x^{(k)}$ is the usual derivative of x of order k.

Lemma 2.4 ([28]). *For $\alpha > 0$ and n be given by (2), the general solution of the fractional differential equation $^{C}D^{\alpha}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$.

From the above lemma, it follows that

$$I^{\alpha\,C}D^{\alpha}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$.

Definition 2.5 ([28]). *The Hadamard fractional integral of order $\alpha > 0$ for a function $x \in L^1(J, E)$ is defined as*

$$^{H}\mathfrak{I}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s},\ \alpha > 0.$$

Set $\delta = \left(t\frac{d}{dt}\right)$, $\alpha > 0$, $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. Define the space

$$AC_{\delta}^{n}(J) = \left\{x : J \to E : \delta^{n-1}x \in AC(J, E)\right\}.$$

Definition 2.6 ([28]). *The Hadamard fractional derivative of order $\alpha > 0$ for a function $x \in AC_{\delta}^{n}(J)$ is defined as*

$$^{H}\mathfrak{D}^{\alpha}x(t) = \delta^{n}\left(^{H}\mathfrak{I}^{n-\alpha}x\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^{n}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}x(s)\frac{ds}{s}.$$

Definition 2.7 ([25]). *The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a function $x \in AC_{\delta}^{n}(J)$ is defined as*

$$^{C}_{H}\mathfrak{D}^{\alpha}x(t) = \left(^{H}\mathfrak{I}^{n-\alpha}\delta^{n}x\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^{n}x(s)\frac{ds}{s}.$$

Lemma 2.8 ([25]). *Let $\alpha > 0$ and $n = [\alpha] + 1$. If $x \in AC_{\delta}^{n}(J)$, then the Caputo-Hadamard fractional differential equation*

$$^{C}_{H}\mathfrak{D}^{\alpha}x(t) = 0,$$

has a solution

$$x(t) = \sum_{k=0}^{n-1}c_{k}\left(\log\frac{t}{a}\right)^{k},$$

and the following formula holds

$$^{H}\mathfrak{I}^{\alpha}\left(^{C}_{H}\mathfrak{D}^{\alpha}x\right)(t) = x(t) + \sum_{k=0}^{n-1}c_{k}\left(\log\frac{t}{a}\right)^{k},$$

where $c_k \in \mathbb{R}$, $k = 1, 2, ..., n-1$.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.9 ([3, 11]). *Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0, \infty)$ defined by*

$$\mu(B) = \inf\left\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n}B_i \text{ and } diam(B_i) \leq \epsilon\right\}, \text{ here } B \in \Omega_E.$$

The measure of noncompactness satisfies some important properties
(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact),
(b) $\mu(B) = \mu\left(\overline{B}\right)$,
(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
(d) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(e) $\mu(cB) = |c|\,\mu(B)$, $c \in \mathbb{R}$,
(f) $\mu(convB) = \mu(B)$.
Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively. The details of μ and its properties can be found in [3, 11].

Definition 2.10. *A map $f : J \times E \to E$ is said to be Caratheodory if*
(i) $t \to f(t, x)$ is measurable for each $x \in E$.
(ii) $x \to f(t, x)$ is continuous for almost all $t \in J$.

Notation 2.11. *For a given set V of function $v : J \to E$, let us denote by*

$$V(t) = \{v(t) : v \in V\}, \; t \in J,$$

and

$$V(J) = \{v(t) : v \in V, \; t \in J\}.$$

Let us now recall Mönch fixed point theorem and an useful lemma.

Theorem 2.12 ([2, 30]). *Let D be a bounded, closed and convex subset of the Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every V of D, then N has a fixed point.

Lemma 2.13 ([37]). *Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$. Let G be a continuous function on $J \times J$ and f a function from $J \times E \to E$, which satisfies the Caratheodory conditions, and assume there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \mu(f(J_{t,h} \times B)) \leq p(t)\, \mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t)\, f(s, y(s))\, ds : y \in V\right\}\right) \leq \int_J \|G(s,t)\|\, p(s)\, \mu(V(s))\, ds.$$

3. Existence results

Let us start by defining what we mean by a solution of the problem (1).

Definition 3.1. *A function $x \in AC^2(J, E)$ is said to be a solution of problem (1) if x satisfies the equation ${}^C D^\beta \left[{}^C_H \mathfrak{D}^\alpha x(t)\right] = f(t, x(t))$ on J and the conditions $x(a) = 0$, $x(b) = \lambda x(\eta)$, $a < \eta < b$.*

For the existence of solutions for the problem (1), we need the following auxiliary lemma.

Lemma 3.2. *Let $\Lambda = \left(\lambda \frac{\left(\log \frac{\eta}{a}\right)^\alpha}{\Gamma(\alpha+1)} - \frac{\left(\log \frac{b}{a}\right)^\alpha}{\Gamma(\alpha+1)}\right) \neq 0$. For any $q \in C(J, E)$, the unique solution of the boundary value problem*

$$\begin{cases} {}^C D^\beta \left[{}^C_H \mathfrak{D}^\alpha x(t)\right] = q(t), \; t \in (a, b), \\ x(a) = 0, \; x(b) = \lambda x(\eta), \; a < \eta < b, \end{cases} \tag{3}$$

is given by

$$\begin{aligned} x(t) &= {}^H \mathfrak{I}^\alpha \left(I^\beta q\right)(t) + \frac{\left(\log \frac{t}{a}\right)^\alpha}{\Gamma(\beta+1)\Lambda} \left({}^H \mathfrak{I}^\alpha \left(I^\beta q\right)(b) - \lambda\, {}^H \mathfrak{I}^\alpha \left(I^\beta q\right)(\eta)\right) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} q(\sigma)\, d\sigma\right) \frac{ds}{s} \\ &+ \frac{\left(\log \frac{t}{a}\right)^\alpha}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} q(\sigma)\, d\sigma\right) \frac{ds}{s}\right. \\ &\left. -\lambda \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} q(\sigma)\, d\sigma\right) \frac{ds}{s}\right). \end{aligned} \tag{4}$$

Proof. Taking the Riemann-Liouville fractional integral of order β to the first equation of (3), we get

$$ {}_{H}^{C}\mathfrak{D}^{\alpha}x(t) = I^{\beta}q(t) + c_0. \tag{5}$$

Again taking the Hadamard fractional integral of order α to the above equation, we obtain

$$x(t) = {}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(t) + \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}c_0 + c_1. \tag{6}$$

Substituting $t = a$ in (5) and applying the first boundary condition of (3), it follows that $c_1 = 0$. For $t = b$ in (5) we get

$$x(b) = {}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(b) + \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}c_0,$$

and for $t = \eta$, we have

$$x(\eta) = {}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(\eta) + \frac{\left(\log\frac{\eta}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}c_0.$$

Using the second boundary condition of (3), we have

$${}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(b) + \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}c_0 = \lambda {}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(\eta) + \lambda\frac{\left(\log\frac{\eta}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}c_0. \tag{7}$$

By solving (7), we find that

$$\begin{aligned} c_0 &= \frac{1}{\left(\lambda\frac{\left(\log\frac{\eta}{a}\right)^{\alpha}}{\Gamma(\alpha+1)} - \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)}\right)}\left({}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(b) - \lambda {}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(\eta)\right) \\ &= \frac{1}{\Lambda}\left({}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(b) - \lambda {}^{H}\mathfrak{J}^{\alpha}\left(I^{\beta}q\right)(\eta)\right). \end{aligned}$$

Replacing the values of c_0 and c_1 into (6), we get the integral equation (4). The converse follows by direct computation which completes the proof. □

In the following we prove existence results for the boundary value problem (1) by using a Mönch of fixed point theorems.

The following assumptions will be used in our main results

(H1) The functions $f : J \times E \to E$ satisfy the Caratheodory conditions.

(H2) There exists $p_f \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$ such that

$$\left\|f(t,x)\right\| \le p_f(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E.$$

(H3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h\to 0^+}\mu\left(f\left(J_{t,h}\times B\right)\right) \le p_f(t)\,\mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J.$$

Theorem 3.3. *Assume that $\Lambda \neq 0$ and the assumptions (H1)-(H3) hold. Let $p^* = \sup_{t\in J} p_f(t)$. If*

$$\frac{p^*(b-a)^{\beta}\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)}\left[1 + \frac{\left(\log\frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right] < 1, \tag{8}$$

then the boundary value problem (1) has at least one solution.

Proof. We transform the problem (1) into a fixed point problem by defining an operator $N : C(J,E) \to C(J,E)$ as

$$
\begin{aligned}
(Nx)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} f(\sigma, x(\sigma))\, d\sigma \right) \frac{ds}{s} \\
&+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)\Lambda} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} f(\sigma, x(\sigma))\, d\sigma \right) \frac{ds}{s} \right. \\
&\left. - \frac{\lambda}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} f(\sigma, x(\sigma))\, d\sigma \right) \frac{ds}{s} \right).
\end{aligned}
$$

By Lemma 3.2, the fixed points of operator N are solutions of the problem (1). Let $R > 0$ and consider the subset

$$
D_R = \{ x \in C(J,E) : \|x\|_\infty \le R \}.
$$

Clearly, the subset D_R is closed, bounded, and convex. We will show that N satisfies the assumptions of Theorem 2.12. The proof will be given in three steps.

Step 1. N maps D_R into itself.

For each $x \in D_R$, by (H2) and (8) we have for each $t \in J$

$$
\begin{aligned}
\|(Nx)(t)\| &\le \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \\
&+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \right. \\
&\left. + \frac{|\lambda|}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \right) \\
&\le \frac{p^* R}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} d\sigma \right) \frac{ds}{s} \\
&+ \frac{\left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \left(\frac{p^* R}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} d\sigma \right) \frac{ds}{s} \right. \\
&\left. + \frac{|\lambda| p^* R}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} d\sigma \right) \frac{ds}{s} \right).
\end{aligned}
$$

Also, note that

$$
\int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} d\sigma \right) \frac{ds}{s} \le \frac{(b-a)^{\beta} \left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)},
$$

where we have used the fact that $(s-a)^{\beta} \le (b-a)^{\beta}$ for $0 < \beta \le 1$. Using the above arguments, we have

$$
\begin{aligned}
\|(Nx)(t)\| &\le \frac{p^* R (b-a)^{\beta} \left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} \left[1 + \frac{\left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} (1 + |\lambda|) \right] \\
&\le R.
\end{aligned}
$$

Step 2. $N(D_R)$ is bounded and equicontinuous.

By Step 1, we have $N(D_R) = \{Nx : x \in D_R\} \subset D_R$. Thus, for each $x \in D_R$, we have $\|Nx\|_\infty \le R$, which means that $N(D_R)$ is bounded.

For the equicontinuity of $N(D_R)$. Let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in D_R$. Then

$$\begin{aligned}
&\|(Nx)(t_2) - (Nx)(t_1)\| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right) \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \\
&+ \frac{\left(\log \frac{t_2}{a}\right)^{\alpha} - \left(\log \frac{t_1}{a}\right)^{\alpha}}{|\Lambda| \Gamma(\alpha+1)\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \right. \\
&\left. + |\lambda| \int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \right) \\
&\leq \frac{p^* R (b-a)^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha)} \left[\int_a^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right) \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \right] \\
&+ \frac{\left(\log \frac{t_2}{a}\right)^{\alpha} - \left(\log \frac{t_1}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \frac{p^* R (b-a)^{\beta} \left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} (1+|\lambda|) \\
&\leq \frac{p^* R (b-a)^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+1)} \left(2\left(\log \frac{t_2}{t_1}\right)^{\alpha} \right) + \frac{\left(\log \frac{t_2}{a}\right)^{\alpha} - \left(\log \frac{t_1}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|} \frac{p^* R (b-a)^{\beta} \left(\log \frac{b}{a}\right)^{\alpha}}{\Gamma(\beta+1)\Gamma(\alpha+1)} (1+|\lambda|).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero and the convergence is independent of $x \in D_R$. Hence, we conclude that $N(D_R)$ is equicontinuous.

Step 3. N is continuous.

Let $\{x_n\}$ be sequence such that $x_n \to x$ in $C(J, E)$. Then, for each $t \in J$, we have

$$\begin{aligned}
&\|(Nx_n)(t) - (Nx)(t)\| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x_n(\sigma)) - f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \\
&+ \frac{\left(\log \frac{t}{a}\right)^{\alpha}}{|\Lambda| \Gamma(\alpha+1)\Gamma(\alpha)\Gamma(\beta)} \left(\int_a^b \left(\log \frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x_n(\sigma)) - f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \right. \\
&\left. + |\lambda| \int_a^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} \left\| f(\sigma, x_n(\sigma)) - f(\sigma, x(\sigma)) \right\| d\sigma \right) \frac{ds}{s} \right).
\end{aligned}$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem, we have

$$\|(Nx_n)(t) - (Nx)(t)\|_{\infty} \to 0 \text{ as } n \to \infty.$$

This shows that (Nx_n) converges pointwise to Nx on J. Moreover, the sequence (Nx_n) is equicontinuous by a similar proof of Step 2. Therefore (Nx_n) converges uniformly to Nx and hence N is continuous.

Now let V be a subset of D_R such that $V \subset \overline{conv}((NV) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $v \to v(t) = \mu(V(t))$ is continuous on J. By assumption (H3), Lemma 2.13 and the properties of the measure μ we have for each $t \in J$

$$\begin{aligned}
v(t) &\leq \mu((NV)(t) \cup \{0\}) \leq \mu((NV)(t)) \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} p_f(\sigma)\,\mu(V(\sigma))\,d\sigma\right)\frac{ds}{s} \\
&+ \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\Gamma(\alpha+1)|\Lambda|}\left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} p_f(\sigma)\,\mu(V(\sigma))\,d\sigma\right)\frac{ds}{s}\right. \\
&\left.+\frac{|\lambda|}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \left(\log\frac{b}{s}\right)^{\alpha-1} \left(\int_a^s (s-\sigma)^{\beta-1} p_f(\sigma)\,\mu(V(\sigma))\,d\sigma\right)\frac{ds}{s}\right) \\
&\leq \|v\|_\infty \frac{p^*(b-a)^\beta\left(\log\frac{b}{a}\right)^\alpha}{\Gamma(\beta+1)\Gamma(\alpha+1)}\left[1+\frac{\left(\log\frac{b}{a}\right)^\alpha}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right].
\end{aligned}$$

This means that

$$\|v\|_\infty\left(1-\frac{p^*(b-a)^\beta\left(\log\frac{b}{a}\right)^\alpha}{\Gamma(\beta+1)\Gamma(\alpha+1)}\left[1+\frac{\left(\log\frac{b}{a}\right)^\alpha}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right]\right) \leq 0.$$

By (8), it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 2.12, we conclude that N has a fixed point, which is a solution of the problem (1) □

4. Example

As an application of our results, we consider the following boundary value problem of a fractional differential equation

$$\begin{cases} {}^{C}D^{\frac{1}{2}}\left[{}^{C}_{H}\mathfrak{D}^{\frac{2}{3}}x(t)\right] = \frac{1}{2t^2+\exp(t^2-1)}x(t),\ t \in (1,2), \\ x(1) = 0,\ x(2) = \frac{1}{10}x\left(\frac{3}{2}\right). \end{cases} \tag{9}$$

Here $a = 1$, $b = 2$, $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$, $\lambda = \frac{1}{10}$ and $\eta = \frac{3}{2}$. With these date we find $\Lambda = -0.80691 \neq 0$. Let

$$E = l^1 = \left\{x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty\right\},$$

equipped with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

Set

$$x = (x_1, x_2, ..., x_n, ...),\ f = (f_1, f_2, ..., f_n, ...),\ f_n(t, x_n) = \frac{1}{2t^2+\exp(t^2-1)}x_n,\ t \in J.$$

For each x_n and $t \in J$, we have

$$\left|f_n(t, x_n)\right| \leq \frac{1}{2t^2+\exp(t^2-1)}|x_n|. \tag{10}$$

Hence conditions (H1) and (H2) are satisfied with $p_f(t) = \frac{1}{2t^2+\exp(t^2-1)}$. By (10) and for any bounded set $B \subset l^1$, we have

$$\mu(f(t,B)) \leq \frac{1}{2t^2+\exp(t^2-1)}\mu(B) \text{ for each } t \in J.$$

Hence (H3) is satisfied. The condition

$$\frac{p^*(b-a)^\beta \left(\log\frac{b}{a}\right)^\alpha}{\Gamma(\beta+1)\Gamma(\alpha+1)}\left[1+\frac{\left[\log\left(\frac{b}{a}\right)\right]^\alpha}{\Gamma(\alpha+1)|\Lambda|}(1+|\lambda|)\right] \simeq 0.72 < 1,$$

is satisfied with $p^* = \sup_{t\in J} p_f(t) = \frac{1}{3}$. Consequently, Theorem 3.3 implies that the problem (9) has a solution defined on J.

Acknowledgements. The authors are grateful to the referees for their valuable comments which have led to improvement of the presentation.

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