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Riesz Fractional Integral Inequalities for Convex Stochastic Processes

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Abstract. The aim of this paper is to establish some integral inequalities for convex stochastic processes in a form of Riesz fractional integrals. These results allow us to obtain a new class of functional inequalities which generalizes known ones.

1. Introduction

In the stochastic context, a stochastic process is a temporal parameterized family of random variables on a probability space. In other words, if $(\Omega, \mathcal{F}, P)$ be a probability space, a function $X : \Omega \to \mathbb{R}$ is called a random variable, if it is $\mathcal{F}$-measurable. Correspondingly, $X : [a, b] \times \Omega \to \mathbb{R}$ is called a stochastic process, if the function $X(t, .)$ is a random variable for all $t \in [a, b]$.
Let P-lim and $E[X(t, .)]$ denote the limit in probability and the expectation value of random variable $X(t, .)$ respectively. Then, a stochastic process $X : [a, b] \times \Omega \to \mathbb{R}$ is

(i) continuous in probability in $[a, b]$, if for all $t_0 \in [a, b]$,

$$P - \lim_{t \mapsto t_0} X(t, .) = X(t_0, .).$$

(ii) mean-square continuous in $[a, b]$, if for all $t_0 \in [a, b]$

$$\lim_{t \mapsto t_0} E[(X(t, .) - X(t_0, .))^2] = 0.$$

(iii) increasing (resp. decreasing) if for all $t_1, t_2 \in [a, b]$ such that $t_1 < t_2$ (resp. $t_1 > t_2$), $X(t_1, .) \leq X(t_2, .)$ (resp. $X(t_1,) \geq X(t_2, .)$), (a.e).

(iv) monotonic if it is increasing or decreasing.

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(v) mean-square differentiable at a point $t \in [a,b]$, if there is a random variable $X^{'}(t,.) : [a,b] \times \Omega \to \mathbb{R}$ such that

$$X^{'}(t,.) = P - \lim_{t \mapsto t_0} \frac{X(t,.) - X(t_0,.)}{t - t_0}.$$

Note that if the stochastic process $X : [a,b] \times \Omega \to \mathbb{R}$ has mean-square continuity, then it has continuity in probability, but the converse is not true.

(vi) mean-square integrable on $[a,b]$ if

$$\lim_{n \mapsto \infty} E\Big[\Big(\sum_{k=1}^{n} X(T_k,.) \cdot (t_k - t_{k-1}) - Y(t,.)\Big)^2\Big] = 0,$$

with $E[X(t)^2] < \infty$ for all $t \in [a,b]$ and $T_k \in [t_k - 1, t_k]$, $k = 1,...,n$ and $a = t_0 < t_1 < ... < t_n = b$ is a partition of $[a,b]$. Then almost everywhere, it can be sometimes showed with

$$\int_a^t X(s,.)ds = Y(t,.).$$

In 1980, Nikodem [1] introduced the notion of convex stochastic processes and proposed the following definition: a stochastic process $X : [a,b] \times \Omega \to \mathbb{R}$ is said to be convex if

$$X(\lambda x + (1-\lambda)y,.) \le \lambda X(x,.) + (1-\lambda)X(y,.),$$

holds almost everywhere for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. For example the stochastic process defined by $X(t,.) = \theta(.)e^{\rho t}$ where $\theta(.)$ is a random variable, is convex since the exponential function is convex.
If we put $\lambda = \frac{1}{2}$ in the above inequality, then the process X is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process X is termed concave if $-X$ is convex.
In 1992, Skowronski [2] obtained some further results on convex stochastic processes which generalize some known properties of convex functions.
Integral inequalities play an important role in the theory of differential equations, functional analysis, linear programming, extreme and optimization problems. They are also useful to show uniqueness of solutions for differential equations, and to estimate integral meanings of real valued functions. For a stochastic process, several integral inequalities with convexity exist in the literature, the most important and well known ones are:
• The Hermite-Hadamard inequality [3, 4]: If $X : [a,b] \times \Omega \to \mathbb{R}$ be a convex and mean square continuous stochastic process, then

$$X\Big(\frac{a+b}{2},.\Big) \le \frac{1}{b-a}\int_a^b X(t,.)dt \le \frac{X(a,.) + X(b,.)}{2}, \quad (a.e). \tag{1}$$

• The Hermite-Hadamard-Fejér inequality [5]: If $X : [a,b] \times \Omega \to \mathbb{R}$ be a convex stochastic process and $Y : [a,b] \times \Omega \to \mathbb{R}$ is a non-negative and mean square integrable stochastic process, symmetric with respect to $\frac{a+b}{2}$, then

$$X\Big(\frac{a+b}{2},.\Big)\int_a^b Y(t,.)dt \le \int_a^b (XY)(t,.)dt \le \frac{X(a,.) + X(b,.)}{2}\int_a^b Y(t,.)dt \quad (a.e). \tag{2}$$

• The Gonzalez-Merentes-Lopez's first inequality [6]: If $X : [a,b] \times \Omega \to \mathbb{R}$ is a mean-square differentiable stochastic process on $[a,b]$ with $|X^{'}|$ is convex on $[a,b]$, then

$$\Big|\frac{X(a,.) + X(b,.)}{2} - \frac{1}{b-a}\int_a^b X(t,.)dt\Big| \le \frac{b-a}{8}\Big[|X^{'}(a,.)| + |X^{'}(b,.)|\Big], \quad (a.e). \tag{3}$$

• The Gonzalez-Merentes-Lopez's second inequality [6]: Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a mean-square differentiable stochastic process and $X(t,.) \in L^1[a,b]$. If $|X|^q$ is convex on $[a,b]$, $p > 1$ such that $q = \frac{p}{p-1}$, then

$$\left|\frac{X(a,.)+X(b,.)}{2} - \frac{1}{b-a}\int_a^b X(t,.)dt\right| \le \frac{b-a}{2(1+p)^{\frac{1}{p}}}\left[\frac{|X'(a,.)|^q + |X'(b,.)|^q}{2}\right]^{\frac{1}{q}}, \quad (a.e). \tag{4}$$

In the last decades, fractional calculus has attracted the attention of many researchers in different areas of science, such as mathematical modelling, physics, biology, and engineering [7]. In the literature, the definition of fractional integral has been treated using different approaches such as Riemann-Liouville, Caputo, Weyl, among others [8].

Many authors have explored certain extensions and generalizations of these inequalities by involving different approaches of fractional operators [9–18]. In [19], Agahi et. al. represented Hermite-Hadamard inequalities in Riemann-Liouville fractional integral as follows:

If $X : [a,b] \times \Omega \to \mathbb{R}$ is a mean-square continuous convex stochastic process, then

$$X\Big(\frac{a+b}{2},.\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big[{}^{RL}\mathbb{J}^{\alpha}_{a+}X(b,.) + {}^{RL}\mathbb{J}^{\alpha}_{b-}X(a,.)\Big] \le \frac{X(a,.)+X(b,.)}{2}, \quad (a.e), \tag{5}$$

with ${}^{RL}\mathbb{J}^{\alpha}_{a+}X$ (resp. ${}^{RL}\mathbb{J}^{\alpha}_{b-}X$) is the left Riemann-Liouville integral (resp. right Riemann-Liouville integral) which are given by [20]

$$\begin{aligned} {}^{RL}\mathbb{J}^{\alpha}_{a+}X(t,.) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}X(s,.)ds, \\ {}^{RL}\mathbb{J}^{\alpha}_{b-}X(t,.) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}X(s,.)ds. \end{aligned}$$

By using the last definitions of fractional integration and the Hermite-Hadamard-Fejér inequality (2), we can obtain the generalized property: If $X : [a,b] \times \Omega \to \mathbb{R}$ is a convex stochastic process with $X(t,.) \in L^1[a,b]$ and $Y(t,.) \in L^1(a,b)$ be a non-negative and symmetric stochastic process with respect to $\frac{a+b}{2}$, then

$$\begin{aligned} X\Big(\frac{a+b}{2},.\Big)({}^{RL}\mathbb{J}^{\alpha}_{a+}Y(b,.) + {}^{RL}\mathbb{J}^{\alpha}_{b-}Y(a,.)) &\le {}^{RL}\mathbb{J}^{\alpha}_{a+}(XY)(b,.) + {}^{RL}\mathbb{J}^{\alpha}_{b-}(XY)(a,.) \\ &\le \frac{X(a,.)+X(b,.)}{2}({}^{RL}\mathbb{J}^{\alpha}_{a+}Y(b,.) + {}^{RL}\mathbb{J}^{\alpha}_{b-}Y(a,.)), \quad (a.e). \end{aligned} \tag{6}$$

Moreover, loads of papers appeared in the literature to generalize some inequalities of some fractional integrals, see for example [21–23].

The α-th order Riesz derivative [24] of a suitably mean square differentiable stochastic process X is given by

$${}^{RZ}D^{\alpha}_{[a,b]}X(t,.) = -\frac{1}{2\cos(\alpha\pi/2)}({}^{RL}D^{\alpha}_{a+} + {}^{RL}D^{\alpha}_{b-})X(t,.), \qquad 0 < \alpha < 1.$$

The α-th Riesz fractional integral (usually called Riesz potential) of a stochastic process $X(t,.) \in L^1_{loc}([a,b])$ be locally integrable is defined as the linear combination

$$\begin{aligned} {}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(t,.) &= \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)}\int_a^b \frac{X(s,.)}{|t-s|^{1-\alpha}}ds \\ &= \frac{1}{2\cos(\alpha\pi/2)}({}^{RL}\mathbb{J}^{\alpha}_{a+} + {}^{RL}\mathbb{J}^{\alpha}_{b-})X(t,.). \end{aligned}$$

We note that, at variance with the Riemann-Liouville integral, the Riesz fractional integral has the semigroup property only in restricted range, e.g.

$${}^{RZ}\mathbb{J}^{\alpha}_{[a,b]} \circ {}^{RZ}\mathbb{J}^{\beta}_{[a,b]} = {}^{RZ}\mathbb{J}^{\alpha+\beta}_{[a,b]} \quad \text{if} \quad \alpha+\beta < 1.$$

The aim of this paper is to establish generalization of these type integral inequalities by using Riesz fractional integral.

2. Main results

Here, we present our main results that are a generalization of (1), (2), (3) and (4) inequalities corresponding to Riesz fractional integral. Then we have the following theorems:

Theorem 2.1. *If X is a convex stochastic process on $[a,b]$, then*

$$^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(t,.) \leq \frac{1+\frac{X(b,.)-X(a,.)}{\Gamma(\alpha+2)}}{2\cos(\alpha\pi/2)}\Big\{(t-a)^{\alpha}+(b-t)^{\alpha}\Big\}, \quad (a.e). \tag{7}$$

Proof. The convexity of X means that, for all $t \in [a,b]$,

$$X(t,.) \leq X(a,.) + \frac{X(b,.)-X(a,.)}{b-a}(t-a), \quad (a.e).$$

By applying the Riesz fractional operator of integration on both sides of the preceding inequality and by using

$$^{RZ}\mathbb{J}^{\alpha}_{[a,b]}[C] = \frac{C}{2\Gamma(\alpha+1)\cos(\alpha\pi/2)}\Big\{(t-a)^{\alpha}+(b-t)^{\alpha}\Big\}, \quad \text{for} \quad C = cte,$$
$$^{RZ}\mathbb{J}^{\alpha}_{[a,b]}[t-a] = \frac{b-a}{2\Gamma(\alpha+2)\cos(\alpha\pi/2)}\Big\{(t-a)^{\alpha}+(b-t)^{\alpha}\Big\},$$

we find the result. □

Example 2.2. *We consider the convex stochastic process defined by $X(t,\omega) = t$, for $(t,\omega) \in [a,b]\times\mathbb{R}$, and such as*

$$^{RL}\mathbb{J}^{\alpha}_{a+}[t] = \frac{1}{\Gamma(\alpha+2)}(t+\alpha a)(t-a)^{\alpha},$$
$$^{RL}\mathbb{J}^{\alpha}_{b-}[t] = \frac{1}{\Gamma(\alpha+2)}(t+\alpha b)(b-t)^{\alpha},$$

and by applying (7) we get

$$(t+\alpha a)(t-a)^{\alpha} + (t+\alpha b)(b-t)^{\alpha} \leq \Big(b-a+\Gamma(\alpha+2)\Big)\Big\{(t-a)^{\alpha}+(b-t)^{\alpha}\Big\}. \tag{8}$$

Theorem 2.3. *Let $X \in L^1_{loc}([a,b])$. If X is convex on $[a,b]$, then the Hermite-Hadamard inequalities for Riesz fractional integral hold almost everywhere*

$$X\Big(\frac{a+b}{2},.\Big) \leq \frac{C(\alpha)}{(b-a)^{\alpha}}[^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(a,.) +^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(b,.)] \leq \frac{X(a,.)+X(b,.)}{2}, \tag{9}$$

where $C(\alpha) = \Gamma(\alpha+1)\cos(\alpha\pi/2)$.

Proof. Such as $^{RL}\mathbb{J}^{\alpha}_{a+}X(a,.) = {}^{RL}\mathbb{J}^{\alpha}_{b-}X(b,.) = 0$, then

$$^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(a,.) +^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(b,.) = \frac{1}{2\cos(\alpha\pi/2)}[^{RL}\mathbb{J}^{\alpha}_{a+}X(b,.) +^{RL}\mathbb{J}^{\alpha}_{b-}X(a,.)].$$

Using inequality (5) completes the proof. □

Remark 2.4. *If we let $\alpha \to 1$, (9) becomes (1).*

Theorem 2.5. *Let $X, Y : [a,b]\times\Omega \to \mathbb{R}$ be two convex stochastic processes with $X \in L^1_{loc}([a,b])$. If Y is non negative, integrable and symmetric with respect to $\frac{a+b}{2}$, then the Hermite-Hadamard-Fejér inequalities for Riesz fractional integral operator hold almost everywhere*

$$X\Big(\frac{a+b}{2},.\Big)\Big({}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}Y(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}Y(a,.)\Big) \leq{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}(XY)(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}(XY)(a,.)$$
$$\leq \frac{X(a,.)+X(b,.)}{2}\Big({}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}Y(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}Y(a,.)\Big). \tag{10}$$

Proof. For the right inequality, using (6) we have

$$\begin{aligned}{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}(XY)(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}(XY)(a,.) &= \frac{1}{2\cos(\alpha\pi/2)}\Big[{}^{RL}\mathbb{J}^{\alpha}_{a+}(XY)(b,.) +{}^{RL}\mathbb{J}^{\alpha}_{b-}(XY)(a,.)\Big]\\
&\leq \frac{X(a,.)+X(b,.)}{4\cos(\alpha\pi/2)}\Big[{}^{RL}\mathbb{J}^{\alpha}_{a+}Y(b,.) +{}^{RL}\mathbb{J}^{\alpha}_{b-}Y(a,.)\Big]\\
&\leq \frac{X(a,.)+X(b,.)}{2}\Big[{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}Y(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}Y(a,.)\Big].\end{aligned}$$

In the same way, using (6) we obtain the left inequality

$$\begin{aligned}{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}(XY)(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}(XY)(a,.) &\geq \frac{1}{2\cos(\alpha\pi/2)}X\Big(\frac{a+b}{2},.\Big)\Big[{}^{RL}\mathbb{J}^{\alpha}_{a+}Y(b,.) +{}^{RL}\mathbb{J}^{\alpha}_{b-}Y(a,.)\Big]\\
&= X\Big(\frac{a+b}{2},.\Big)\Big[{}^{RZ}\mathbb{J}^{\alpha}_{a+}Y(b,.) +{}^{RZ}\mathbb{J}^{\alpha}_{b-}Y(a,.)\Big].\end{aligned}$$

□

Remark 2.6. *If we let $\alpha \to 1$, (10) becomes (2).*

Lemma 2.7. *[25] Let $X : [a,b] \to \mathbb{R}$ be a mean-square differentiable stochastic process on $(a,b)\times\Omega$. If $X' \in L^1[a,b]$, then the following equality for fractional Riemann-Liouville integrals holds almost everywhere*

$$\frac{X(a,.)+X(b,.)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big[{}^{RL}\mathbb{J}^{\alpha}_{a+}X(b,.) +{}^{RL}\mathbb{J}^{\alpha}_{b-}X(a,.)\Big]$$
$$= \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]X'(ta+(1-t)b,.)dt. \tag{11}$$

Lemma 2.8. *Let $X : [a,b] \to \mathbb{R}$ be a mean-square differentiable stochastic process on $(a,b)\times\Omega$ with $X \in L^1_{loc}([a,b])$. If $X' \in L^1[a,b]$, then the following equality for Riesz fractional integral holds almost everywhere*

$$\begin{aligned}{}^{RZ}\Delta^{\alpha}_{[a,b]}(X) &= \frac{X(a,.)+X(b,.)}{2} - \frac{C(\alpha)}{(b-a)^{\alpha}}\Big[{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(a,.) +{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X(b,.)\Big]\\
&= \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]X'(ta+(1-t)b,.)dt.\end{aligned}$$

Proof. Using (2.7), the proof holds. □

Theorem 2.9. *Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a mean-square differentiable stochastic process on $(a,b) \times \Omega$ with $X(t,.) \in L^1_{loc}(a,b)$. If $|X'|$ is convex on $[a,b]$, then the following inequality for Riesz fractional integrals holds almost everywhere*

$$|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq D(\alpha)\Big(\frac{(b-a)\Big[|X'(a,.)| + |X'(b,.)|\Big]}{2}\Big), \tag{12}$$

where $D(\alpha) = \dfrac{1 - \left(\frac{1}{2}\right)^{\alpha}}{1+\alpha}$.

Proof. Using Lemma 2.7 and the convexity of $|X'|$, we have

$$|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq \frac{b-a}{2}\int_0^1 \Big|(1-t)^{\alpha} - t^{\alpha}\Big| \cdot \Big|X'(ta + (1-t)b,.)\Big| dt.$$

As $|X'|$ is convex, therefore

$$\begin{aligned}
|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq & \frac{b-a}{2}\Big[\int_0^{1/2}\Big((1-t)^{\alpha} - t^{\alpha}\Big)\Big(t|X'(a,.)| + (1-t)|X'(b,.)|\Big)dt \\
& + \int_{1/2}^{1}\Big(t^{\alpha} - (1-t)^{\alpha}\Big)\Big(t|X'(a,.)| + (1-t)|X'(b,.)|\Big)dt\Big] \\
\leq & \frac{b-a}{2}\Big[|X'(a,.)|\Big(\int_0^{1/2} t(1-t)^{\alpha}dt - \int_0^{1/2} t^{\alpha+1}dt\Big) \\
& + |X'(b,.)|\Big(\int_0^{1/2}(1-t)^{\alpha+1}dt - \int_0^{1/2}(1-t)t^{\alpha}dt\Big) \\
& + |X'(a,.)|\Big(\int_{1/2}^{1} t^{\alpha+1}dt - \int_{1/2}^{1} t(1-t)^{\alpha}dt\Big) \\
& + |X'(b,.)|\Big(\int_{1/2}^{1}(1-t)t^{\alpha}dt - \int_{1/2}^{1}(1-t)^{\alpha+1}dt\Big)\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq & \frac{b-a}{2}\Big[|X'(a,.)|\Big(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}\Big) + |X'(b,.)|\Big(\frac{1}{\alpha+2} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}\Big) \\
& + |X'(a,.)|\Big(\frac{1}{\alpha+2} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}\Big) + |X'(b,.)|\Big(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1}\Big)\Big] \\
\leq & \frac{b-a}{2}\Big[\Big(\frac{1}{(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+2} - \frac{\left(\frac{1}{2}\right)^{\alpha}}{\alpha+1}\Big)\Big[|X'(a,.)| + |X'(b,.)|\Big]\Big] \\
\leq & \Big(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{1+\alpha}\Big)\frac{(b-a)\Big[|X'(a,.)| + |X'(b,.)|\Big]}{2}.
\end{aligned}$$

□

Remark 2.10. *If we let $\alpha \to 1$, (12) becomes (3).*

Theorem 2.11. *Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a mean-square differentiable stochastic process such that $X(t,.) \in L^1_{loc}(a,b)$ and $X' \in L^1[a,b]$. If $|X'|^q$ is convex on $[a,b]$ for some fixed $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for* *fractional integrals holds almost everywhere*

$$|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq E(\alpha,p,q)\Big(\frac{(b-a)\Big[|X'(a,.)|^q + |X'(b,.)|^q\Big]^{1/q}}{2}\Big), \tag{13}$$

where $E(\alpha,p,q) = \dfrac{1}{2^{1/q}(1+\alpha p)^{1/p}}$.

Proof. From Lemma 2.7, the convexity of $|X'|^q$ and Holder inequality leads to

$$\begin{aligned}|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| &\leq \frac{b-a}{2}\int_0^1 \Big|(1-t)^{\alpha} - t^{\alpha}\Big| \cdot \Big|X'(ta+(1-t)b,.)\Big| dt \\ &\leq \frac{b-a}{2}\Big(\int_0^1 \Big|(1-t)^{\alpha} - t^{\alpha}\Big|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|X'(ta+(1-t)b,.)\Big|^q dt\Big)^{\frac{1}{q}} \\ &\leq \frac{b-a}{2}\Big(\int_0^1 \Big|(1-t)^{\alpha} - t^{\alpha}\Big|^p dt\Big)^{\frac{1}{p}}\Big(\Big|X'(a,.)\Big|^q \int_0^1 t dt + \Big|X'(b,.)\Big|^q \int_0^1 (1-t) dt\Big)^{\frac{1}{q}}.\end{aligned}$$

As $\alpha \in (0,1)$ and $\forall x, y \in [0,1]$, $|x^{\alpha} - y^{\alpha}| \leq |x-y|^{\alpha}$ then

$$|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq \frac{b-a}{2}\Big(\int_0^1 |1-2t|^{p\alpha} dt\Big)^{\frac{1}{p}}\Big(\frac{|X'(a,.)|^q + |X'(b,.)|^q}{2}\Big)^{\frac{1}{q}}.$$

As $\forall t \in [0,1]$, $1-2t \in [-1,1]$ then

$$\int_0^1 |1-2t|^{p\alpha} dt = \int_0^{\frac{1}{2}} (1-2t)^{p\alpha} dt + \int_{1/2}^1 (2t-1)^{p\alpha} dt = \frac{1}{1+p\alpha},$$

which completes the proof. □

Remark 2.12. *If we take* $\alpha \to 1$ *in (13), we get (4).*

Another similar result may be extended in the following remark.

Remark 2.13. *As for all* $0 < s < 1$, $(x+y)^s \leq x^s + y^s$ *for all* x, y *positive numbers,* $\Big[|X'(a,.)|^q + |X'(b,.)|^q\Big]^{1/q} \leq |X'(a,.)| + |X'(b,.)|$.
By using $\frac{x+y}{2} \geq \sqrt{xy}$, *we get* $E(\alpha,p,q) \leq \frac{1}{(\alpha p)^{1/2p}}$. *Then the inequality in the above theorem becomes*

$$|^{RZ}\Delta^{\alpha}_{[a,b]}(X)| \leq \frac{1}{(\alpha p)^{1/2p}}\Big(\frac{(b-a)\Big[|X'(a,.)| + |X'(b,.)|\Big]}{2}\Big), \quad (a.e). \tag{14}$$

In the next section, we cite an important application of our results named fractional moment estimates.

3. Application: Moment Estimates

Let

$$\begin{array}{cccc} X & : & [a,b]\times\Omega & \to & \mathbb{R} \\ & & (t,\omega) & \mapsto & X(t,\omega) \end{array}$$

be a random variable. For simplicity, we denote for a fixed $\omega \in \Omega$:

$$X \equiv X(t) = X(t,.).$$

In this section, we assume that $X(t)$ is an $\{F_t\}$-adapted, positive, non-decreasing convex stochastic process, where $F_t = \sigma\{X(s), 0 \leq s \leq t\}$ for each $t \in [0,T]$, is the filtration.

Theorem 3.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex, non-decreasing function and X a one dimensional adapted, positive, non-decreasing convex process such that $E[\phi(X^2(t)] < \infty$ for all $t \in [a,b]$. Then*

$$\begin{aligned}\phi\Big(\frac{a+b}{2}\Big) &\leq \frac{C(\alpha)}{(b-a)^{\alpha}}\Big\{E\Big[{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}\phi\Big(X^2(a)\Big)\Big] + E\Big[{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}\phi\Big(X^2(b)\Big)\Big]\Big\}\\ &\leq \frac{E\Big[\phi\Big(X^2(a)\Big)\Big] + E\Big[\phi\Big(X^2(b)\Big)\Big]}{2}, \quad (a.e). \end{aligned} \tag{15}$$

Proof. Note that, if X is a positive convex stochastic process, X^2 is also convex (and non-decreasing). Then the composition $\phi \circ X^2$ of a non-decreasing convex function ϕ and X^2 is also convex and non-decreasing stochastic process. Following (9) for convex function $\phi \circ X^2$, we have

$$\begin{aligned}\phi\Big(X^2\Big(\frac{a+b}{2}\Big)\Big) &\leq \frac{C(\alpha)}{(b-a)^{\alpha}}\Big\{{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}\phi\Big(X^2(a)\Big) + {}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}\phi\Big(X^2(b)\Big)\Big\}\\ &\leq \frac{\phi\Big(X^2(a)\Big) + \phi\Big(X^2(b)\Big)}{2}, \quad (a.e).\end{aligned}$$

Applying the linearity of the operator E and Jensen's inequality

$$\phi\Big(E\Big[X^2\Big(\frac{a+b}{2}\Big)\Big]\Big) \leq E\Big[\phi\Big(X^2\Big(\frac{a+b}{2}\Big)\Big)\Big],$$

the result follows. □

Corollary 3.2. *If $\phi(X^2(t)) = X^2(t)$, we obtain*

$$E\Big[{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X^2(a)\Big] + E\Big[{}^{RZ}\mathbb{J}^{\alpha}_{[a,b]}X^2(b)\Big] = \frac{(a+b)(b-a)^{\alpha}}{2C(\alpha)}, \quad (a.e). \tag{16}$$

4. Conclusion

In this paper, the generalisations of some inequalities involving Riesz fractional integrals for convex stochastic processes have been established. These inequalities are also useful in some techniques that are using in different proofs of existence and uniqueness problems. Finally, the obtained results may stimulate further research in the theory of fractional integrals and generalized convex stochastic processes.

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