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Conformable Special Curves in Euclidean 3-Space

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Abstract. In this study, the effect of fractional derivatives on curves, whose application area is increasing day by day, is investigated. While investigating this effect, the conformable fractional derivative, which best suits the algebraic structure of differential geometry, is selected. As a result, many special curves and Frenet frame previously obtained using classical derivatives have been redefined with the help of conformable fractional derivatives.

1. Introduction

Fractional analysis means derivative and integral accounts that are not integers. The concept of arbitrary derivatives and integration first introduced by Leibniz and L'Hospital in 1965. Today, the subject of fractional analysis has become very popular and studied by many researchers in different fields [3, 4, 11, 14, 15, 25]. Although there are different definitions of fractional derivatives, the most commonly used of these derivatives are Caputo and Riemann-Liouville fractional derivatives, which respectively include integral representation and Gamma function. All fractional derivative definitions in literature, especially Caputo and Riemann-Liouville fractional derivatives, have one thing in common with the classical derivative definition, providing the linearity characteristics of both fractional derivatives and classical derivatives. Regarding features other than linearity, there is no tangible cohesion between fractional derivative and classical derivative. For example, for fractional derivative type except for Caputo fractional derivative, the derivative of the constant is not zero. Similarly, the classical derivatives of the product and quotient of the two functions and their fractional derivatives do not exhibit any harmony. Due to this mismatch, a new definition of fractional derivatives is recently introduced, which is called conformable fractional derivative and based on the classical derivative definition [19]. By providing the natural properties of the classical derivative, the comformable fractional derivative aims to expand the definition of derivative in the known sense and to give a new perspective to the theory of differential equations with the help of harmonious fractional differential equations obtained using this derivative definition [9, 21]. Recently, it is desirable to make a geometric interpretation of the fractional derivative through differential geometry, a subclass of geometry. Various studies have started to be carried out on this subject and it is attracting attention day by day [1, 13, 16, 17, 22, 23]. However, the biggest problem is not provided by many fractional derivatives for product derivatives, quotient derivatives and chain rule fractional derivatives that are

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very important for differential geometry. Therefore, conformable fractional derivative is the most suitable fractional derivative type for geometric approach to fractional derivatives. In addition, the conformable fractional derivative is the local fractional derivative, unlike the Riemann-Liouville and Caputo fractional derivative. However, the effect of conformable fractional derivatives and integrals on some physical phenomena is worth investigating. As mentioned before, it is very important physically to obtain more numerical results with the help of fractional derivatives. Therefore, researchers are working on obtaining physical results with the help of fractional derivatives [2, 6, 7]. It will be interesting that fractional derivatives do not have a geometric interpretation as in the classical sense. However, there are many mathematicians investigating the effect of fractional calculus on differential geometry.

In this study, some basic information of differential geometry is introduced with conformable fractional derivatives. The main reason for this is that fractional derivatives and integrals give more precise numerical results at the solution point of systems than the classically known derivatives and integrals. As can be easily seen in Conclusion 3.3, the conformable derivative does not change the Frenet vectors at a point on the curve. However, the curvature and torsion of the curve are modified by the effect of conformable fractional derivative. In this article, the effect of conformable fractional derivative on the curvature and torsion of a curve is examined. In addition, many special types of curves known in classical differential geometry such as helix, slant helix and special plane curves are redefined by obtaining new characterizations with conformable fractional derivatives.

2. Preliminaries

2.1. Basic Definitions and Theorems of Conformable Fractional Derivative and Conformable Fractional Integral

In this section, some basic definitions and theorems of the appropriate fractional derivative and integral are given.

Definition 2.1. *Let us give a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" for f of order α is defined by*

$$D_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for each $t>0$, $0<\alpha<1$. If f is α-differentiable in some $(0,\alpha)$, $\alpha>0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exist, then $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$ is define [19].

Theorem 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a function α-differentiable at $t_0>0$, $0<\alpha<1$, then f is continuous at t_0 [19].*

Accordingly, it is easily visible that the conformable fractional derivative provides all the properties given in the theorem below.

Theorem 2.3. *Let f,g be α-differentiable for each $t>0$, $0<\alpha<1$. Then*
(1) $D_\alpha(af+bg)(t)=aD_\alpha(f)(t)+bD_\alpha(g)(t)$, *for all $a,b\in\mathbb{R}$.*
(2) $D_\alpha(t^p)=pt^{p-\alpha}$, *for all $p\in\mathbb{R}$.*
(3)$D_\alpha(\lambda)=0$, *for all constant functions $f(t)=\lambda$.*
(4) $D_\alpha(fg)(t)=f(t)D_\alpha(g)(t)+g(t)D_\alpha(f)(t)$.
(5) $D_\alpha(\frac{f}{g})(t)=\frac{f(t)D_\alpha(g)(t)-g(t)D_\alpha(f)(t)}{g^2(t)}$.
(6) *If f is a differentiable function, then* $D_\alpha(f)(t)=t^{1-\alpha}\frac{df(t)}{dt}$ [19].

Theorem 2.4. *Let $f,g:[0,\infty)\to\mathbb{R}$ be α-differentiable at $t_0>0$, $0<\alpha<1$. If $f\circ g$ is $\alpha-$differentiable and for all t with $t\neq 0$ and $f(t)\neq 0$, we have [21],*

$$D_\alpha(g\circ f)(t)=f(t)^{\alpha-1}D_\alpha(f)(t)D_\alpha(g)(f(t)).$$

Definition 2.5. *Let $f:[0,\infty)\to\mathbb{R}$ be a function. The expression $I_\alpha^a f(t)=I_1^a f(t^{\alpha-1}f)=\int_a^t \frac{f(x)}{x^{1-\alpha}}dx$ is called a conformable fractional integral, where $a>0$.*

Theorem 2.6. *Let $f:[0,\infty)\to\mathbb{R}$ be a function. $D_\alpha I_\alpha^a f(t)=f(t)$, for $t\geq a$.*

2.2. Basic Definitions and Theorems of Differential Geometry

In section's definitions and theorems, the curves in $\mathbb{R}^3$ will be introduced in a nutshell.

Definition 2.7. *Let the curve $x(t)$ be given in n-dimensional Euclidean space with (I, α) coordinate neighborhood. The arc length of the curve x from a to b, is calculated as*

$$s = \int_a^b \|x'(t)\| dt, \quad t \in I$$

which is the length between the points $x(a)$ and $x(b)$ of the curve. The parameter s is said to be arc-length, [10].

Theorem 2.8. *Let $x = x(s)$ be a regular unit speed curve in the Euclidean 3−space where s measures its arc length. Also, let $t = x'$ be its unit tangent vector, $n = \frac{t'}{\|t'\|}$ be its principal normal vector and $b = t \times n$ be its binormal vector. The triple $\{t, n, b\}$ be the Frenet frame of the curve x. Then the Frenet formula of the curve is given by*

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{1}$$

where $\kappa(s) = \left\| \frac{d^2x}{ds^2} \right\|$ and $\tau(s) = \left\langle \frac{dn}{ds}, b \right\rangle$ are curvature and torsion of x, respectively [8].

Definition 2.9. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space E^3. If any U fixed direction with the unit tangent vector of the curve x makes a fixed angle, the curve x is called the general helix [18]. The most well-known characterization of the helix curve is $\frac{\kappa}{\tau}$ = constant (Lancret theorem) [8].*

Definition 2.10. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space E^3. If any U fixed direction with the principal unit normal vector of the curve x makes a fixed angle, the curve x is called the slant helix. Izumiya and Takeuchi obtain a necessary and sufficient condition for a curve to be slant helix, a curve is an oblique propeller if its geodesic curvature and the principal normal satisfy the expression,*

$$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)'$$

is constant function [20].

Definition 2.11. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space E^3. The curve is called rectifying curve for all $s \in I$ if the orthogonal complement of $n(s)$ contains a fixed point. Since the orthogonal complement of n is $n^\perp = \{v \in T_\alpha E^3 \mid \langle v, n \rangle = 0\}$, the position vector of rectifying curve x in E^3 can be written as*

$$x(s) = \lambda t(s) + \mu b(s),$$

where λ, μ are differentiable function [5].

Definition 2.12. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space E^3. Then it is said that x is an osculating curve if its position vector is in the orthogonal complement of the binormal vector for all $s \in I$ and consequently, the osculating curve x is*

$$x(s) = \lambda t(s) + \mu n(s),$$

where λ, μ are differentiable function [12].

Definition 2.13. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space E^3. The curves for which the position vector always lie in their normal plane, is for simplicity called normal curve. The following equation is provided in normal curve for each $s \in I$,*

$$x(s) = \lambda n(s) + \mu b(s),$$

where λ, μ are differentiable function [12].

2.3. Basic Definitions and Theorems of Conformable Curves

In this part of the preliminaries section, basic definitions and theorems about conformable curves are given.

Definition 2.14. *Let x be a curve. If the curve* $x:(0,\infty)\to\mathbb{R}^3$ *is* $\alpha-$*differentiable, the curve x is called a conformable curve, [24].*

Definition 2.15. *Let* $x:(0,\infty)\to\mathbb{R}^3$ *be a conformable curve in* $\mathbb{R}^3$*. Velocity vector of x is determined by*

$$\frac{D_\alpha(x)(t)}{t^{1-\alpha}}$$

for all $t\in(0,\infty)$ *[24].*

Definition 2.16. *Let* $x:(0,\infty)\to\mathbb{R}^3$ *be a conformable curve in* $\mathbb{R}^3$*. Then the velocity function v of x is defined by*

$$v(t)=\frac{\|D_\alpha(x)(t)\|}{t^{1-\alpha}}$$

for all $t\in(0,\infty)$*. If* $v(t)=1$ *for all* $t\in(0,\infty)$*, it is said that x has unit speed [24].*

Definition 2.17. *Let* $x:(0,\infty)\to\mathbb{R}^3$ *be a conformable curve in* $\mathbb{R}^3$*. The arc length function s of x is defined by*

$$s(t)=I_\alpha^0\,\|D_\alpha(x)(t)\|$$

for all $t\in(0,\infty)$ *[24].*

Definition 2.18. *Let x be a conformable curve. If* $D_\alpha(x)(t)\neq 0$ *for all* $t\in(0,\infty)$*, x is called a conformable regular curve [24].*

Theorem 2.19. *Let x be a unit speed conformable curve according to frame* $\{E_1,E_2,E_3\}$ *with curvature* $\kappa>0$ *and torsion* τ*. Then the conformable Frenet formulas of the curve is given by [24],*

$$\begin{aligned} D_\alpha E_1 &= \kappa E_2\\ D_\alpha E_2 &= -\kappa E_1+\tau E_3\\ D_\alpha E_3 &= -\tau E_2 \end{aligned}$$

Theorem 2.20. *Let* $x=x(s)$ *be a regular conformable curve with arbitrary speed in the Euclidean* $3-$*space where s measures its arc length. Also, let* $E_1=\frac{D_\alpha(x)(s)}{\|D_\alpha(x)(s)\|}$ *be its unit tangent vector,* $E_3=\frac{D_\alpha(x)(s)\times D_\alpha^2(x)(s)}{\left\|D_\alpha(x)(s)\times D_\alpha^2(x)(s)\right\|}$ *be its binormal vector and* $E_2=E_3\times E_1$ *be its principal normal vector. The triple* $\{E_1,E_2,E_3\}$ *be the conformable Frenet frame of the curve x. Then the conformable Frenet formula of the curve is given by*

$$\begin{pmatrix} D_\alpha E_1\\ D_\alpha E_2\\ D_\alpha E_3\end{pmatrix}=\begin{pmatrix} 0 & \kappa_\alpha v\lambda^{1-\alpha} & 0\\ -\kappa_\alpha v\lambda^{1-\alpha} & 0 & \tau_\alpha v\lambda^{1-\alpha}\\ 0 & -\tau_\alpha v\lambda^{1-\alpha} & 0\end{pmatrix}\begin{pmatrix} E_1\\ E_2\\ E_3\end{pmatrix} \tag{2}$$

where $s\neq 0$ *for* $t\in(0,\infty)$*,* $\lambda=\frac{t}{s(t)}$ *and* $\kappa_\alpha=\left(\frac{t}{\lambda}\right)^{1-\alpha}\frac{\left\|D_\alpha(x)\times D_\alpha^2(x)\right\|}{\|D_\alpha(x)\|^3}$*,* $\tau_\alpha=\left(\frac{t}{\lambda}\right)^{1-\alpha}\frac{\left(D_\alpha(x)\times D_\alpha^2(x)\right).D_\alpha^3(x)}{\left\|D_\alpha(x)\times D_\alpha^2(x)\right\|^2}$ *are curvature and torsion of x, respectively [24].*

3. Conformable Special Curves in the Euclidean 3−Space

In this section, some special curves previously defined with classical derivative will be obtained with conformable fractional derivative.

Definition 3.1. *Let $x = x(s)$ be a regular unit speed conformable curve in the Euclidean 3−space where s measures its arc length. Also, let $t = D_\alpha(x)(s)s^{\alpha-1}$ be its unit tangent vector, $n = \frac{D_\alpha(t)(s)}{\|D_\alpha(t)(s)\|}$ be its principal normal vector and $b = t \times n$ be its binormal vector. The triple $\{t, n, b\}$ be the conformable Frenet frame of the curve x. Then the conformable Frenet formula of the curve is given by*

$$\begin{pmatrix} D_\alpha(t)(s) \\ D_\alpha(n)(s) \\ D_\alpha(b)(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{3}$$

where $\kappa_\alpha(s) = \|D_\alpha(t)(s)\|$ and $\tau_\alpha(s) = \langle D_\alpha(n)(s),\ b\rangle$ are curvature and torsion of x, respectively.

Conclusion 3.2. *Let $x = x(s)$ be a regular unit speed conformable curve in the Euclidean 3−space where s measures its arc length. The following relation exists between the curvature and torsion of the curve x and the conformable curvature and torsion*

$$\kappa_\alpha = s^{1-\alpha}\kappa, \tag{4}$$
$$\tau_\alpha = s^{1-\alpha}\tau. \tag{5}$$

Conclusion 3.3. *Let $x = x(s)$ be a regular unit speed conformable curve where s measures its arc length. As can be seen from equation (3), when x is a unit speed curve, the conformable derivative has no effect on the Frenet vectors, so the Frenet vectors do not undergo any change. However, considering equations (4) and (5), the curvature and torsion of the curve x has changed under the conformable fractional derivative.*

Definition 3.4. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the unit tangent vector of the curve x according to conformable frame makes a fixed angle with any u fixed direction, the curve x is called a conformable helix.*

Theorem 3.5. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is the conformable helix according to conformable frame, the following equation exists,*

$$\frac{\kappa_\alpha}{\tau_\alpha} = constant.$$

Proof. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If any u fixed direction with the unit tangent vector of the curve x makes a fixed angle, the curve x is called the general helix. Accordingly, the following equation can be written,

$$\langle t,\ u\rangle = \cos\theta, \quad \cos\theta = \text{constant}.$$

If conformable fractional derivative of the above equation is taken according to s,

$$\langle D_\alpha(t)(s),\ u\rangle + \langle t,\ D_\alpha(u)(s)\rangle = 0.$$

Conformable fractional derivative of u is zero because u is a fixed direction. Then,

$$\langle D_\alpha(t)(s),\ u\rangle = 0.$$

If equation (3) is used in this equation, we get

$$\kappa_\alpha \langle n,\ u\rangle = 0. \tag{6}$$

When $\kappa_\alpha = 0$, the result is obvious. So, if $\kappa_\alpha \neq 0$ is selected, $\langle n,\ u\rangle = 0$. Taking the conformable fractional derivative of equation (6) according to the s again, then

$$D_\alpha(\kappa_\alpha)(s)\langle n,\ u\rangle + \kappa_\alpha\langle D_\alpha(n)(s),\ u\rangle + \kappa_\alpha\langle n,\ D_\alpha(u)(s)\rangle = 0,$$

and

$$\kappa_\alpha\langle D_\alpha(n)(s),\ u\rangle = 0$$

are obtained. If the conformable frame formulas in equation (3) are used in this equation, we can write

$$-\kappa_\alpha^2\langle t,\ u\rangle + \kappa_\alpha\tau_\alpha\langle b,\ u\rangle = 0.$$

If the equality of $\langle t,\ u\rangle = \cos\theta$ is used in the above equation, we obtain

$$\langle b,\ u\rangle = \frac{\kappa_\alpha}{\tau_\alpha}\cos\theta.$$

If conformable fractional derivative of this expression is taken according to s again, we get

$$-\tau_\alpha\langle n,\ u\rangle = D_\alpha\left(\frac{\kappa_\alpha}{\tau_\alpha}\right)(s)\cos\theta.$$

Because it is $\langle n,\ u\rangle = 0$, we can write

$$D_\alpha\left(\frac{\kappa_\alpha}{\tau_\alpha}\right)(s) = 0.$$

So, we can easily see that

$$\frac{\kappa_\alpha}{\tau_\alpha} = c,\ \ c \in \mathbb{R}.$$

That way, the proof is complete. □

Definition 3.6. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the unit principal normal vector of x according to conformable frame makes a fixed angle with any u fixed direction, the curve x is called a conformable slant helix.*

Theorem 3.7. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is the conformable slant helix according to the conformable frame, the following equation exists.*

$$D_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)(s)\frac{1}{\kappa_\alpha} = constant.$$

Proof. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If any u fixed direction with the unit principal normal vector of the curve x makes a fixed angle, the curve x is called the slant helix. Accordingly, the following equation can be written

$$\langle n,\ u\rangle = \cos\theta.$$

If both sides of the above equation are taken conformable fractional derivative according to s, we get

$$\langle D_\alpha(n)(s),\ u\rangle + \langle n,\ D_\alpha(u)(s)\rangle = 0.$$

If equation (3) is used in above equation, we obtain that

$$-\kappa_\alpha\langle t,\ u\rangle + \tau_\alpha\langle b,\ u\rangle = 0,$$

As can be seen from the rotation matrix, if $\langle n,\ u\rangle = \cos\theta$, then $\langle b,\ u\rangle = \sin\theta$ equality is obtained. If this equation is used in the above equation, we can write as

$$\langle t,\ u\rangle = \frac{\tau_\alpha}{\kappa_\alpha}\sin\theta.$$

If once again conformable fractional derivative is taken in above equation according to s and used conformable frame formulas, we can say

$$\kappa_\alpha \langle n,\ u\rangle = D_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)(s)\sin\theta.$$

If the equality of $\langle n,\ u\rangle = \cos\theta$ is written here in place,

$$D_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)(s)\frac{1}{\kappa_\alpha} = \cot\theta$$

is obtained. Because θ is selected as a constant, we get

$$D_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)(s)\frac{1}{\kappa_\alpha} = c,\ \ c \in \mathbb{R}$$

That way, the proof is complete. □

Definition 3.8. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the position vector of x always lies in its rectifying plane according to the conformable frame, the curve x is called a conformable rectifying curve.*

Theorem 3.9. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is the conformable rectifying curve according to conformable frame, the following equation exists.*

$$\frac{\tau_\alpha}{\kappa_\alpha} = as + b,\ \ a, b \in \mathbb{R}.$$

Proof. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is a conformable rectifying curve, the following equation exists, as previously seen in the definition,

$$x = \lambda t + \mu b,$$

where $\lambda,\ \mu$ are differentiable function. If the conformable fractional derivative of the above equation is taken from the order $\alpha-$th according to the s, we obtain

$$D_\alpha(x) = D_\alpha\left(\lambda t + \mu b\right),$$

and

$$t_\alpha = D_\alpha(\lambda)t + \lambda D_\alpha(t) + D_\alpha(\mu)b + \mu D_\alpha(b)$$

where $t_\alpha = s^{1-\alpha}t$ and t_α and t are linearly dependent. If the features of the conformable frame are used in above equation, we get

$$t_\alpha = D_\alpha(\lambda)t + (\lambda\kappa_\alpha - \mu\tau_\alpha)n + D_\alpha(\mu)b.$$

From the reciprocal equality in the above equation, we can write

$$D_\alpha(\lambda) = s^{1-\alpha},\quad D_\alpha(\mu) = 0,\quad \lambda\kappa_\alpha - \mu\tau_\alpha = 0. \tag{7}$$

By solving the above equations, we have

$$\lambda = s + c_1, \tag{8}$$

$$\mu = c_2, \quad c_2 \in \mathbb{R}. \tag{9}$$

If equations (8) and (9) are used in the third equation of equation (7), we get

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{\lambda}{\mu} = \frac{s + c_1}{c_2}.$$

If $\frac{1}{c_2} = a$ and $\frac{c_1}{c_2} = b$ is selected in the last equality, finally

$$\frac{\tau_\alpha}{\kappa_\alpha} = as + b, \;\; a, b \in \mathbb{R}$$

is obtained. □

Definition 3.10. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the position vector of x always lies in its normal plane according to the conformable frame, the curve x is called a conformable normal curve.*

Theorem 3.11. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is the conformable normal curve according to the conformable frame, the following equation exists.*

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{D_\alpha(\mu)}{s^{1-\alpha}}$$

or

$$\lambda^2 + \mu^2 = c, \quad c \in \mathbb{R}$$

where λ, μ are differentiable function.

Proof. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is a conformable normal curve, the following equation exists

$$x = \lambda n + \mu b$$

where λ, μ are differentiable function. If the conformable fractional derivative of the above equation is taken from the order α−th according to the s, we have

$$t_\alpha = D_\alpha(\lambda)n + \lambda D_\alpha(n) + D_\alpha(\mu)b + \mu D_\alpha(b).$$

where $t_\alpha = s^{1-\alpha}t$ and t_α and t are linearly dependent. If the features of the conformable frame are used in this above equation, we get

$$t_\alpha = -\lambda\kappa_\alpha t + (D_\alpha(\lambda) - \mu\tau_\alpha)n + (D_\alpha(\mu) + \lambda\tau_\alpha)\, b. \tag{10}$$

From the above equation to the reciprocal equality, we can write

$$\lambda\kappa_\alpha = -s^{1-\alpha}$$

and

$$\kappa_\alpha = -\frac{s^{1-\alpha}}{\lambda}. \tag{11}$$

On the other hand, from the following equation,

$$D_\alpha(\mu) + \lambda\tau_\alpha = 0,$$

we have

$$\tau_\alpha = -\frac{1}{\lambda} D_\alpha(\mu). \tag{12}$$

If equations (11) and (12) are used in this equation, we can easily,

$$\frac{\tau_\alpha}{\kappa_\alpha} = \frac{D_\alpha(\mu)}{s^{1-\alpha}}.$$

On the other hand, considering the equation (10), we have

$$\begin{aligned} D_\alpha(\lambda) - \mu\tau_\alpha &= 0, \\ D_\alpha(\mu) + \lambda\tau_\alpha &= 0, \end{aligned}$$

and

$$\lambda D_\alpha(\lambda) + \mu D_\alpha(\mu) = 0,$$

$$D_\alpha(\lambda^2) + D_\alpha(\mu^2) = 0,$$

If the conformable fractional integral is taken in above equation considering *Theorem* 2.6, we can easily see that

$$\lambda^2 + \mu^2 = c, \quad c \in \mathbb{R}.$$

□

Definition 3.12. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the position vector of x always lies in its osculating plane according to the conformable frame, the curve x is called a conformable osculating curve.*

Theorem 3.13. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is the conformable osculating curve compared to the conformable frame, the following equation exists.*

$$\lambda^2 + \mu^2 = I_0^\alpha s^{1-\alpha}\lambda$$

where λ, μ are differentiable function, or the curve x is planar.

Proof. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in Euclidean 3-space E^3. If the curve x is a conformable normal curve, the following equation exists, as previously seen in the definition,

$$x = \lambda t + \mu n,$$

where λ, μ are differentiable function. If the conformable fractional derivative of the above equation is taken from the order $\alpha-$th according to the s, following equation is obtained

$$t_\alpha = D_\alpha(\lambda)t + \lambda D_\alpha(t) + D_\alpha(\mu)n + \mu D_\alpha(n)$$

where $t_\alpha = s^{1-\alpha}t$ and t_α and t are linearly dependent. If the features of the conformable frame are used in above equation, we get

$$t_\alpha = (D_\alpha(\lambda) - \mu\kappa_\alpha)\, t + (D_\alpha(\mu) + \lambda\kappa_\alpha)n + \mu\tau_\alpha b. \tag{13}$$

From the above equation to the reciprocal equality, we have

$$D_\alpha(\lambda) - \mu\kappa_\alpha = s^{1-\alpha},$$

$$D_\alpha(\mu) + \lambda\kappa_\alpha = 0.$$

From the above equations, we get

$$\lambda D_\alpha(\lambda) + \mu D_\alpha(\mu) = s^{1-\alpha}\lambda.$$

If the conformable integral of both sides of the equation is taken and considering the *Theorem* 2.6,

$$\lambda^2 + \mu^2 = I_0^\alpha s^{1-\alpha}\lambda.$$

On the other hand, considering equation (13), we get

$$\mu\tau_\alpha = 0. \tag{14}$$

Since $\mu = 0$ is the obvious solution in the above equation, let $\mu \neq 0$. In this case, it can be easily seen that $\tau_\alpha = 0$, so x is a planar curve. □

Example 3.14. *Let* $x : I \subset \mathbb{R} \to E^3$ *be a unit speed conformable curve in* $\mathbb{R}^3$ *parameterized by*

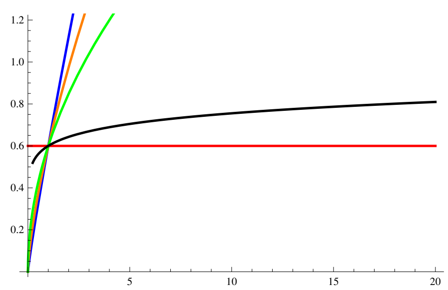
$$x(s) = \left(\frac{3}{5}\cos s,\ \frac{3}{5}\sin s,\ \frac{4}{5}s\right).$$

From equation (3) *that s is the arc-length parameter and*

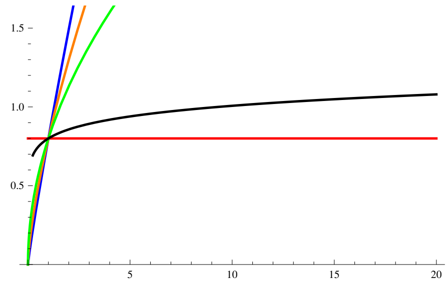
$$\begin{aligned} t(s) &= \left(-\frac{3}{5}\sin s,\ \frac{3}{5}\cos s,\ \frac{4}{5}\right), \\ n(s) &= (-\cos s,\ -\sin s,\ 0), \\ b(s) &= \left(\frac{4}{5}\sin s,\ -\frac{4}{5}\cos s,\ \frac{3}{5}\right), \end{aligned}$$

$$\begin{aligned} \kappa_\alpha &= \frac{3}{5}s^{1-\alpha}, \\ \tau_\alpha &= \frac{4}{5}s^{1-\alpha}. \end{aligned}$$

For different values of α the graphs of the curvature κ_α *and torsion* τ_α *with fractional-order as in following Figure 1 and Figure 2.*



For $\alpha = 0.1$ (*Blue*), $\alpha = 0.3$ (*Orange*), $\alpha = 0.5$ (*Green*), $\alpha = 0.9$ (*Black*) *and* $\alpha \to 1$ (*Red*), curvature of curve $x(s)$.

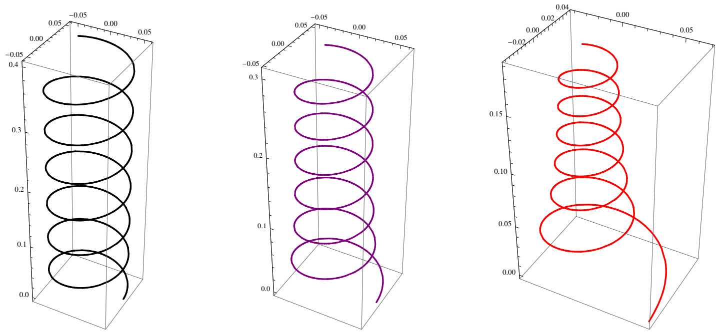


For $\alpha = 0.1$ (*Blue*), $\alpha = 0.3$ (*Orange*), $\alpha = 0.5$ (*Green*), $\alpha = 0.9$ (*Black*) *and* $\alpha \to 1$ (*Red*), torsion of curve $x(s)$.

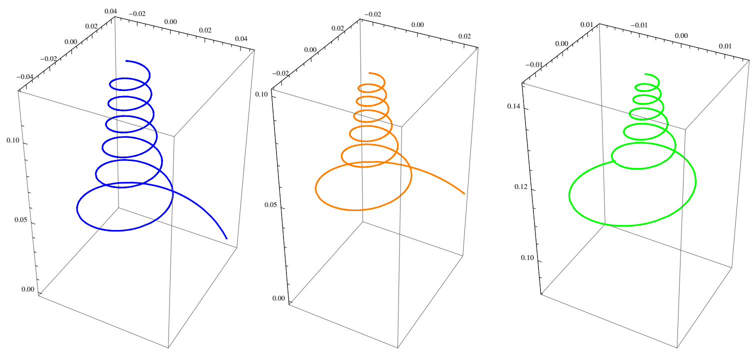
Example 3.15. *Let* $x : I \subset \mathbb{R} \to E^3$ *be a regular with arbitrary speed conformable curve in* $\mathbb{R}^3$ *parameterized by*

$$x(s) = \left(-\frac{3}{50}\int s^{\alpha-1}\sin s ds,\ \frac{3}{50}\int s^{\alpha-1}\cos s ds,\ \frac{1}{100}\int s^{\alpha-1} ds\right).$$

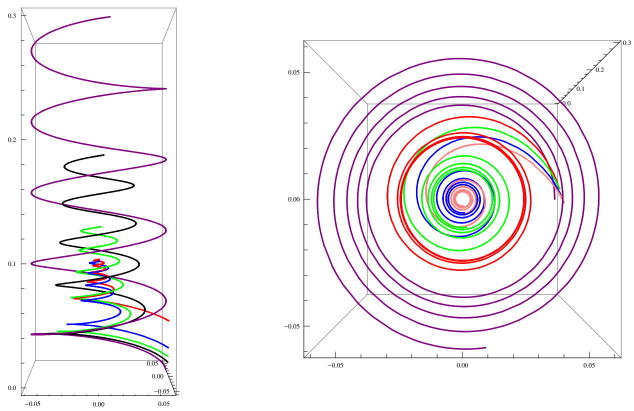
The view of the curve for different α *values are given below.*



The curve $x(s)$ for $\alpha \to 1$, $\alpha = 0.9$ *and* $\alpha = 0.7$, respectively.



The curve $x(s)$ for $\alpha = 0.5$, $\alpha = 0.3$ and $\alpha = 0.1$, respectively.



The curve $x(s)$ for $\alpha \to 1$ $(Purple)$, $\alpha = 0.8$ (Red), $\alpha = 0.6$ $(Green)$, $\alpha = 0.4$ $(Blue)$ *and* $\alpha = 0.2$ $(Pink)$.

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