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Sojourns of a Two-Dimensional Fractional Bronwian Motion Risk Process

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Abstract. This paper derives the asymptotic behavior of

$$\mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(B_H(s)-c_1s>q_1u, B_H(s)-c_2s>q_2u\Big)ds>T_u\right\},\quad u\to\infty,$$

where B_H is a fractional Brownian motion, $c_1,c_2,q_1,q_2>0,\ H\in(0,1),\ T_u\ge 0$ is a measurable function and $\mathbb{I}(\cdot)$ is the indicator function.

1. Introduction & Preliminaries

Consider the risk model defined by

$$R(t)=u+\rho t-X(t),\quad t\ge 0, \tag{1}$$

where $X(t)$ is a centered Gaussian risk process with a.s. continuous sample paths, $\rho>0$ is the net profit rate and $u>0$ is the initial capital. This model is relevant to insurance and financial applications, see, e.g., [1]. A question of numerous investigations (see [2–17]) is the study of the asymptotics of the classical ruin probability

$$\lambda(u):=\mathbb{P}\{\exists t\ge 0: R(t)<0\} \tag{2}$$

as $u\to\infty$ under different levels of generality. It turns out, that only for X being a Brownian motion (later on BM) $\lambda(u)$ can be calculated explicitly: if X is a standard BM, then $\lambda(u)=e^{-2\rho u},\ u,\rho>0$, see [18]. Since it seems impossible to find the exact value of $\lambda(u)$ in other cases, the approximations of $\lambda(u)$ as $u\to\infty$ is dealt with. Some contributions (see, e.g., [19, 20]), extend the classical ruin problem to the so-called sojourn problem, i.e., approximation of the sojourn probability defined by

$$\mathbb{P}\left\{\int_0^\infty \mathbb{I}(R(s)<0)ds>T_u\right\}, \tag{3}$$

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where $T_u \geq 0$ is a measurable function of u. As in the classical case, only for X being a BM the probability above can be calculated explicitly, see [20]:

$$\mathbb{P}\left\{\int_0^\infty \mathbb{I}(B(s) - cs > u)ds > T\right\} = \left(2(1 + c^2T)\Psi(c\sqrt{T}) - \frac{c\sqrt{2T}}{\sqrt{\pi}}e^{\frac{-c^2T}{2}}\right)e^{-2cu}, \quad c > 0,\ T, u \geq 0,$$

where Ψ is the survival function of a standard Gaussian random variable, B is a standard BM and $\mathbb{I}(\cdot)$ is the indicator function. Motivated by [21] (see also [5, 22, 23]), we study a generalization of the main problem in [21] for the sojourn case, i.e., we shall study the asymptotics of

$$\mathbb{C}_{T_u}(u) := \mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(B_H(s) - c_1 s > q_1 u, B_H(s) - c_2 s > q_2 u\Big)ds > T_u\right\}, \quad u \to \infty,$$

where B_H is a standard fractional Brownian motion (later on fBM), i.e., a Gaussian process with zero expectation and covariance defined by

$$\mathrm{cov}(B_H(s), B_H(t)) = \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}, \qquad t, s \in \mathbb{R}.$$

The ruin probability above is of interest for reinsurance models, see [21] and references therein. By the self-similarity of fBM we have

$$\begin{aligned}
\mathbb{C}_{T_u}(u) &= \mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(B_H(su) > c_1 su + q_1 u, B_H(su) > c_2 su + q_2 u\Big)d(su) > T_u\right\} \\
&= \mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(u^H B_H(s) > (c_1 s + q_1)u, u^H B_H(s) > (c_2 s + q_2)u\Big)ds > T_u/u\right\} \\
&= \mathbb{P}\left\{\int_0^\infty \mathbb{I}\left(\frac{B_H(s)}{\max(c_1 s + q_1, c_2 s + q_2)} > u^{1-H}\right)ds > T_u/u\right\}.
\end{aligned}$$

In order to prevent the problem of degenerating to the one-dimensional sojourn problem discussed in [19, 20] (i.e., to impose the denominator in the line above be nonlinear function) we assume that

$$c_1 > c_2, \quad q_2 > q_1. \tag{4}$$

The variance of the process two lines above can achieve its unique maxima only at one of the following points:

$$t_* = \frac{q_2 - q_1}{c_1 - c_2}, \quad t_1 = \frac{q_1 H}{(1-H)c_1}, \quad t_2 = \frac{q_2 H}{(1-H)c_2}. \tag{5}$$

From (4) it follows that $t_1 < t_2$; as we shall see later, the order between t_1, t_2 and t_* determines the asymptotics of $\mathbb{C}_{T_u}(u)$ as $u \to \infty$. As mentioned in [9], for the approximation of the one-dimensional Parisian ruin probability we need to control the growth of T_u as $u \to \infty$. As in [9], we impose the following condition:

$$\lim_{u\to\infty} T_u u^{1/H-2} = T \in [0, \infty),\ H \in (0, 1). \tag{6}$$

Note that T_u satisfying (6) may go to ∞ for $H > 1/2$, converges to non-negative limit for $H = 1/2$ and approaches 0 for $H < 1/2$ as $u \to \infty$. We see later on in Proposition 2.2 that the condition above is necessary and it seems very difficult to derive the exact asymptotics of $\mathbb{C}_{T_u}(u)$ as $u \to \infty$ without it.

The rest of the paper is organized in the following way. In the next section we present the main results of the paper, in Section 3 we give all proofs, while technical calculations are deferred to the Appendix.

2. Main Results

Define for some function h and $K \geq 0$ the sojourn Piterbarg constant by

$$\mathcal{B}_K^h = \int_{\mathbb{R}} \mathbb{P}\left\{ \int_{-\infty}^{\infty} \mathbb{I}\Big(\sqrt{2}B(s) - |s| + h(s) > x\Big) ds > K \right\} e^x dx$$

when the integral above is finite and Berman's constant by

$$\mathcal{B}_{2H}(x) = \lim_{S\to\infty} \frac{1}{S} \int_{\mathbb{R}} \mathbb{P}\left\{ \int_0^S \mathbb{I}(\sqrt{2}B_H(t) - t^{2H} + z > 0) dt > x \right\} e^{-z} dz, \quad x \geq 0.$$

It is known (see, e.g., [20]) that $\mathcal{B}_{2H}(x) \in (0, \infty)$ for all $x \geq 0$; we refer to [20] and references therein for the properties of relevant Berman's constants. Let for $i = 1, 2$

$$\mathbb{D}_H = \frac{c_1 t_* + q_1}{t_*^H}, \ K_H = \frac{2^{\frac{1}{2} - \frac{1}{2H}} \sqrt{\pi}}{\sqrt{H(1-H)}}, \ \mathbb{C}_H^{(i)} = \frac{c_i^H q_i^{1-H}}{H^H (1-H)^{1-H}}, \ D_i = \frac{c_i^2 (1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}} H^2}. \tag{7}$$

Now we are ready to give the asymptotics of $\mathbb{C}_{T_u}(u)$ as $u \to \infty$.

Theorem 2.1. *Assume that* (4) *holds and* T_u *satisfies* (6).
1)If $t_* \notin (t_1, t_2)$*, then as* $u \to \infty$

$$\mathbb{C}_{T_u}(u) \sim (\frac{1}{2})^{\mathbb{I}(t_* = t_i)} \times \begin{cases} \Big(2(1 + c_i^2 T)\Psi(c_i\sqrt{T}) - \frac{c_i\sqrt{2T}}{\sqrt{\pi}} e^{\frac{-c_i^2 T}{2}}\Big) e^{-2c_i q_i u}, & H = 1/2 \\ K_H \mathcal{B}_{2H}(TD_i)(\mathbb{C}_H^{(i)} u^{1-H})^{\frac{1}{H}-1} \Psi(\mathbb{C}_H^{(i)} u^{1-H}), & H \neq 1/2, \end{cases} \tag{8}$$

where $i = 1$ *if* $t_* \leq t_1$ *and* $i = 2$ *if* $t_* \geq t_2$.
2) If $t_* \in (t_1, t_2)$ *and* $\lim_{u\to\infty} T_u u^{2-1/H} = 0$ *for* $H > 1/2$*, then as* $u \to \infty$

$$\mathbb{C}_{T_u}(u) \sim \Psi(\mathbb{D}_H u^{1-H}) \times \begin{cases} 1, & H > 1/2 \\ \mathcal{B}_{T'}^d, & H = 1/2 \\ \mathcal{B}_{2H}(\overline{D}T) A u^{(1-H)(1/H-2)}, & H < 1/2, \end{cases} \tag{9}$$

where $\mathcal{B}_{T'}^d \in (0, \infty)$,

$$T' = T\frac{(c_1 q_2 - q_1 c_2)^2}{2(c_1 - c_2)^2}, \ d(s) = s\frac{c_1 q_2 + c_2 q_1 - 2c_2 q_2}{c_1 q_2 - q_1 c_2}\mathbb{I}(s < 0) + s\frac{2c_1 q_1 - c_1 q_2 - q_1 c_2}{c_1 q_2 - q_1 c_2}\mathbb{I}(s \geq 0) \tag{10}$$

and

$$A = \Big(|H(c_1 t_* + q_1) - c_1 t_*|^{-1} + |H(c_2 t_* + q_2) - c_2 t_*|^{-1}\Big) \frac{t_*^H \mathbb{D}_H^{\frac{1}{H}-1}}{2^{\frac{1}{2H}}}, \quad \overline{D} = \frac{(c_1 t_* + q_1)^{\frac{1}{H}}}{2^{\frac{1}{2H}} t_*^2}. \tag{11}$$

Note that if $T = 0$, then the result above reduces to Theorem 3.1 in [21]. As already mentioned in the introduction (6) is a necessary condition for the theorem above. To illustrate situation when it is not satisfied we consider a "simple" scenario with T_u being a positive constant.

Proposition 2.2. *If* $H < 1/2$, $T_u = T > 0$ *and* $t_* \in (t_1, t_2)$*, then*

$$\begin{aligned} \bar{C}\Psi(\mathbb{D}_H u^{1-H}) e^{-C_{1,\alpha} u^{2-4H} - C_{2,\alpha} u^{2(1-3H)}} &\leq \mathbb{C}_{T_u}(u) \\ &\leq (2 + o(1))\Psi(\mathbb{D}_H u^{1-H})\Psi\left(u^{1-2H}\frac{T^H \mathbb{D}_H}{2t_*^H}\right), \quad u \to \infty, \end{aligned}$$

where $\bar{C} \in (0, 1)$ *is a fixed constant that does not depend on* u *and*

$$\alpha = \frac{T^{2H}}{2t_*^{2H}}, \qquad C_{i,\alpha} = \frac{\alpha^i}{i}\mathbb{D}_H^2, \quad i = 1, 2.$$

Note that instead of the exact asymptotics as in Theorem 2.1 here we observe lower and upper bounds, that decays to zero with different speed as $u \to \infty$. Moreover, asymptoitcs in (9) is exponentially bigger than the upper bound in Proposition 2.2

3. Proofs

First we give the following auxiliary results. As shown, e.g., in Lemma 2.1 in [24]

$$(1 - \frac{1}{u^2})\frac{1}{\sqrt{2\pi}u}e^{-u^2/2} \le \Psi(u) \le \frac{1}{\sqrt{2\pi}u}e^{-u^2/2}, \quad u > 0. \tag{12}$$

Recall that K_H, D_1 and $\mathbb{C}_H^{(1)}$ are defined in (7). A proof of the proposition below is given in the Appendix.

Proposition 3.1. *Assume that T_u satisfies* (6). *Then as $u \to \infty$*

$$\mathbb{P}\left\{\int_0^\infty \mathbb{I}(B_H(t) - c_1 t > q_1 u)dt > T_u\right\} \sim \begin{cases} \left(2(1+c_1^2T)\Psi(c_1\sqrt{T}) - \frac{c_1\sqrt{2T}}{\sqrt{\pi}}e^{\frac{-c_1^2T}{2}}\right)e^{-2c_1q_1u}, & H = 1/2,\\ K_H\mathcal{B}_{2H}(TD_1)(\mathbb{C}_H^{(1)}u^{1-H})^{\frac{1}{H}-1}\Psi(\mathbb{C}_H^{(1)}u^{1-H}), & H \neq 1/2.\end{cases}$$

Now we are ready to perform our proofs.

Proof of Theorem 2.1. Case (1). Assume that $t_* < t_1$. Let

$$V_i(t) = \frac{B_H(t)}{c_i t + q_i} \quad \text{and} \quad \psi_i(T_u, u) = \mathbb{P}\left\{\int_0^\infty \mathbb{I}(B_H(t) - c_i t > q_i u)ds > T_u\right\}, \quad i = 1,2.$$

For $0 < \varepsilon < t_1 - t_*$ by the self-similarity of fBM we have

$$\psi_1(T_u,u) \ge \mathbb{C}_{T_u}(u) \ge \mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon} \mathbb{I}(V_1(t) > u^{1-H}, V_2(t) > u^{1-H})dt > \frac{T_u}{u}\right\} = \mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon}\mathbb{I}(V_1(t) > u^{1-H})dt > \frac{T_u}{u}\right\}.$$

We have by Borel-TIS inequality, see [14] (details are in the Appendix)

$$\psi_1(T_u,u) \sim \mathbb{P}\left\{\int_{t_1-\varepsilon}^{t_1+\varepsilon}\mathbb{I}(V_1(t) > u^{1-H})ds > T_u/u\right\}, \quad u \to \infty \tag{13}$$

implying $\mathbb{C}_{T_u}(u) \sim \psi_1(T_u,u)$ as $u \to \infty$. The asymptotics of $\psi_1(T_u,u)$ is given in Proposition 3.1, thus the claim follows.

Assume that $t_* = t_1$. We have

$$\begin{aligned}\mathbb{P}\left\{\int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H})ds > T_u\right\} &\le \mathbb{C}_{T_u}(u)\\ &\le \mathbb{P}\left\{\int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H})ds > T_u\right\} + \mathbb{P}\left\{\exists t \in [0,t_1] : V_2(t) > u^{1-H}\right\}.\end{aligned}$$

From the proof of Theorem 3.1, case (4) in [21] it follows that the second term in the last line above is negligible comparing with the final asymptotics of $\mathbb{C}_{T_u}(u)$ given in (8), hence

$$\mathbb{C}_{T_u}(u) \sim \mathbb{P}\left\{\int_{t_1}^\infty \mathbb{I}(V_1(s) > u^{1-H})ds > T_u\right\}, \quad u \to \infty.$$

Since t_1 is the unique maxima of $\mathrm{Var}\{V_1(t)\}$ from the proof of Theorem 2.1, case i) in [20] we have

$$\mathbb{P}\left\{\int_{t_1}^{\infty} \mathbb{I}(V_1(t) > u^{1-H})dt > T_u/u\right\} \sim \frac{1}{2}\mathbb{P}\left\{\int_{0}^{\infty} \mathbb{I}(V_1(t) > u^{1-H})dt > T_u/u\right\}$$
$$= \frac{1}{2}\mathbb{P}\left\{\int_{0}^{\infty} \mathbb{I}(B_H(t) - c_1 t > q_1 u)dt > T_u\right\}, \quad u \to \infty.$$

The asymptotics of the last probability above is given in Proposition 3.1 establishing the claim. Case $t_* \geq t_2$ follows by the same arguments.

Case (2). Assume that $H > 1/2$. We have by Theorem 2.1 in [22] and Theorem 3.1 in [21] with

$$\mathcal{R}_{T_u}(u) = \mathbb{P}\{\exists t \geq 0 : B_H(t) - c_1 t > q_1 u, B_H(t) - c_2 t > q_2 u\},$$
$$\mathcal{P}_{T_u}(u) = \mathbb{P}\left\{\exists t \geq 0 : \inf_{s\in[t,t+T_u]}(B_H(s) - c_1 s) > q_1 u, \inf_{s\in[t,t+T_u]}(B_H(s) - c_2 s) > q_2 u\right\}$$

that

$$\Psi(\mathbb{D}_H u^{1-H}) \sim \mathcal{P}_{T_u}(u) \leq \mathbb{C}_{T_u}(u) \leq \mathcal{R}_{T_u}(u) \sim \Psi(\mathbb{D}_H u^{1-H}), \quad u \to \infty,$$

and the claim follows.

Assume that $H = 1/2$. First let (6) holds with $T_u = T > 0$. We have as $u \to \infty$ and then $S \to \infty$ (proof is in the Appendix)

$$\mathbb{C}_{T_u}(u) \sim \mathbb{P}\left\{\int_{ut_*-S}^{ut_*+S} \mathbb{I}\Big(B(s) - c_1 s > q_1 u, B(s) - c_2 s > q_2 u\Big)ds > T\right\} =: \kappa_S(u). \tag{14}$$

Next with ϕ_u the density of $B(ut_*)$, $\eta = c_1 t_* + q_1 = c_2 t_* + q_2$ and $\eta_* = \eta/t_* - c_2 = q_2/t_*$ we have

$$\begin{aligned}
\kappa_S(u) &= \int_{\mathbb{R}} \mathbb{P}\left\{\int_{ut_*-S}^{ut_*} \mathbb{I}(B(s) - c_2 s > q_2 u)ds + \int_{ut_*}^{ut_*+S} \mathbb{I}(B(s) - c_1 s > q_1 u)ds > T \Big| B(ut_*) = \eta u - x\right\}\phi_u(\eta u - x)dx \\
&= \int_{\mathbb{R}} \mathbb{P}\Big\{\int_{ut_*-S}^{ut_*} \mathbb{I}(B(s) - c_2 s > q_2 u)ds \\
&\quad + \int_{ut_*}^{ut_*+S} \mathbb{I}(B(s) - B(ut_*) - c_1(s - ut_*) - c_1 ut_* > q_1 u - \eta u + x)ds > T \Big| B(ut_*) = \eta u - x\Big\}\phi_u(\eta u - x)dx \\
&= \int_{\mathbb{R}} \mathbb{P}\left\{\int_{ut_*-S}^{ut_*} \mathbb{I}(B(s) - c_2 s > q_2 u)ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x)ds > T \Big| B(ut_*) = \eta u - x\right\}\phi_u(\eta u - x)dx \\
&= \frac{e^{-\frac{\eta^2 u}{2t_*}}}{\sqrt{2\pi u t_*}} \int_{\mathbb{R}} \mathbb{P}\left\{\int_{-S}^{0} \mathbb{I}\Big(Z_u(s) + \eta_* s > x\Big)ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x)ds > T\right\} e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}}dx,
\end{aligned}$$

where $Z_u(t)$ is a Gaussian process with expectation and covariance defined below:

$$\mathbb{E}\{Z_u(t)\} = \frac{-x}{ut_*}t, \qquad \mathrm{cov}(Z_u(s), Z_u(t)) = \frac{-st}{ut_*} - t, \qquad s \leq t \leq 0. \tag{15}$$

Since $Z_u(t)$ converges to BM in the sense of convergence finite-dimensional distributions for any fixed $x \in \mathbb{R}$ as $u \to \infty$ we have (details are in the Appendix)

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{P}\left\{\int_{-S}^{0} \mathbb{I}\Big(Z_u(s) + \eta_* s > x\Big)ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x)ds > T\right\} e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx \\
\sim\ &\int_{\mathbb{R}} \mathbb{P}\left\{\int_{-S}^{0} \mathbb{I}\Big(B(s) + \eta_* s > x\Big)ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x)ds > T\right\} e^{\frac{\eta x}{t_*}} dx \qquad (16)\\
=:\ &K(S).
\end{aligned}$$

Since $\mathbb{P}\{\exists t \ge 0 : B(t) - ct > x\} = e^{-2cx},\ c, x > 0$ (see, e.g., [18]) we have

$$\begin{aligned}
K(S) &\le \int_0^\infty \Big(\mathbb{P}\{\exists s < 0 : B(s) + \eta_* s > x\} + \mathbb{P}\{\exists s \ge 0 : B_*(s) - c_1 s > x\}\Big) e^{\frac{\eta x}{t_*}} dx + \int_{-\infty}^{0} e^{\frac{\eta x}{t_*}} dx \\
&= \int_0^\infty \Big(e^{(-2\eta_* + \eta/t_*)x} + e^{(-2c_1 + \eta/t_*)x}\Big)dx + t_*/\eta < \infty
\end{aligned}$$

provided that $t_* \in (t_1, t_2)$. Since $K(S)$ is an increasing function and $\lim_{S\to\infty} K(S) < \infty$ we have as $S \to \infty$

$$\begin{aligned}
K(S) &\to \int_{\mathbb{R}} \mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(B(s) - \eta_* s > x\Big)ds + \int_0^\infty \mathbb{I}(B_*(s) - c_1 s > x)ds > T\right\} e^{\frac{\eta x}{t_*}} dx \\
&= \frac{t_*}{\eta} \int_{\mathbb{R}} \mathbb{P}\left\{\int_0^\infty \mathbb{I}\Big(B(s) - \frac{\eta_* t_*}{\eta} s > x\Big)ds + \int_0^\infty \mathbb{I}(B_*(s) - \frac{c_1 t_*}{\eta} s > x)ds > \frac{\eta^2 T}{t_*^2}\right\} e^x dx \\
&= \frac{t_*}{\eta} \int_{\mathbb{R}} \mathbb{P}\left\{\int_{-\infty}^{\infty} \mathbb{I}\Big(\sqrt{2}B(s) - |s| + d(s) > x\Big)ds > \frac{\eta^2 T}{2t_*^2}\right\} e^x dx \\
&= \frac{t_*}{\eta} \mathcal{B}^d_{T'} \in (0, \infty),
\end{aligned}$$

where T' and $d(s)$ are defined in (10). Finally, combining (16) with the line above we have as $u \to \infty$ and then $S \to \infty$

$$\kappa_S(u) \sim \mathcal{B}^d_{T'} \Psi(\mathbb{D}_{1/2}\sqrt{u})$$

and by (14) the claim follows. If (6) holds with $T_u = 0$, then we obtain the claim immediately by Theorem 3.1 in [21] and observation that $\mathcal{B}^d_0$ coincides with the corresponding Piterbarg constant introduced in [21].

Now assume that (6) holds with any possible T_u. If (6) holds with $T > 0$, then for large u and any $\varepsilon > 0$ it holds, that $\mathbb{C}_{(1+\varepsilon)T}(u) \le \mathbb{C}_{T_u}(u) \le \mathbb{C}_{(1-\varepsilon)T}(u)$ and hence

$$(1 + o(1))\mathcal{B}^d_{T'(1+\varepsilon)} \Psi(\mathbb{D}_{1/2}\sqrt{u}) \le \mathbb{C}_{T_u}(u) \le \mathcal{B}^d_{T'(1-\varepsilon)} \Psi(\mathbb{D}_{1/2}\sqrt{u})(1 + o(1)), \qquad u \to \infty.$$

By Lemma 4.1 in [20] $\mathcal{B}^d_x$ is a continuous function with respect to x and thus letting $\varepsilon \to 0$ we obtain the claim. If (6) holds with $T = 0$, then for large u and any $\varepsilon > 0$ we have

$$\mathcal{B}^d_\varepsilon \Psi(\mathbb{D}_{1/2}\sqrt{u}) \le \mathbb{C}_{T_u}(u) \le \mathcal{B}^d_0 \Psi(\mathbb{D}_{1/2}\sqrt{u})$$

and again letting $\varepsilon \to 0$ we obtain the claim by continuity of $\mathcal{B}^d_{(\cdot)}$.

Assume that $H < 1/2$. First we have with $\delta_u = u^{2H-2}\ln^2 u$ as $u \to \infty$ (proof is in Appendix)

$$\begin{aligned}\mathbb{C}_{T_u}(u) &\sim \mathbb{P}\left\{\int_{ut_*-u\delta_u}^{ut_*} \mathbb{I}(B_H(t)-c_2t>q_2u)dt > T_u\right\} + \mathbb{P}\left\{\int_{ut_*}^{ut_*+u\delta_u} \mathbb{I}(B_H(t)-c_1t>q_1u)dt > T_u\right\}\\ &=: g_1(u)+g_2(u). \end{aligned} \tag{17}$$

Assume that (6) holds with $T > 0$. Using the approach from [20] we have with $\mathbb{I}_a(b) = \mathbb{I}(b > a),\ a, b \in \mathbb{R}$

$$\begin{aligned} g_2(u) &= \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u} \mathbb{I}_{M(u)}\Big(\frac{B_H(ut_*+tT_u)}{u(q_1+c_1t_*)+c_1tT_u}M(u)\Big)dt > 1\right\}\\ &=: \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} u} \mathbb{I}_{M(u)}(Z_u^{(1)}(t))dt > 1\right\}\\ &= \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} uK_1} \mathbb{I}_{M(u)}(Z_u^{(1)}(tK_1^{-1}))dt > K_1\right\}\\ &=: \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} uK_1} \mathbb{I}_{M(u)}(Z_u^{(2)}(t))dt > K_1\right\}, \end{aligned}$$

where

$$K_1 = \frac{T\mathbb{D}_H^{1/H}}{2^{\frac{1}{2H}}t_*}, \quad M(u) = \inf_{t\in[t_*,\infty)} \frac{u(c_1t+q_1)}{\mathrm{Var}\{B_H(ut)\}} = \mathbb{D}_H u^{1-H}.$$

For variance $\sigma^2_{Z_u^{(2)}}(t)$ and correlation $r_{Z_u^{(2)}}(s,t)$ of $Z_u^{(2)}$ for $t, s \in [0, \delta_u T_u^{-1} uK_1]$ it holds, that as $u \to \infty$

$$\begin{aligned} 1-\sigma_{Z_u^{(2)}}(t) &= \frac{2^{\frac{1}{2H}} t_*^H \mathbb{D}_H^{1-1/H}|q_1H-(1-H)c_1t_*|}{(q_1+c_1t_*)^2} tu^{1-1/H} + O(t^2u^{2(1-1/H)}),\\ 1-r_{Z_u^{(2)}}(s,t) &= \mathbb{D}_H^{-2}u^{2H-2}|t-s|^{2H} + O(u^{2H-2}|t-s|^{2H}\delta_u). \end{aligned}$$

Now we apply Theorem 2.1 in [20]. All conditions of the theorem are fulfilled with parameters

$$\begin{aligned} &\omega(x) = x,\ \overleftarrow{\omega}(x) = x,\ \beta = 1,\ g(u) = \frac{2^{\frac{1}{2H}} t_*^H \mathbb{D}_H^{1-1/H}|q_1H-(1-H)c_1t_*|}{(q_1+c_1t_*)^2} u^{1-1/H},\\ &\eta_\varphi(t) = B_H(t),\ \sigma_\eta^2(t) = t^{2H},\ \Delta(u) = 1,\ \varphi = 1,\\ &n(u) = \mathbb{D}_H u^{1-H},\ a_1(u) = 0,\ a_2(u) = \delta_u T_u^{-1} uK_1,\ \gamma = 0,\ x_1 = 0,\ x_2 = \infty,\ y_1 = 0,\ y_2 = \infty,\ x = K_1,\\ &\theta(u) = u^{(1/H-2)(1-H)}\mathbb{D}_H^{-1+1/H}|q_1H-(1-H)c_1t_*|^{-1}t_*^H 2^{-\frac{1}{2H}}, \end{aligned}$$

and thus as $u \to \infty$

$$g_2(u) = \mathbb{P}\left\{\int_0^{\delta_u T_u^{-1} uK_1} \mathbb{I}_{M(u)}(Z_u^{(2)}(t))dt > K_1\right\} \sim \mathcal{B}_{2H}\Big(\frac{T\mathbb{D}_H^{\frac{1}{H}}}{2^{\frac{1}{2H}}t_*}\Big)u^{(\frac{1}{H}-2)(1-H)} \frac{t_*^H \mathbb{D}_H^{-1+1/H}}{2^{\frac{1}{2H}}|q_1H-(1-H)c_1t_*|}\Psi(\mathbb{D}_H u^{1-H}).$$

Similarly we obtain

$$g_1(u) \sim \mathcal{B}_{2H}\Big(\frac{T\mathbb{D}_H^{1/H}}{2^{\frac{1}{2H}}t_*}\Big)u^{(1/H-2)(1-H)} \frac{t_*^H \mathbb{D}_H^{-1+1/H}}{2^{\frac{1}{2H}}|q_2H-(1-H)c_2t_*|}\Psi(\mathbb{D}_H u^{1-H}), \quad u \to \infty$$

and the claim follows if in (6) $T > 0$. Now let (6) holds with $T = 0$. Since $\mathcal{P}_{T_u}(u) \le \mathbb{C}_{T_u}(u) \le \mathcal{R}_{T_u}(u)$ we obtain the claim by Theorem 2.1 in [22] and Theorem 3.1 in [21]. □

Proof of Proposition 2.2. The proof of this proposition is the same as the proof of Proposition 2.2 in [22], thus we refer to [22] for the proof. □

4. Appendix

Proof of (13). To establish the claim we need to show that

$$\mathbb{P}\left\{ \int\limits_{[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} \mathbb{I}\Big(V_1(s) > u^{1-H}\Big) ds > T_u/u \right\} = o(\psi_1(T_u,u)), \quad u \to \infty.$$

Applying Borell-TIS inequality (see, e.g., [14]) we have as $u \to \infty$

$$\begin{aligned} \mathbb{P}\left\{ \int\limits_{[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} \mathbb{I}(V_1(s) > u^{1-H}) ds > T_u/u \right\} &\le \mathbb{P}\left\{\exists t \in [0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon] : V_1(t) > u^{1-H}\right\} \\ &\le e^{-\frac{(u^{1-H}-M)^2}{2m^2}}, \end{aligned}$$

where

$$M = \mathbb{E}\left\{ \sup_{\exists t\in[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} V_1(t) \right\} < \infty, \quad m^2 = \max_{\exists t\in[0,\infty)\setminus[t_1-\varepsilon,t_1+\varepsilon]} \mathrm{Var}\{V_1(t)\}.$$

Since $\mathrm{Var}\{V_1(t)\}$ achieves its unique maxima at t_1 we obtain by (12) that

$$e^{-\frac{(u^{1-H}-M)^2}{2m^2}} = o(\mathbb{P}\left\{V_1(t_1) > u^{1-H}\right\}), \quad u \to \infty$$

and the claim follows from the asymptotics of $\psi_1(T_u, u)$ given in Proposition 3.1. □

Proof of (14). To prove the claim it is enough to show that as $u \to \infty$ and then $S \to \infty$

$$\mathbb{P}\left\{ \int\limits_{[0,\infty)\setminus[ut_*-S,ut_*+S]} \mathbb{I}\Big(B(t) - c_1 t > q_1 u, B(t) - c_2 t > q_2 u\Big) dt > T \right\} = o(\mathbb{C}_{T_u}(u)), \quad u \to \infty.$$

We have that the probability above does not exceed

$$\mathbb{P}\left\{\exists t \in [0,\infty)\setminus[ut_* - S, ut_* + S] : B(t) - c_1 t > q_1 u, B(t) - c_2 t > q_2 u\right\}.$$

From the proof of Theorem 3.1 in [21], Case (3) and the final asymptotics of $\mathbb{C}_{T_u}(u)$ given in (9) it follows that the expression above equals $o(\mathbb{C}_{T_u}(u))$, as $u \to \infty$ and then $S \to \infty$. □

Proof of (16). Define

$$G(u,x) = \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}\Big(Z_u(s) + \eta_* s > x\Big) ds + \int_{0}^{S} \mathbb{I}\Big(B_*(s) - c_1 s > x\Big) ds > T \right\}.$$

First we show that

$$\int\limits_{\mathbb{R}} G(u,x) e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx = \int_{-M}^{M} G(u,x) e^{\frac{\eta x}{t_*}} dx + A_{M,u}, \tag{18}$$

where $A_{M,u} \to 0$ as $u \to \infty$ and then $M \to \infty$. We have

$$\begin{aligned} |A_{M,u}| &= |\int_{\mathbb{R}} G(u,x) e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} dx - \int_{-M}^{M} G(u,x) e^{\frac{\eta x}{t_*}} dx| \\ &\le |\int_{-M}^{M} G(u,x)(e^{\frac{\eta x}{t_*} - \frac{x^2}{2ut_*}} - e^{\frac{\eta x}{t_*}}) dx| + \int_{|x|>M} G(u,x) e^{\frac{\eta x}{t_*}} dx =: |I_1| + I_2. \end{aligned}$$

Since the variance of Z_u (see (15)) converges to those of BM we have by Borell-TIS inequality for $x > 0$, large u and some $C > 0$

$$\begin{aligned} G(u,x) &\le \mathbb{P}\{\exists t \in [-S,0) : (Z_u(t) + \eta_* t) > x\} + \mathbb{P}\{\exists t \in [0,S] : (B_*(t) - c_1 t) > x\} \\ &\le \mathbb{P}\{\exists t \in [-S,0] : (Z_u(t) - \mathbb{E}\{Z_u(t)\}) > x\} + \mathbb{P}\{\exists t \in [0,S] : B_*(t) > x\} \le e^{-x^2/C}. \end{aligned} \tag{19}$$

Let $u > M^4$. For $x \in [-M, M]$ it holds, that $1 - e^{-\frac{x^2}{2ut_*}} \le \frac{x^2}{2ut_*} \le \frac{1}{M}$ and hence for $u > M^4$ by (19) we have as $M \to \infty$

$$|I_1| \le \int_{-M}^{0} e^{\frac{\eta x}{t_*}} (1 - e^{-\frac{x^2}{2ut_*}}) dx + \int_{0}^{M} e^{-x^2/C + \frac{\eta x}{t_*}} (1 - e^{-\frac{x^2}{2ut_*}}) dx \le \frac{1}{M} \Big(\int_{-\infty}^{0} e^{\frac{\eta x}{t_*}} dx + \int_{0}^{\infty} e^{-x^2/C + \frac{\eta x}{t_*}} dx \Big) \to 0.$$

For I_2 we have

$$I_2 \le \int_{-\infty}^{-M} e^{\frac{\eta x}{t_*}} dx + \int_{M}^{\infty} e^{-x^2/C} e^{\frac{\eta x}{t_*}} dx \to 0, \quad M \to \infty,$$

hence (18) holds. Next we show that

$$G(u,x) \to \mathbb{P}\left\{ \int_{-S}^{0} \mathbb{I}\Big(B(s) + \eta_* s > x\Big) ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x) ds > T \right\}, \quad u \to \infty$$

that is equivalent with

$$\lim_{u \to \infty} \mathbb{P}\left\{ \int_{-S}^{S} \mathbb{I}\Big(X_u(s) > x\Big) ds > T \right\} = \mathbb{P}\left\{ \int_{-S}^{S} \mathbb{I}\Big(B(s) + k(s) > x\Big) ds > T \right\},$$

where $k(s) = \mathbb{I}(s < 0)\eta_* s - \mathbb{I}(s \ge 0) c_1 s$ and

$$X_u(t) = (Z_u(t) + \eta_* t)\mathbb{I}(t < 0) + (B_*(t) - c_1 t)\mathbb{I}(t \ge 0).$$

We have for large u

$$\mathbb{E}\left\{ (X_u(t) - X_u(s))^2 \right\} = \begin{cases} |t-s| + |t-s|^2 & t,s \ge 0 \\ -\frac{(s-t)^2}{ut_*} + |t-s| + \frac{x^2(t-s)^2}{u^2 t_*^2} - \frac{2x(t-s)^2 \eta_*}{ut_*} + \eta_*^2 (t-s)^2 & t,s \le 0 \\ |t-s| - \frac{s^2}{ut_*} + \frac{x^2 s^2}{u^2 t_*^2} - \frac{2xs(\eta_* s + c_1 t)}{ut_*} + (\eta_* s + c_1 t)^2 & s < 0 < t \end{cases}$$

implying for all u large enough, some $C > 0$ and $t, s \in [-S, S+T]$ that

$$\mathbb{E}\left\{ (X_u(t) - X_u(s))^2 \right\} \le C|t-s|.$$

Next, by Proposition 9.2.4 in [14] the family $X_u(t),\ u > 0,\ t \in [-S, S+T]$ is tight in $\mathcal{B}(C([-S, S+T]))$ (Borell σ-algebra in the space of the continuous functions on $[-S, S+T]$ generated by the cylindric sets).

As follows from (15), $Z_u(t)$ converges to $B(t)$ in the sense of convergence finite-dimensional distributions as $u \to \infty$, $t \in [-S, S+T]$. Thus, by Theorems 4 and 5 in Chapter 5 in [25] the tightness and convergence of finite-dimensional distributions imply weak convergence

$$X_u(t) \Rightarrow B(t) + k(t) =: W(t), \quad t \in [-S, S+T].$$

By Skorohod representation theorem (Theorem 11, Chapter 5 in [25]) we can assume that the convergence is almost surely. Thus, we assume that $X_u(t) \to W(t)$ a.s. as $u \to \infty$ as elements of $C[-S, S]$ space with the uniform metric. We prove that for all $x \in \mathbb{R}$

$$\mathbb{P}\left\{\lim_{u\to\infty} \int_{-S}^{S} \mathbb{I}(X_u(t) > x)dt = \int_{-S}^{S} \mathbb{I}(W(t) > x)dt\right\} = 1. \tag{20}$$

Fix $x \in \mathbb{R}$. We shall show that as $u \to \infty$ with probability 1

$$\mu_\Lambda\{t \in [-S,S] : X_u(\omega,t) > x > W(\omega,t)\} + \mu_\Lambda\{t \in [-S,S] : W(\omega,t) > x > X_u(\omega,t)\} \to 0, \tag{21}$$

where μ_Λ is the Lebesgue measure. Since for any fixed $\varepsilon > 0$ for large u and $t \in [-S, S]$ with probability one $|W(t) - X_u(t)| < \varepsilon$ we have that

$$\begin{aligned} &\mu_\Lambda\{t \in [-S,S] : X_u(\omega,t) > x > W(\omega,t)\} + \mu_\Lambda\{t \in [-S,S] : W(\omega,t) > x > X_u(\omega,t)\} \\ \leq\ &\mu_\Lambda\{t \in [-S,S] : W(\omega,t) \in [-\varepsilon + x, \varepsilon + x]\}. \end{aligned}$$

Thus, (21) holds if

$$\mathbb{P}\left\{\lim_{\varepsilon\to 0} \mu_\Lambda\{t \in [-S,S] : W(t) \in [-\varepsilon + x, x + \varepsilon]\} = 0\right\} = 1. \tag{22}$$

Consider the subset $\Omega_* \subset \Omega$ consisting of all ω_* such that

$$\lim_{\varepsilon\to 0} \mu_\Lambda\{t \in [-S,S] : W(\omega_*,t) \in [-\varepsilon + x, x + \varepsilon]\} > 0.$$

Then for each ω_* there exists the set $\mathcal{A}(\omega_*) \subset [-S, S]$ such that $\mu_\Lambda\{\mathcal{A}(\omega_*)\} > 0$ and for $t \in \mathcal{A}(\omega_*)$ it holds, that $W(\omega_*, t) = x$. Thus,

$$\mathbb{P}\{\Omega_*\} = \mathbb{P}\{\mu_\Lambda\{t \in [-S,S] : W(t) = x\} > 0\},$$

the right side of the equation above equals 0 by Lemma 4.2 in [26]. Hence we conclude that (22) holds, consequently (21) and (20) are true. Since convergence almost sure implies convergence in distribution we have by (20) that for any fixed $x \in \mathbb{R}$

$$\lim_{u\to\infty} \mathbb{P}\left\{\int_{-S}^{S} \mathbb{I}(X_u(t) > x)dt > T\right\} = \mathbb{P}\left\{\int_{-S}^{S} \mathbb{I}(W(t) > x)dt > T\right\}.$$

By the dominated convergence theorem we obtain

$$\int_{-M}^{M} G(u,x)e^{\frac{\eta x}{t_*}}dx \to \int_{-M}^{M} \mathbb{P}\left\{\int_{-S}^{0} \mathbb{I}\Big(B(s) + \eta_* s > x\Big)ds + \int_{0}^{S} \mathbb{I}(B_*(s) - c_1 s > x)ds > T\right\} e^{\frac{\eta x}{t_*}}dx, \quad u \to \infty.$$

Thus, the claim follows from the line above and (18). □

Proof of (17)**.** We have by the proof of Theorem 3.1 in [21], Case (3) and the final asymptotics of $\mathbb{C}_{T_u}(u)$ given in (9)

$$\begin{aligned}
&\mathbb{P}\left\{ \int\limits_{[0,\infty)\setminus[ut_*-u\delta_u,ut_*+u\delta_u]} \mathbb{I}(B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u)dt > T_u \right\} \\
\leq\;& \mathbb{P}\{\exists t\in[0,\infty)\setminus[ut_*-u\delta_u,ut_*+u\delta_u] : B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u\} \\
=\;& o(\mathbb{C}_{T_u}(u)), \quad u\to\infty
\end{aligned}$$

and hence

$$\mathbb{P}\left\{ \int\limits_{[ut_*-u\delta_u,ut_*+u\delta_u]} \mathbb{I}(B_H(t)-c_1t>q_1u, B_H(t)-c_2t>q_2u)dt > T_u \right\} \sim \mathbb{C}_{T_u}(u), \quad u\to\infty.$$

The last probability above is equivalent with $g_1(u)+g_2(u)$ as $u\to\infty$, this observation follows from the application of the double-sum method, see the proofs of Theorem 3.1, Case (3) $H<1/2$ in [21] and Theorem 2.1 in [20] case i). □

Proof of Proposition 3.1. If $H=1/2$, then an equality takes place, see [20], Eq. [5]. Assume from now on that $H\neq 1/2$. First let (6) holds with $T>0$. We have for $c>0$ with $\widetilde{M}(u)=u^{1-H}\frac{c^H}{(1-H)^{1-H}H^H}$ (recall, $\mathbb{I}_a(b)=\mathbb{I}(b>a)$, $a,b\in\mathbb{R}$)

$$\begin{aligned}
h_{T_u}(u) &:= \mathbb{P}\left\{ \int\limits_0^\infty \mathbb{I}(B_H(t)-ct>u)dt > T_u \right\} \\
&= \mathbb{P}\left\{ u\left(u^{\frac{1}{H}-2}\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}\right) \int\limits_0^\infty \mathbb{I}_{\widetilde{M}(u)}\left(\frac{B_H(tu)\widetilde{M}(u)}{u(1+ct)}\right)dt > T\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2} \right\}.
\end{aligned}$$

Next we apply Theorem 3.1 in [20] to calculate the asymptotics of the last probability above as $u\to\infty$. For the parameters in the notation therein we have

$$\begin{aligned}
&\alpha_0=\alpha_\infty=H,\ \sigma(t)=t^H,\ \overleftarrow{\sigma}(t)=t^{\frac{1}{H}},\ t^*=\frac{H}{c(1-H)},\ A=\frac{c^H}{H^H(1-H)^{1-H}},\ x=T\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2} \\
&B=\frac{c^{2+H}(1-H)^{2+H}}{H^{H+1}},\ M(u)=u^{1-H}\frac{c^H}{(1-H)^{1-H}H^H},\ v(u)=u^{\frac{1}{H}-2}\frac{c^2(1-H)^{2-\frac{1}{H}}}{2^{\frac{1}{2H}}H^2}.
\end{aligned}$$

and hence we obtain

$$h_{T_u}(u)\sim K_H\mathcal{B}_{2H}(TD)(C_Hu^{1-H})^{\frac{1}{H}-1}\Psi(C_Hu^{1-H}), \quad u\to\infty, \tag{23}$$

where

$$C_H=\frac{c^H}{H^H(1-H)^{1-H}} \quad \text{and} \quad D=2^{-\frac{1}{2H}}c^2H^{-2}(1-H)^{2-1/H}.$$

Assume that (6) holds with $T=0$. For $\varepsilon>0$ for all large u we have $h_{\varepsilon u^{1/H-2}}(u)\leq h_{T_u}(u)\leq h_0(u)$ and thus

$$K_H\mathcal{B}_{2H}(\varepsilon D)(C_Hu^{1-H})^{\frac{1}{H}-1}\Psi(C_Hu^{1-H})\leq h_{T_u}(u)\leq K_H\mathcal{B}_{2H}(0)(C_Hu^{1-H})^{\frac{1}{H}-1}.$$

Since $\mathcal{B}_{2H}(\cdot)$ is a continuous function (Lemma 4.1 in [20]) letting $\varepsilon\to 0$ we obtain (23) for any T_u satisfying (6). Replacing in (23) u and c by q_1u and c_1 we obtain the claim. □

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