Filomat 36:14 (2022), 4665–4674
https://doi.org/10.2298/FIL2214665C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Trace Formulae for a Conformable Fractional Diffusion Operator

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Abstract. In this paper, we obtain the regularized trace formulae for a diffusion operator, which includes conformable fractional derivatives of order α $(0 < \alpha \leq 1)$ instead of the ordinary derivatives in a traditional diffusion operator by the contour integration method. The results of this paper are of great importance in solving inverse problems and can be considered as partial fractional generalizations.

1. Introduction

The fractional derivative has an important place in the applied mathematics. Since 1695, several definitions of fractional derivatives such as the Riemann–Liouville, Caputo, and Grünwald–Letnikov derivatives have been introduced by many authors (see [1]-[4]). In 2014, Khalil et al. defined the conformable fractional derivative by modifying the limit definition of the classical derivative (see [5]). Shortly after, Abdeljawad showed the elementary properties of this derivative (see [6]). Some other important ideas on the conformable derivative were given in ([7], [8], [9]). In 2017, Zhao and Luo gave a physical interpretation of this derivative (see [10]). There are many advantages of Conformable fractional derivative with respect to fractional derivatives. For example, while some mathematical properties such as the product, chain, and division rules are not provided in fractional derivatives, these properties are provided in the conformable fractional derivative. In [11], it has been realized that this derivative proves to be essential and useful in generating new types of fractional operators. The Conformable fractional derivative arises in various fields such as anomalous diffusion, variational calculus, mechanics, arbitrary time scale problems (see [12]-[17]). Recently, important studies for various operators and differential systems which include the conformable fractional derivatives have been published (see [18]-[30] and references therein).

The trace of a matrix with finite-dimensional is the sum of the elements on the main diagonal and is finite. However, the trace of ordinary differential operators with infinite-dimensional, which is the sum of all eigenvalues, is not finite. Therefore, the concept regularize trace, which is finite, is mentioned for these type operators. The regularized trace formulae have great importance, especially, in the solution of the inverse problem according to two spectra.

For about seventy years, the regularized trace formulae for the different types of differential operators have been investigated. Firstly, in 1953, Gelfand and Levitan obtained the regularized trace formula for the classical Sturm-Liouville operator with Neumann conditions (see [31]). The study led to the birth of a great and very important theory. After this study, the theory of regularized trace has been continued for

2020 *Mathematics Subject Classification*. Primary 26A33, 34B24, 11F72, 34L05, 34L20
Keywords. Diffusion operator, trace formula, conformable fractional derivative
Received: 14 February 2021; Revised: 01 May 2021; Accepted: 29 September 2021
Communicated by Snežana Č. Živković-Zlatanović
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the various operators by many researchers (see [32]-[48], and references therein). In 2010, Yang obtained the regularized trace formulas for the classical diffusion operator, i.e., a quadratic pencil of the Schrödinger operator (see [49]). For the conformable fractional differential operators, there is only one study on the trace formulation. In 2019, Mortazaasl and Jodayree Akbarfam calculated the regularized trace formula for a conformable fractional Sturm-Liouville problem (see [50]).

We consider the boundary value problem $L_\alpha(p(x), q(x), h, H) = L_\alpha$, called as the conformable fractional diffusion operator (CFDO), of the form

$$\ell_\alpha y := -T_\alpha T_\alpha y + [2\lambda p(x) + q(x)]\, y = \lambda^2 y, \quad 0 < x < \pi \tag{1}$$

$$U_\alpha(y) := T_\alpha y(0) - hy(0) = 0 \tag{2}$$

$$V_\alpha(y) := T_\alpha y(\pi) + Hy(\pi) = 0 \tag{3}$$

where λ is the spectral parameter, $h, H \in \mathbb{R}$, $q(x) \in W^1_{2,\alpha}[0,\pi]$, $p(x) \in W^2_{2,\alpha}[0,\pi]$ are real valued functions, $p(x) \neq const.$, $\alpha \in (0,1]$ and $T_\alpha y$ is a conformable fractional derivative of $y = y(x)$ of order α which is defined by

$$T_\alpha y = \lim_{h\to 0} \frac{y(x + hx^{1-\alpha}) - y(x)}{h}, \; x > 0 \text{ and } T_\alpha y(0) = \lim_{x\to 0^+} T_\alpha y(x).$$

We note that more detailed knowledge about the conformable fractional calculus can be seen in [5]-[7], [27]-[29].

In the present paper, we obtained the asymptotics of the eigenvalues and eigenfunctions of the operator L_α in the section Preliminaries and the regularized trace formulae for the operator in section Main Result.

2. Preliminaries

In this section, we will obtain the asymptotics of the eigenvalues and eigenfunctions of the operator L_α.

Let the functions $\varphi := \varphi(x, \lambda; \alpha)$ and $\psi := \psi(x, \lambda; \alpha)$ be the solutions of the equation (1) satisfying the initial conditions

$$\varphi(0, \lambda; \alpha) = 1, \; T_\alpha \varphi(0, \lambda; \alpha) = h, \tag{4}$$

$$\psi(\pi, \lambda; \alpha) = 1, \; T_\alpha \psi(\pi, \lambda; \alpha) = -H, \tag{5}$$

respectively. It is clear that $U_\alpha(\varphi) = 0$, $V_\alpha(\psi) = 0$.

Denote

$$\Delta_\alpha(\lambda) = W_\alpha[\psi, \varphi] = \begin{vmatrix} \psi & \varphi \\ T_\alpha\psi & T_\alpha\varphi \end{vmatrix} = \psi T_\alpha \varphi - \varphi T_\alpha \psi, \tag{6}$$

where $\Delta_\alpha(\lambda)$ is called as the characteristic function of the operator L_α and $W_\alpha[\psi, \varphi]$ is the fractional Wronskian of the functions ψ and φ. Furthermore, it is proven in [50] that $W_\alpha[\psi, \varphi]$ does not depend on x for each fixed α and can be written as

$$\Delta_\alpha(\lambda) = V_\alpha(\varphi) = -U_\alpha(\psi). \tag{7}$$

Lemma 2.1. *For $|\lambda| \to \infty$ and each fixed α, the following asymptotic formulae hold:*

$$\varphi(x, \lambda; \alpha) = \cos\left(\tfrac{\lambda}{\alpha}x^\alpha - P(x)\right) + O\left(\tfrac{1}{|\lambda|}\exp\left(\tfrac{|\tau|}{\alpha}x^\alpha\right)\right), \tag{8}$$

$$T_\alpha\varphi(x, \lambda; \alpha) = -(\lambda - p(x))\sin\left(\tfrac{\lambda}{\alpha}x^\alpha - P(x)\right) + O\left(\exp\left(\tfrac{|\tau|}{\alpha}x^\alpha\right)\right) \tag{9}$$

where $\lambda = \sigma + i\tau$ and

$$P(x) := \int_0^x p(t) d_\alpha t. \tag{10}$$

Proof. Based on the proof of Lemma 1 for $\alpha = 1$ in [51], we can rewrite equation (1) as follows

$$T_\alpha T_\alpha y + \frac{T_\alpha p(x)}{\lambda - p(x)} T_\alpha y + [\lambda - p(x)]^2 y = \left[q(x) + p^2(x)\right] y + \frac{T_\alpha p(x)}{\lambda - p(x)} T_\alpha y. \tag{11}$$

It is clear for each fixed α that the system of functions $\left\{\cos\left(\frac{\lambda}{\alpha}x^\alpha - P(x)\right),\ \sin\left(\frac{\lambda}{\alpha}x^\alpha - P(x)\right)\right\}$ is a fundamental system for the following differential equation

$$T_\alpha T_\alpha y + \frac{T_\alpha p(x)}{\lambda - p(x)} T_\alpha y + [\lambda - p(x)]^2 y = 0. \tag{12}$$

Thus, by the variation of parameters the solution of equation (1) satisfying the initial conditions (4) provides the following integral equations

$$\begin{aligned}\varphi(x,\lambda;\alpha) &= \cos\left(\tfrac{\lambda}{\alpha}x^\alpha - P(x)\right) + \tfrac{h}{\lambda - p(0)}\sin\left(\tfrac{\lambda}{\alpha}x^\alpha - P(x)\right)\\ &+ \int_0^x \tfrac{\sin\left[\frac{\lambda}{\alpha}(x^\alpha - t^\alpha) - P(x) + P(t)\right]}{\lambda - p(t)}\left[\left(q(t) + p^2(t)\right)\varphi(t,\lambda;\alpha) + \tfrac{T_\alpha p(t)}{\lambda - p(t)} T_\alpha \varphi(t,\lambda;\alpha)\right] d_\alpha t\end{aligned} \tag{13}$$

and by $\alpha-$differentiating (13) with respect to x,

$$\begin{aligned}T_\alpha\varphi(x,\lambda;\alpha) &= -(\lambda - p(x))\left\{\sin\left(\tfrac{\lambda}{\alpha}x^\alpha - P(x)\right) - \tfrac{h}{\lambda - p(0)}\cos\left(\tfrac{\lambda}{\alpha}x^\alpha - P(x)\right)\right.\\ &\left.+ \int_0^x \tfrac{\cos\left[\frac{\lambda}{\alpha}(x^\alpha - t^\alpha) - P(x) + P(t)\right]}{\lambda - p(t)}\left[\left(q(t) + p^2(t)\right)\varphi(t,\lambda;\alpha) + \tfrac{T_\alpha p(t)}{\lambda - p(t)} T_\alpha \varphi(t,\lambda;\alpha)\right] d_\alpha t\right\}.\end{aligned} \tag{14}$$

For each fixed α, denote

$$\mu_1(\lambda) := \max_{0\le x\le\pi}\left|\varphi(x,\lambda;\alpha)\exp\left(-\tfrac{|\tau|}{\alpha}x^\alpha\right)\right|,\ \mu_2(\lambda) := \max_{0\le x\le\pi}\left|T_\alpha\varphi(x,\lambda;\alpha)\exp\left(-\tfrac{|\tau|}{\alpha}x^\alpha\right)\right|.$$

Since $\left|\cos\left(\frac{\lambda}{\alpha}x^\alpha - P(x)\right)\right| \le \exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)$ and $\left|\sin\left(\frac{\lambda}{\alpha}x^\alpha - P(x)\right)\right| \le \exp\left(\frac{|\tau|}{\alpha}x^\alpha\right)$, we deduce from (13) and (14) that

$$\mu_1(\lambda) \le C\left(1 + \tfrac{|h|}{|\lambda|} + \tfrac{\mu_1(\lambda)}{|\lambda|} + \tfrac{\mu_2(\lambda)}{|\lambda|^2}\right),\ \mu_2(\lambda) \le C\left(|\lambda| + |h| + \mu_1(\lambda) + \tfrac{\mu_2(\lambda)}{|\lambda|}\right).$$

Hence, we get

$$\mu_1(\lambda) \le C,\ \mu_2(\lambda) \le C|\lambda|$$

or

$$\varphi(x,\lambda;\alpha) = O\left(\exp\left(\tfrac{|\tau|}{\alpha}x^\alpha\right)\right),\ T_\alpha\varphi(x,\lambda;\alpha) = O\left(\lambda\exp\left(\tfrac{|\tau|}{\alpha}x^\alpha\right)\right),\ |\lambda| \to \infty.$$

Substituting these into (13) and (14) we obtain (8) and (9). □

Applying successive approximations method to the equations (13), we can have the more detalied asymptotic of the function $\varphi(x,\lambda;\alpha)$ as follows

$$\begin{aligned}
\varphi(x,\lambda;\alpha) &= \cos\left(\tfrac{\lambda}{\alpha}x^{\alpha}-P(x)\right)+\tfrac{p(x)-p(0)}{2\lambda}\cos\left(\tfrac{\lambda}{\alpha}x^{\alpha}-P(x)\right)\\
&+\tfrac{1}{\lambda}\left(h+\tfrac{1}{2}\int_0^x\left(q(t)+p^2(t)\right)d_\alpha t\right)\sin\left(\tfrac{\lambda}{\alpha}x^{\alpha}-P(x)\right)\\
&+\tfrac{1}{2\lambda}\int_0^x\left(q(t)+p^2(t)\right)\sin\left(\tfrac{\lambda}{\alpha}\left(x^{\alpha}-2t^{\alpha}\right)-P(x)+2P(t)\right)d_\alpha t\\
&-\tfrac{1}{2\lambda}\int_0^x T_\alpha p(t)\cos\left(\tfrac{\lambda}{\alpha}\left(x^{\alpha}-2t^{\alpha}\right)-P(x)+2P(t)\right)d_\alpha t\\
&+\tfrac{1}{\lambda^2}\left[\tfrac{h\left(p(x)+p(0)\right)}{2}+\tfrac{1}{4}\int_0^x\left(q(t)+p^2(t)\right)\left(p(x)-p(0)+2p(t)\right)d_\alpha t\right]\sin\left(\tfrac{\lambda}{\alpha}x^{\alpha}-P(x)\right)\\
&+\tfrac{1}{\lambda^2}\left[\tfrac{p^{1+\alpha}(x)-p^{1+\alpha}(0)}{2}+\tfrac{\left(p(x)-p(0)\right)^{1+\alpha}}{4(1+\alpha)}-\tfrac{h}{2}\int_0^x\left(q(t)+p^2(t)\right)d_\alpha t-\tfrac{1}{8}\left(\int_0^x\left(q(t)+p^2(t)\right)d_\alpha t\right)^2\right]\cos\left(\tfrac{\lambda}{\alpha}x^{\alpha}-P(x)\right)\\
&+O\left(\tfrac{1}{|\lambda|^3}\exp\left(\tfrac{|\tau|}{\alpha}x^{\alpha}\right)\right),\ |\lambda|\to\infty,
\end{aligned}\tag{15}$$

uniformly with respect to $x\in[0,\pi]$, for each fixed α.

The eigenvalues of L_α coincide with the zeros of its characteristic function $\Delta_\alpha(\lambda)=V_\alpha(\varphi)=T_\alpha\varphi(\pi,\lambda;\alpha)+H\varphi(\pi,\lambda;\alpha)$. Thus, using the formula (15), we can establish the following asymptotic

$$\begin{aligned}
\Delta_\alpha(\lambda) &= -\lambda\sin\left(\tfrac{\lambda}{\alpha}\pi^{\alpha}-c_0\right)+\tfrac{\left(p(\pi)+p(0)\right)}{2}\sin\left(\tfrac{\lambda}{\alpha}\pi^{\alpha}-c_0\right)+c_1\cos\left(\tfrac{\lambda}{\alpha}\pi^{\alpha}-c_0\right)+\tfrac{c_2}{\lambda}\sin\left(\tfrac{\lambda}{\alpha}\pi^{\alpha}-c_0\right)\\
&+\tfrac{c_3}{\lambda}\cos\left(\tfrac{\lambda}{\alpha}\pi^{\alpha}-c_0\right)+\tfrac{1}{2}\int_0^{\pi}\left(q(t)+p^2(t)\right)\cos\left[\tfrac{\lambda}{\alpha}\left(\pi^{\alpha}-2t^{\alpha}\right)-c_0+2P(t)\right]d_\alpha t\\
&+\tfrac{1}{2}\int_0^{\pi}T_\alpha p(t)\sin\left[\tfrac{\lambda}{\alpha}\left(\pi^{\alpha}-2t^{\alpha}\right)-c_0+2P(t)\right]d_\alpha t+O\left(\tfrac{1}{|\lambda|^2}\exp\left(\tfrac{|\tau|}{\alpha}\pi^{\alpha}\right)\right),\ |\lambda|\to\infty,
\end{aligned}\tag{16}$$

where,

$$c_0=P(\pi)=\int_0^{\pi}p(t)d_\alpha t,$$

$$c_1=h+H+\tfrac{1}{2}\int_0^{\pi}\left(q(t)+p^2(t)\right)d_\alpha t,$$

$$c_2=\tfrac{p(\pi)\left(p(\pi)-p(0)\right)}{2}-\tfrac{p^{1+\alpha}(\pi)-p^{1+\alpha}(0)}{2}-\tfrac{\left(p(\pi)-p(0)\right)^{1+\alpha}}{4(1+\alpha)}+hH+\tfrac{h+H}{2}\int_0^{\pi}\left(q(t)+p^2(t)\right)d_\alpha t+\tfrac{1}{8}\left(\int_0^{\pi}\left(q(t)+p^2(t)\right)d_\alpha t\right)^2,$$

$$c_3=\tfrac{(H-h)\left(p(\pi)-p(0)\right)}{2}+\tfrac{1}{4}\int_0^{\pi}\left(q(t)+p^2(t)\right)\left(2p(t)-p(\pi)-p(0)\right)d_\alpha t.$$

Take a circle $\Gamma_N=\left\{\lambda \mid |\lambda|=\tfrac{\alpha}{\pi^{\alpha-1}}\left(N+\tfrac{1}{2}\right),\ N=0,1,2,\ldots\right\}$ in the $\lambda-$plane for each fixed α. By the standard method using (16) and Rouche's theorem (see [52]) and taking $\Delta_\alpha(\lambda_n)=0$, one can prove that in the circle Γ_N, there exist exactly $|n|$ eigenvalues λ_n and have the form

$$\lambda_n=\frac{n\alpha}{\pi^{\alpha-1}}+\frac{\alpha c_0}{\pi^{\alpha}}+\frac{c_1+A_n}{n\pi}+O\left(\frac{1}{n^2}\right),\ |n|\to\infty,\tag{17}$$

where $n\in\mathbb{Z}$ and

$$A_n=\tfrac{1}{2}\int_0^{\pi}\left(q(t)+p^2(t)\right)\cos\left(\tfrac{2nt^{\alpha}}{\pi^{\alpha-1}}+\tfrac{2c_0t^{\alpha}}{\pi^{\alpha}}-2P(t)\right)d_\alpha t-\tfrac{1}{2}\int_0^{\pi}T_\alpha p(t)\sin\left(\tfrac{2nt^{\alpha}}{\pi^{\alpha-1}}+\tfrac{2c_0t^{\alpha}}{\pi^{\alpha}}-2P(t)\right)d_\alpha t.$$

Corollary 2.2. *According to (17), for each fixed α and sufficiently large $|n|$, the eigenvalues λ_n are real and simple.*

3. Main Result

In this section, we will find two regularized trace formulae for the operator L_α using the contour integration method as in [49].

Theorem 3.1. *Let $\{\lambda_n\}_{n\geq 0}$ be the sequence of the eigenvalues of the operator L_α. Then, for each fixed α, the following trace formulae are valid:*

$$2(\lambda_0 - c_0) + \sum_{n=1}^{\infty}\left[\lambda_n + \lambda_{-n} - 2c_0 - \frac{1}{\alpha\pi^{1-\alpha}}\frac{B_n}{n}\right] = \frac{p(\pi)+p(0)-2c_0}{2}$$
$$-\frac{1}{2}\int_0^\pi \left(1 - \frac{2t^\alpha}{\alpha}\right)\left(q(t)+p^2(t)\right)\sin 2P(t)d_\alpha t + \frac{1}{2}\int_0^\pi\left(1-\frac{2t^\alpha}{\alpha}\right)T_\alpha p(t)\cos 2P(t)d_\alpha t \tag{18}$$

and

$$2(\lambda_0 - c_0)^2 + \sum_{n=1}^{\infty}\left[(\lambda_n - c_0)^2 + (\lambda_{-n}-c_0)^2 - 2\left(\frac{n\alpha}{\pi^{1-\alpha}}\right)^2 - \frac{4\alpha c_1}{\pi^\alpha} - \frac{2\alpha C_n}{\pi^\alpha}\right]$$
$$= \frac{2\alpha c_1}{\pi^\alpha} + \frac{\alpha}{\pi^\alpha}\int_0^\pi\left(q(t)+p^2(t)\right)\cos 2P(t)d_\alpha t + \frac{\alpha}{\pi^\alpha}\int_0^\pi T_\alpha p(t)\sin 2P(t)d_\alpha t \tag{19}$$
$$+(p(\pi)-c_0)(p(\pi)-p(0)) - (p(\pi)-c_0)^{1+\alpha} + (p(0)-c_0)^{1+\alpha} - \frac{\left(p(\pi)-p(0)\right)^{1+\alpha}}{2(1+\alpha)}$$
$$+2hH + (h+H)\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t + \frac{1}{4}\left(\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right)^2,$$

where

$B_n = \int_0^\pi \left(q(t)+p^2(t)\right)\sin\left(\frac{2n\alpha t^\alpha}{\pi^{2\alpha-1}}\right)\sin 2P(t)d_\alpha t - \int_0^\pi T_\alpha p(t)\sin\left(\frac{2n\alpha t^\alpha}{\pi^{2\alpha-1}}\right)\cos 2P(t)d_\alpha t,$

$C_n = \int_0^\pi \left(q(t)+p^2(t)\right)\cos\left(\frac{2n\alpha t^\alpha}{\pi^{2\alpha-1}}\right)\cos 2P(t)d_\alpha t + \int_0^\pi T_\alpha p(t)\cos\left(\frac{2n\alpha t^\alpha}{\pi^{2\alpha-1}}\right)\sin 2P(t)d_\alpha t.$

Proof. Firstly, we consider the case $c_0 = 0$.

Denote

$$\Delta_\alpha^0(\lambda) = -\lambda\sin\left(\frac{\lambda}{\alpha}\pi^\alpha\right). \tag{20}$$

It is clear for each fixed α that the zeros of the function $\Delta_\alpha^0(\lambda)$ is

$$\mu_n = \frac{n\alpha}{\pi^{\alpha-1}},\ n\in\mathbb{Z},$$

where only $\mu_0 = 0$ is double. We note that for each fixed α and sufficiently large N, the eigenvalues λ_n which are the zeros of $\Delta_\alpha(\lambda)$ are inside Γ_N and the numbers μ_n do not lie on the contour Γ_N.

Let $\Delta_\alpha(\lambda) = \lambda(\lambda - \lambda_n)$ and $\Delta_\alpha^0(\lambda) = \lambda\left(\lambda - \frac{n\alpha}{\pi^{\alpha-1}}\right)$, then we have from the logaritmic derivatives of the functions $\Delta_\alpha(\lambda)$ and $\Delta_\alpha^0(\lambda)$ that

$$\lambda\frac{\dot{\Delta}_\alpha(\lambda)}{\Delta_\alpha(\lambda)} = \frac{2\lambda-\lambda_n}{\lambda-\lambda_n} \text{ and } \lambda\frac{\dot{\Delta}_\alpha^0(\lambda)}{\Delta_\alpha^0(\lambda)} = \frac{2\lambda - \frac{n\alpha}{\pi^{\alpha-1}}}{\lambda - \frac{n\alpha}{\pi^{\alpha-1}}},$$

respectively where $\dot{\Delta} = \frac{d}{d\lambda}$.

Thus, from the residue theorem, the following equalities are valid:

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\lambda\frac{\dot{\Delta}_\alpha(\lambda)}{\Delta_\alpha(\lambda)}d\lambda = \sum_{n=-N}^{N}\operatorname{Res}\left(\lambda\frac{\dot{\Delta}_\alpha(\lambda)}{\Delta_\alpha(\lambda)},\lambda_n\right) = \sum_{n=0}^{N}(\lambda_n + \lambda_{-n}) \tag{21}$$

and similarly

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\lambda\frac{\dot{\Delta}^0_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)}d\lambda=\sum_{n=-N}^{N}\operatorname{Res}\left(\lambda\frac{\dot{\Delta}^0_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)},\frac{n\alpha}{\pi^{\alpha-1}}\right)=\sum_{n=0}^{N}\left[\frac{n\alpha}{\pi^{\alpha-1}}+\left(-\frac{n\alpha}{\pi^{\alpha-1}}\right)\right]. \tag{22}$$

Subtracting (21) and (22) side by side, we get

$$\sum_{n=0}^{N}(\lambda_n+\lambda_{-n})=\frac{1}{2\pi i}\oint_{\Gamma_N}\lambda\left(\frac{\dot{\Delta}_\alpha(\lambda)}{\Delta_\alpha(\lambda)}-\frac{\dot{\Delta}^0_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)}\right)d\lambda=\frac{1}{2\pi i}\oint_{\Gamma_N}\lambda d\left(\ln\frac{\Delta_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)}\right)=-\frac{1}{2\pi i}\oint_{\Gamma_N}\ln\frac{\Delta_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)}d\lambda. \tag{23}$$

On the other hand, it follows from (16) and (20) that

$$\frac{\Delta_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)}=1-\frac{p(\pi)+p(0)+2A(\lambda)}{2\lambda}-\frac{c_1+B(\lambda)}{\lambda}\cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right)-\frac{c_2}{\lambda^2}-\frac{c_3}{\lambda^2}\cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right)+O\left(\frac{1}{\lambda^3}\exp\left(\frac{|\tau|}{\alpha}\pi^\alpha\right)\right),\ \text{on }\Gamma_N, \tag{24}$$

where

$A(\lambda)=\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)\sin\left(\frac{2\lambda t^\alpha}{\alpha}-2P(t)\right)d_\alpha t+\frac{1}{2}\int_0^\pi T_\alpha p(t)\cos\left(\frac{2\lambda t^\alpha}{\alpha}-2P(t)\right)d_\alpha t,$

$B(\lambda)=\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)\cos\left(\frac{2\lambda t^\alpha}{\alpha}-2P(t)\right)d_\alpha t-\frac{1}{2}\int_0^\pi T_\alpha p(t)\sin\left(\frac{2\lambda t^\alpha}{\alpha}-2P(t)\right)d_\alpha t.$

Taking Taylor's expansion formula for the $\ln(1-u)$ into account, we get

$$\ln\left(\frac{\Delta_\alpha(\lambda)}{\Delta^0_\alpha(\lambda)}\right)=-\frac{p(\pi)+p(0)+2A(\lambda)}{2\lambda}-\frac{c_1+B(\lambda)}{\lambda}\cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right)+O\left(\frac{1}{\lambda^2}\right),\ \text{on }\Gamma_N. \tag{25}$$

Thus, substituting (25) into (23), we find

$$\sum_{n=0}^{N}(\lambda_n+\lambda_{-n})=\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{p(\pi)+p(0)+2A(\lambda)}{2\lambda}d\lambda+\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{c_1+B(\lambda)}{\lambda}\cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right)d\lambda+\frac{1}{2\pi i}\oint_{\Gamma_N}O\left(\frac{1}{\lambda^2}\right)d\lambda. \tag{26}$$

By the well-known formulas such as the generalized Cauchy integral formula, the residue theorem and $\cot z=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}$ (see [53] for more details), the contour integrals in (26) calculate that

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{p(\pi)+p(0)+2A(\lambda)}{2\lambda}d\lambda=\frac{p(\pi)+p(0)}{2}+A(0), \tag{27}$$

$$\frac{1}{2\pi i}\oint_{\Gamma_N}\frac{c_1+B(\lambda)}{\lambda}\cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right)d\lambda=\frac{\alpha\dot{B}(0)}{\pi^\alpha}+\frac{1}{\alpha\pi^{1-\alpha}}\sum_{n=1}^{N}\frac{1}{n}\left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)-B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right] \tag{28}$$

and for sufficiently large N and each fixed α

$$\left|\oint_{\Gamma_N}O\left(\frac{1}{\lambda^2}\right)d\lambda\right|=O\left(\frac{1}{N}\right). \tag{29}$$

From (26)-(29), we get

$$2\lambda_0+\sum_{n=1}^{N}\left\{\lambda_n+\lambda_{-n}-\frac{1}{n\alpha\pi^{1-\alpha}}\left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)-B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right]\right\}=\frac{p(\pi)+p(0)}{2}+A(0)+\frac{\alpha\dot{B}(0)}{\pi^\alpha}+O\left(\frac{1}{N}\right). \tag{30}$$

For $N\to\infty$ in (30),

$$2\lambda_0+\sum_{n=1}^{\infty}\left\{\lambda_n+\lambda_{-n}-\frac{1}{n\alpha\pi^{1-\alpha}}\left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)-B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right]\right\}=\frac{p(\pi)+p(0)}{2}+A(0)+\frac{\alpha\dot{B}(0)}{\pi^\alpha} \tag{31}$$

is obtained.

Now we consider the case $c_0\neq 0$.

It is obvious that we can rewrite the equation (1) as

$$-T_\alpha T_\alpha y + \left[2(\lambda - c_0)(p(x) - c_0) + q(x) + 2c_0 p(x) - c_0^2\right] y = (\lambda - c_0)^2 y.$$

Denote $\lambda - c_0 = \widetilde{\lambda}$, $q(x) + 2c_0 p(x) - c_0^2 = \widetilde{q}(x)$ and $p(x) - c_0 = \widetilde{p}(x)$. Thus we get that

$$-T_\alpha T_\alpha y + \left[2\widetilde{\lambda}\widetilde{p}(x) + \widetilde{q}(x)\right] y = \widetilde{\lambda}^2 y. \tag{32}$$

For the equation (32), in the case $\widetilde{c_0} = \int_0^\pi \widetilde{p}(t) d_\alpha t = 0$, according to (31), we obtain

$$2\widetilde{\lambda}_0 + \sum_{n=1}^{\infty} \left[\widetilde{\lambda}_n + \widetilde{\lambda}_{-n} - \frac{B_n}{n\alpha\pi^{1-\alpha}}\right] = \frac{\widetilde{p}(\pi) + \widetilde{p}(0)}{2} + \widetilde{A}(0) + \frac{\alpha \dot{\widetilde{B}}(0)}{\pi^\alpha}, \tag{33}$$

where

$$A\left(\widetilde{\lambda}\right) := \widetilde{A}(\lambda) = A(\lambda), B\left(\widetilde{\lambda}\right) := \widetilde{B}(\lambda) = B(\lambda) \text{ and } \widetilde{B}\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) - \widetilde{B}\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) = B_n.$$

Substituting the expressions of $\widetilde{\lambda}_n$, $\widetilde{q}(x)$ and $\widetilde{p}(x)$ into (33), we arrive at (18).

Similarly, we prove that the formula (19) is true.

We consider the case $c_0 = 0$ again. Denote $\Delta_\alpha(\lambda) = \lambda^2(\lambda - \lambda_n)$ and $\Delta_\alpha^0(\lambda) = \lambda^2\left(\lambda - \frac{n\alpha}{\pi^{\alpha-1}}\right)$. Then, we have

$$\sum_{n=0}^{N} \left[\lambda_n^2 + \lambda_{-n}^2 - \left(\frac{n\alpha}{\pi^{\alpha-1}}\right)^2 - \left(-\frac{n\alpha}{\pi^{\alpha-1}}\right)^2\right] = \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda^2 \left(\frac{\dot{\Delta}_\alpha(\lambda)}{\Delta_\alpha(\lambda)} - \frac{\dot{\Delta}_\alpha^0(\lambda)}{\Delta_\alpha^0(\lambda)}\right) d\lambda$$

$$= \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda^2 d\left(\ln \frac{\Delta_\alpha(\lambda)}{\Delta_\alpha^0(\lambda)}\right) = -\frac{1}{2\pi i} \oint_{\Gamma_N} 2\lambda \ln \frac{\Delta_\alpha(\lambda)}{\Delta_\alpha^0(\lambda)} d\lambda. \tag{34}$$

From (24), we obtain

$$\ln\left(\frac{\Delta_\alpha(\lambda)}{\Delta_\alpha^0(\lambda)}\right) = -\frac{p(\pi) + p(0) + 2A(\lambda)}{2\lambda} - \frac{c_1 + B(\lambda)}{\lambda} \cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right) - \frac{c_2}{\lambda^2} - \frac{c_3}{\lambda^2} \cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right) + O\left(\frac{1}{\lambda^3}\right), \text{ on } \Gamma_N. \tag{35}$$

Thus, the integral on the right side of (34) is written as

$$-\frac{1}{2\pi i} \oint_{\Gamma_N} 2\lambda \ln \frac{\Delta_\alpha(\lambda)}{\Delta_\alpha^0(\lambda)} d\lambda = \frac{1}{2\pi i} \oint_{\Gamma_N} [p(\pi) + p(0) + 2A(\lambda)] d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_N} 2[c_1 + B(\lambda)] \cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right) d\lambda$$

$$+ \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2c_2}{\lambda} d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2c_3}{\lambda} \cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right) d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_N} O\left(\frac{1}{\lambda^2}\right) d\lambda\ . \tag{36}$$

For the contour integrals in (36), we have

$$\frac{1}{2\pi i} \oint_{\Gamma_N} [p(\pi) + p(0) + 2A(\lambda)] d\lambda = 0\ , \tag{37}$$

$$\frac{1}{2\pi i} \oint_{\Gamma_N} 2[c_1 + B(\lambda)] \cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right) d\lambda = \frac{2\alpha c_1}{\pi^\alpha} + \frac{4\alpha c_1}{\pi^\alpha} N + \frac{2\alpha B(0)}{\pi^\alpha} + \frac{2\alpha}{\pi^\alpha} \sum_{n=1}^{N} \left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) + B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right], \tag{38}$$

$$\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2c_2}{\lambda} d\lambda = 2c_2\ , \tag{39}$$

$$\frac{1}{2\pi i} \oint_{\Gamma_N} \frac{2c_3}{\lambda} \cot\left(\frac{\lambda}{\alpha}\pi^\alpha\right) d\lambda = 0\ . \tag{40}$$

Substituting the expressions of (29) and (37)-(40) into (36), we arrive

$$-\frac{1}{2\pi i}\oint_{\Gamma_N} 2\lambda \ln \frac{\Delta_\alpha(\lambda)}{\Delta_\alpha^0(\lambda)} d\lambda = \frac{2\alpha c_1}{\pi^\alpha} + \frac{4\alpha c_1}{\pi^\alpha}N + \frac{2\alpha B(0)}{\pi^\alpha} + \frac{2\alpha}{\pi^\alpha}\sum_{n=1}^{N}\left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) + B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right] + 2c_2 + O\left(\frac{1}{N}\right)$$

and from (34)

$$2\lambda_0^2 + \sum_{n=0}^{N}\left\{\lambda_n^2 + \lambda_{-n}^2 - 2\left(\frac{n\alpha}{\pi^{\alpha-1}}\right)^2 - \frac{4\alpha c_1}{\pi^\alpha} - \frac{2\alpha}{\pi^\alpha}\left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) + B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right]\right\} = \frac{2\alpha c_1}{\pi^\alpha} + \frac{2\alpha B(0)}{\pi^\alpha} + 2c_2 + O\left(\frac{1}{N}\right). \quad (41)$$

For $N \to \infty$ in (41),

$$2\lambda_0^2 + \sum_{n=0}^{\infty}\left\{\lambda_n^2 + \lambda_{-n}^2 - 2\left(\frac{n\alpha}{\pi^{\alpha-1}}\right)^2 - \frac{4\alpha c_1}{\pi^\alpha} - \frac{2\alpha}{\pi^\alpha}\left[B\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) + B\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)\right]\right\} = \frac{2\alpha c_1}{\pi^\alpha} + \frac{2\alpha B(0)}{\pi^\alpha} + 2c_2 \quad (42)$$

is obtained.

In the case $c_0 \neq 0$, from (42), we can write that

$$2\widetilde{\lambda}_0^2 + \sum_{n=0}^{\infty}\left[\widetilde{\lambda}_n^2 + \widetilde{\lambda}_{-n}^2 - 2\left(\frac{n\alpha}{\pi^{\alpha-1}}\right)^2 - \frac{4\alpha\widetilde{c}_1}{\pi^\alpha} - \frac{2\alpha C_n}{\pi^\alpha}\right] = \frac{2\alpha\widetilde{c}_1}{\pi^\alpha} + \frac{2\alpha\widetilde{B}(0)}{\pi^\alpha} + 2\widetilde{c}_2 \,. \quad (43)$$

Substituting the known expressions of $\widetilde{\lambda}_n$, $\widetilde{q}(x)$ and $\widetilde{p}(x)$ into (43), we arrive the formula (19), where

$C_n = \widetilde{B}\left(\frac{n\alpha^2}{\pi^{2\alpha-1}}\right) + \widetilde{B}\left(-\frac{n\alpha^2}{\pi^{2\alpha-1}}\right)$,

$\widetilde{c}_1 = c_1$,

$\widetilde{c}_2 = \frac{\left(p(\pi)-c_0\right)\left(p(\pi)-p(0)\right)}{2} - \frac{\left(p(\pi)-c_0\right)^{1+\alpha}+\left(p(0)-c_0\right)^{1+\alpha}}{2} - \frac{\left(p(\pi)-p(0)\right)^{1+\alpha}}{4(1+\alpha)} + hH + \frac{(h+H)}{2}\int_0^\pi \left(q(t)+p^2(t)\right)d_\alpha t$

$+\frac{1}{8}\left(\int_0^\pi \left(q(t)+p^2(t)\right)d_\alpha t\right)^2$. □

4. Conclusion

In this paper, we suggested a diffusion operator with conformable fractional derivatives and obtained the trace formulae for this operator. This results can be considered as a fractional generalization of the trace formulae for the classical diffusion operator in [49]. We believe that the researchers can study inverse problems according to two spectra for the given operator by using these formulae.

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