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Characterization of Inner Product Spaces by Unitary Carlsson Type Orthogonalities

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Abstract. In this study, we consider the Hermite-Hadamard type of unitary Carlsson's orthogonality (UHH-C-orthogonality) to characterize real inner product spaces. We give a necessary and sufficient condition weaker than the homogeneity of symmetric HH-C-orthogonalities which characterizes inner product spaces among normed linear spaces of dimension at least three. In conclusion, some more characterizations of real inner product spaces are provided.

1. Introduction

It is well-known that when the norm of a normed linear space is induced by an inner product, the orthogonality notion is defined in a unique way. In fact, in an inner product space $(X, \langle\cdot,\cdot\rangle)$ over the field of real numbers $\mathbb{R}$, a vector $x \in X$ is said to be orthogonal to a vector $y \in X$, denoted by $x \perp y$, if $\langle x, y\rangle = 0$. However, there is no unique way to define the concept of orthogonality in general normed linear spaces. During the 20th century, many generalized orthogonality notions have been introduced and studied in normed linear spaces.

Birkhoff-James orthogonality is one of the most important orthogonality types that was introduced by Birkhoff in [5], and then was developed by James in [13]. Throughout this paper, $(X, \|\cdot\|)$ always denotes a real normed linear space. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ in the sense of Birkhoff-James, written as $x \perp_B y$, if

$$\|x\| \le \|x + \lambda y\| \quad (\forall \lambda \in \mathbb{R}).$$

Another well-known concepts of orthogonality is the family of Carlsson's orthogonality [6]. A vector $x \in X$ is called Carlsson's orthogonal to a vector $y \in X$, denoted by $x \perp_C y$, if $\sum_{i=1}^{m} \alpha_i \|\beta_i x + \gamma_i y\|^2 = 0$, where $m \in \mathbb{N}$, α_i, β_i and γ_i $(i = 1, \dots, m)$, are given real numbers satisfying

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1. \tag{1}$$

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It is known that Pythagorean orthogonality and isosceles orthogonality (see [12]) are special cases of Carlsson's orthogonality; see e.g., [1, 2]. In 2010, Kikianty and Dragomir [15] introduced the p-HH-norms $(1 \leq p < \infty)$ on the Cartesian square of a real normed linear space. Since then they introduced Hermite-Hadamard type of Pythagorean and isosceles orthogonalities by utilizing the 2-HH-norm in [11]. Precisely,

- A vector $x \in X$ is called HH-I-orthogonal to a vector $y \in X$, denoted by $x \perp_{HH-I} y$, if and only if

$$\int_0^1 \|(1-t)x - ty\|^2 dt = \int_0^1 \|(1-t)x + ty\|^2 dt.$$

- A vector $x \in X$ is called HH-P-orthogonal to a vector $y \in X$, denoted by $x \perp_{HH-P} y$, if and only if

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \frac{1}{3}(\|x\|^2 + \|y\|^2).$$

Generally, they introduced and studied Hermite-Hadamard type of Carlsson's orthogonality in [17]:

- A vector $x \in X$ is called HH-C-orthogonal to a vector $y \in X$, denoted by $x \perp_{HH-C} y$, if and only if

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 dt = 0,$$

where $m \in \mathbb{N}$, α_i, β_i and γ_i $(i = 1, \ldots, m)$, are given real numbers satisfying (1).

For more information about different types of orthogonality in normed linear spaces the reader is referred to [1–4, 7, 19, 20] and the references therein.

It is known that all of the relationships presented above coincide with the standard orthogonality given by the inner product. The most geometric properties of inner product spaces like strict convexity and smoothness may fail to hold in a general normed linear spaces. Also, some main properties of orthogonality in inner product spaces do not always carry over to generalized orthogonalities. Taking these into account different types of orthogonality notions provide good tools for studying the geometric properties of normed linear spaces. In particular, there are interesting characterizations of inner product spaces connected with the notions of orthogonality in normed linear spaces; see e.g., [1–4, 6–8, 12–14]. For instance, James [12] proved that Pythagorean (isosceles) orthogonality is homogeneous only in real inner product spaces. Some other characterizations of inner product spaces by using some weakened hypothesis of the homogeneity of the Pythagorean and isosceles orthogonalities were presented by Alonso in [1]. More generally, Carlsson [6] proved that Carlsson's orthogonality is homogeneous in a real normed linear space X if and only if X is an inner product space. It was proved in [11] that HH-P- and HH-I-orthogonalities are homogeneous (additive) only in real inner product spaces. Generally, it was proved in [17] that HH-C-orthogonality in a real normed linear space X is homogeneous (additive) if and only if X is an inner product space. Some more characterizations of the real inner product spaces using the notion of HH-C-orthogonality and by considering their relationships with Birkhoff–James orthogonality have been provided in [10].

Alonso and Benítez [1, 2] investigated the family of unitary Carlsson's orthogonality. Some particular members of the unitary Carlsson's orthogonality have been considered separately in [9, 18]. Precisely, unitary isosceles (Singer) orthogonality introduced by Singer in [18]. Also, unitary Pythagorean orthogonality considered by Diminnie, Andalafte and Freese in [9]. Analogously, in this paper, we consider the family of Hermite-Hadamard type of unitary Carlsson's orthogonality, namely, UHH-C-orthogonality and its special cases UHH-I- and UHH-P-orthogonalities in real normed linear spaces. First, some basic properties of UHH-C-orthogonality are studied. In particular, we prove that UHH-C-orthogonality has the existence property. As a main result, we give a necessary and sufficient condition weaker than the homogeneity of symmetric HH-C-orthogonality relationships in real normed linear space X with dimension at least 3, under which the norm of X comes from an inner product. Furthermore, some more characterizations of real inner product spaces are provided.

2. Main results

The main properties of HH-C-orthogonality and its symmetric special cases, HH-P-orthogonality and HH-I-orthogonality were obtained by Dragomir and Kikianty in [11, 17]. In particular, it was proved that HH-C-orthogonality satisfies nondegeneracy and continuity property. In addition, HH-C-orthogonality has the existence property; cf. [17]. According to [16, P. 113], the family of Hermite-Hadamard type of unitary Carlsson's orthogonality is defined as follows:

Definition 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then a vector $x \in X$ is called UHH-C-orthogonal to a vector $y \in X$, denoted by $x \perp_{UHH-C} y$, if $\|x\|\,\|y\| = 0$ or $\frac{x}{\|x\|} \perp_{HH-C} \frac{y}{\|y\|}$, that is*

$$x \perp_{UHH-C} y \Leftrightarrow \|x\|\,\|y\| = 0 \text{ or } \sum_{i=1}^{m} \alpha_i \int_0^1 \Big\|(1-t)\beta_i \frac{x}{\|x\|} + t\gamma_i \frac{y}{\|y\|}\Big\|^2 dt = 0,$$

where α_i, β_i and γ_i are given real numbers satisfying (1). *In particular, unitary version of HH-I- and HH-P-orthogonality are defined analogously as follows:*

$$x \perp_{UHH-I} y \Leftrightarrow \|x\|\,\|y\| = 0 \quad or \quad \frac{x}{\|x\|} \perp_{HH-I} \frac{y}{\|y\|}.$$
$$x \perp_{UHH-P} y \Leftrightarrow \|x\|\,\|y\| = 0 \quad or \quad \frac{x}{\|x\|} \perp_{HH-P} \frac{y}{\|y\|}.$$

First, we note that UHH-C-orthogonality is nondegenerate. Indeed, if $x \in X$, $x \perp_{UHH-C} x$ and $x \neq 0$, then

$$0 = \sum_{i=1}^{m} \alpha_i \int_0^1 \Big\|(1-t)\beta_i \frac{x}{\|x\|} + t\gamma_i \frac{x}{\|x\|}\Big\|^2 dt = \sum_{i=1}^{m} \alpha_i \int_0^1 ((1-t)\beta_i + t\gamma_i)^2 dt = \sum_{i=1}^{m} \alpha_i\beta_i\gamma_i \int_0^1 2t(1-t)dt = \frac{1}{3},$$

which is impossible.

Also, the continuity property of UHH-C-orthogonality follows directly from the continuity of HH-C-orthogonality (see [17]). It was noticed in [17] that HH-C-orthogonality is not homogeneous in normed linear spaces. However, it is easy to see that UHH-C-orthogonality is positively homogeneous. In fact, assume that $x, y \in X$ are nonzero vectors such that $x \perp_{UHH-C} y$ and $\lambda > 0$. Then $\frac{x}{\|x\|} \perp_{HH-C} \frac{y}{\|y\|} = \frac{\lambda y}{\|\lambda y\|}$. Therefore $x \perp_{UHH-C} \lambda y$. Moreover, note that if $x \perp_{UHH-I} y$, then $x \perp_{UHH-I} -y$. It follows that UHH-I-orthogonality is homogeneous.

The first result of this section guarantees that UHH-C-orthogonality has the existence property.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a normed linear space and $x, y \in X$ are linearly independent. Then there exists $\alpha \in \mathbb{R}$ such that $x \perp_{UHH-C} (\alpha x + y)$.*

Proof. Define the continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\lambda) := \sum_{i=1}^{m} \alpha_i \int_0^1 \Big\|(1-t)\beta_i \frac{x}{\|x\|} + t\gamma_i \frac{\lambda x + y}{\|\lambda x + y\|}\Big\|^2 dt,$$

where $m \in \mathbb{N}$ and α_i, β_i and γ_i satisfying (1). Then for each $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}\varphi(\lambda) &= \sum_{i=1}^{m} \alpha_i \int_0^1 (1-t)^2 \Big\|\beta_i \frac{x}{\|x\|} + \frac{t}{1-t} \times \frac{(\lambda x + y)\gamma_i}{\|\lambda x + y\|}\Big\|^2 dt \\ &= \sum_{i=1}^{m} \alpha_i \int_0^1 (1-t)^2 \Big\|\Big(\frac{\beta_i}{\|x\|} + \frac{t\gamma_i}{1-t} \times \frac{\lambda}{\|\lambda x + y\|}\Big)x + \frac{t\gamma_i}{1-t} \times \frac{y}{\|\lambda x + y\|}\Big\|^2 dt.\end{aligned}$$

Hence

$$\lim_{\lambda\to\pm\infty}\varphi(\lambda)=\sum_{i=1}^{m}\alpha_i\int_0^1(1-t)^2\Big\|(\beta_i\pm\frac{t}{1-t}\gamma_i)\frac{x}{\|x\|}\Big\|^2dt=\sum_{i=1}^{m}\alpha_i\int_0^1((1-t)\beta_i\pm t\gamma_i)^2dt$$
$$=\pm\sum_{i=1}^{m}\alpha_i\beta_i\gamma_i\int_0^1 2t(1-t)dt=\pm\frac{1}{3}.$$

Therefore, it follows from the mean value theorem that there is $\alpha\in\mathbb{R}$ such that $\varphi(\alpha)=0$, that is $x\perp_{UHH-C}(\alpha x+y)$. □

Before stating our main results, we remind the notion of Gâteaux left and right derivatives of the norm and their relationship with Birkhoff-James orthogonality, which have a fundamental role in proving our main theorem.

Let $(X,\|\cdot\|)$ be a normed linear space and let $x,y\in X$. The functionals

$$\tau_-(x,y)=\lim_{t\to0^-}\frac{\|x+ty\|-\|x\|}{t}\ ,\ \tau_+(x,y)=\lim_{t\to0^+}\frac{\|x+ty\|-\|x\|}{t}$$

are called Gâteaux left and right derivatives of the norm at x in direction y, respectively. The norm $\|\cdot\|$ is Gâteaux differentiable at x in direction y if $\tau_-(x,y)=\tau_+(x,y):=\tau(x,y)$. If the norm $\|\cdot\|$ is Gâteaux differentiable at x in all directions y, then we say that the norm $\|\cdot\|$ is Gâteaux differentiable at x.

Some properties of these functionals are as follows:

(1) $\tau_-(x,y)\le\tau_+(x,y)$ and $|\tau_\pm(x,y)|\le\|y\|$.

(2) $\tau_\pm(x,\alpha y)=\begin{cases}\alpha\tau_\pm(x,y) & \alpha>0\\ \alpha\tau_\mp(x,y) & \alpha\le0\end{cases}$.

(3) $\tau_\pm(x,\alpha x+y)=\alpha\|x\|+\tau_\pm(x,y)$.

We refer the reader to [3, 4] for more information about norm derivatives.

Lemma 2.3. *[6] Let $(X,\|\cdot\|)$ be a normed linear space. If there exist two real numbers λ and μ with $\lambda+\mu\neq0$ such that $\lambda\tau_+(x,y)+\mu\tau_-(x,y)$ is a continuous function of $x,y\in X$, then the norm of X is Gâteaux differentiable.*

Lemma 2.4. *[13] Let $(X,\|\cdot\|)$ be a normed linear space and let $x,y\in X$. Suppose that the norm of X is Gâteaux differentiable. Then $x\perp_B y$ if and only if $\tau(x,y)=0$.*

In addition, we recall that the following well-known characterization of inner product spaces based on the symmetric property of Birkhoff–James orthogonality.

Theorem 2.5. *[8, 14] A normed linear space X, whose dimension is at least three, is an inner product space if and only if Birkhoff-James orthogonality is symmetric in X.*

Some characterizations of inner product spaces by using some weakened hypothesis of the homogeneity of the isosceles and Pythagorean orthogonalities were presented by Alonso in [1]. Precisely, Alonso proved that a normed linear space $(X,\|\cdot\|)$ with unit sphere $\mathbb{S}_X$, is an inner product space if and only if there exists $\delta>0$ such that

$$x,y\in\mathbb{S}_X, x\perp_\Diamond y, |\lambda|<\delta\Rightarrow x\perp_\Diamond\lambda y,$$

for which $\Diamond\in\{I,P\}$ (see, [1, Proposition 2.27] and [1, Proposition 2.31]). Furthermore, some characterizations of inner product spaces by using some local homogeneity property of the HH-P- and HH-I-orthogonalities were obtained in [10].

Recall that the orthogonality relation $\perp$ on an inner product $(X, \langle \cdot, \cdot \rangle)$ is symmetric, i.e.,

$$x \perp y \Rightarrow y \perp x \quad (\forall x, y \in X).$$

But, as pointed out in [17], HH-C-orthogonality is not symmetric, in general. However, some special cases of HH-C-orthogonality, such as, HH-I-orthogonality, HH-P-orthogonality and HH-α-orthogonality ($\alpha \neq 1$) are symmetric cf. [11, 16, 17]. The next result, present a new characterization of real inner product space X, whose dimension is at least 3, based on a weakened hypothesis of the homogeneity of symmetric HH-C-orthogonality relationships in X.

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a normed linear space, whose dimension is at least three. For given symmetric HH-C-orthogonality in X, the following statements are equivalent:*

(1) For each $x, y \in \mathbb{S}_X$ there is $\delta = \delta(x, y) > 0$ such that

$$x \perp_{HH-C} y \Longrightarrow x \perp_{HH-C} \lambda y \quad (\forall \lambda \in (0, \delta)).$$

(2) X is an inner product space.

Proof. The implication (2)⇒(1) is clear.

To prove (1)⇒(2), assume that $x, y \in \mathbb{S}_X$ and $x \perp_{HH-C} y$. Then there is $\delta = \delta(x, y) > 0$ such that $x \perp_{HH-C} \lambda y$ for all $\lambda \in (0, \delta)$. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\lambda) = \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i \lambda y\|^2 dt.$$

Hence $\Phi(\lambda) = 0$ for all $\lambda \in (0, \delta)$ and $\Phi(0) = \frac{1}{3} \sum_{i=1}^{m} \alpha_i \beta_i^2 = 0$. Therefore

$$\Phi'_+(0) = \lim_{\lambda \to 0^+} \frac{\Phi(\lambda)}{\lambda} = 0.$$

On the other hand, we have

$$\frac{\Phi(\lambda)}{\lambda} = \sum_{\beta_i \neq 0} \Big(\alpha_i \beta_i^2 \int_0^1 \varphi_i(\lambda, t) dt \Big),$$

where

$$\varphi_i(\lambda, t) := \frac{\|(1-t)x + \lambda t \beta_i^{-1} \gamma_i y\|^2 - \|(1-t)x\|^2}{\lambda}.$$

Since

$$\varphi_i(\lambda, t) = \Big(\frac{\|(1-t)x + \lambda t \beta_i^{-1} \gamma_i y\| - \|(1-t)x\|}{\lambda} \Big) \Big(\|(1-t)x + \lambda t \beta_i^{-1} \gamma_i y\| + \|(1-t)x\| \Big),$$

we conclude that $\lim_{\lambda \to 0^+} \varphi_i(\lambda, t) = 2t(1-t)^2 \tau_+(x, \beta_i^{-1}\gamma_i y)$. It follows from Lebesgue dominated convergence theorem that

$$0 = \Phi'_+(0) = \lim_{\lambda \to 0^+} \frac{\Phi(\lambda)}{\lambda} = \sum_{\beta_i \neq 0} \alpha_i \beta_i^2 \int_0^1 \lim_{\lambda \to 0^+} \varphi_i(\lambda, t) dt = \frac{1}{6} (p\tau_+(x, y) + (1-p)\tau_-(x, y))$$

for which $p = \sum_{\beta_i \gamma_i > 0} \alpha_i \beta_i \gamma_i$. Thus $p\tau_+(x, y) + (1-p)\tau_-(x, y) = 0$.

Now, suppose that $x, y \in X$ are arbitrary linearly independent vectors. The existence property of UHH-C-orthogonality yields that there is $\alpha = \alpha(x, y) \in \mathbb{R}$ such that $x \perp_{UHH-C} (\alpha x + y)$, and so $\frac{x}{\|x\|} \perp_{HH-C} \frac{\alpha x+y}{\|\alpha x+y\|}$. According to what we proven, we obtain

$$\begin{aligned}
0 &= p\tau_+(\frac{x}{\|x\|}, \frac{\alpha x + y}{\|\alpha x + y\|}) + (1-p)\tau_-(\frac{x}{\|x\|}, \frac{\alpha x + y}{\|\alpha x + y\|}) \\
&= \frac{1}{\|x\|\,\|\alpha x + y\|}\Big(p(\alpha\|x\| + \tau_+(x, y)) + (1-p)(\alpha\|x\| + \tau_-(x, y))\Big) \\
&= \frac{1}{\|x\|\,\|\alpha x + y\|}\Big(\alpha\|x\| + p\tau_+(x, y) + (1-p)\tau_-(x, y)\Big).
\end{aligned}$$

Hence

$$\alpha = \alpha(x, y) = -\frac{p\tau_+(x, y) + (1-p)\tau_-(x, y)}{\|x\|}.$$

In fact, we have proved

$$x \perp_{UHH-C} (\alpha x + y) \Leftrightarrow \alpha = \alpha(x, y) = -\frac{p\tau_+(x, y) + (1-p)\tau_-(x, y)}{\|x\|}. \tag{2}$$

Now, assume that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$, $y_n \to y$. Then for each $n \in \mathbb{N}$ there is $\alpha_n = \alpha(x_n, y_n)$ such that $x_n \perp_{UHH-C} (\alpha_n x_n + y_n)$. It follows from (2) that

$$\alpha_n = \alpha(x_n, y_n) = -\frac{p\tau_+(x_n, y_n) + (1-p)\tau_-(x_n, y_n)}{\|x_n\|} \quad (\forall n \in \mathbb{N}).$$

Since $|\tau_\pm(x_n, y_n)| \leq \|y_n\|$, we get $\{\alpha_n\}$ is Cauchy and so is bounded. Hence it has a convergent subsequence $\{\alpha'_n\}$. Suppose that $\alpha'_n \to \beta$. It follows that $\alpha_n \to \beta$. Thus the continuity property of UHH-C-orthogonality implies that $x \perp_{UHH-C} (\beta x + y)$. So (2) yields that $\beta = \alpha(x, y)$. Therefore

$$\lim_{n\to\infty} \alpha(x_n, y_n) = \lim_{n\to\infty} \alpha_n = \beta = \alpha(x, y).$$

Consequently, $\alpha(x, y)$, and so $p\tau_+(x, y) + (1-p)\tau_-(x, y)$ is continuous with respect to x and y. Then Lemma 2.3 implies that the norm of X is Gâteaux differentiable, and therefore $\tau(x, y) = \tau_-(x, y) = \tau_+(x, y)$. It follows from Lemma 2.4 that

$$x \perp_{UHH-C} y \Leftrightarrow \tau(x, y) = 0 \Leftrightarrow x \perp_B y.$$

It follows that Birkhoff–James orthogonality is symmetric. Therefore from Theorem 2.5, we conclude that X is an inner product space. □

Considering the homogeneity property of HH-I-orthogonality, the following characterization of real inner product spaces was obtained in [11].

Theorem 2.7. *[11, Theorem 3.6] A normed linear space* $(X, \|\cdot\|)$ *is an inner product space if and only if HH-I-orthogonality is homogeneous in X.*

Finally, as some direct consequences of Theorem 2.6, we give the following characterizations of real inner product spaces in terms of some symmetric orthogonality of Carlsson type such as HH-I-orthogonality, HH-P-orthogonality and some symmetric unitary Carlsson type such as UHH-I-orthogonality and UHH-P-orthogonality. First, note that, UHH-I-orthogonality and HH-I-orthogonality are equivalent in normed linear space X if and only if X is an inner product space. In fact, if UHH-I-orthogonality is equivalent to HH-I-orthogonality in a normed linear space X, then HH-I-orthogonality is homogeneous in X, and so X is an inner product space, by Theorem 2.7.

The next example also shows that UHH-P-orthogonality and HH-P-orthogonality are incomparable in general normed linear spaces.

Example 2.8. *Consider the normed linear space* $X = \mathbb{R}^3$ *with the norm* $\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|$. *Let* $x = (1, 0, 0)$ *and* $y = (\alpha, 0, -1)$, *where* $\alpha < 0$. *Then*

$$\left\|(1-t)\frac{x}{\|x\|} + t\frac{y}{\|y\|}\right\| = |1 - \frac{1-2\alpha}{1-\alpha}t| + \frac{t}{1-\alpha} \quad (\forall t \in [0,1]).$$

Hence $x \perp_{UHH-P} y$ *if and only if*

$$\int_0^1 \left\|(1-t)\frac{x}{\|x\|} + t\frac{y}{\|y\|}\right\|^2 dt = \int_0^{\frac{1-\alpha}{1-2\alpha}} (1 + \frac{2\alpha}{1-\alpha}t)^2 dt + \int_{\frac{1-\alpha}{1-2\alpha}}^1 (2t-1)^2 dt = \frac{2}{3}.$$

Thus $x \perp_{UHH-P} y$ *if and only if* α *is negative root of the quaratic equation* $4\alpha^2 - 2\alpha - 1 = 0$. *Therefore* $x \perp_{UHH-P} y$ *if and only if* $\alpha = \frac{1-\sqrt{5}}{4}$. *However,* $x \not\perp_{HH-P} y$. *Also, according to [11], for* $x = (0, 0, -1)$ *and* $y = (1, 0, \sqrt[3]{2} - 1)$, *we have* $x \perp_{HH-P} y$. *But*

$$\int_0^1 \left\|(1-t)\frac{x}{\|x\|} + t\frac{y}{\|y\|}\right\|^2 dt = \int_0^1 (2t-1)^2 dt = \frac{1}{3} \neq \frac{2}{3},$$

which follows that $x \not\perp_{UHH-P} y$.

Corollary 2.9. *Let* $(X, \|\cdot\|)$ *be a normed linear space whose dimension is at least three. Then the following statements are equivalent:*

(1) $\perp_{UHH-P} \subseteq \perp_{HH-P}$.

(2) X is an inner product space.

Proof. The implication (2)⇒(1) is clear. To prove (1)⇒(2), suppose that $x, y \in \mathbb{S}_X$ and $x \perp_{HH-P} y$. Then $x \perp_{UHH-P} y$, and so $x \perp_{UHH-P} \lambda y$ for all $\lambda > 0$, since UHH-P-orthogonality is positively homogeneous. It follows from (1) that $x \perp_{HH-P} \lambda y$ for all $\lambda > 0$. Therefore we have proved that for each $x, y \in \mathbb{S}_X$,

$$x \perp_{HH-P} y \Rightarrow x \perp_{HH-P} \lambda y \quad (\forall \lambda > 0),$$

for symmetric orthogonality relation, HH-P-orthogonality. Consequently, X is an inner product space, by Theorem 2.6. □

Example 2.10. *Consider the normed linear space* $X = \mathbb{R}^3$ *with the norm* $\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}$. *Let* $x = (2, 0, 1)$ *and* $y = (\frac{1}{2}, 0, -1)$. *Then* $\|x\| = 2$, $\|y\| = 1$,

$$\int_0^1 \left\|(1-t)\frac{x}{\|x\|} - t\frac{y}{\|y\|}\right\|^2 dt = \int_0^{\frac{1}{4}} (1 - \frac{3}{2}t)^2 dt + \frac{1}{4}\int_{\frac{1}{4}}^1 (1+t)^2 dt = \frac{43}{64}$$

and

$$\int_0^1 \left\|(1-t)\frac{x}{\|x\|} + t\frac{y}{\|y\|}\right\|^2 dt = \int_0^{\frac{3}{4}} (1 - \frac{t}{2})^2 dt + \frac{1}{4}\int_{\frac{3}{4}}^1 (1-3t)^2 dt = \frac{43}{64}.$$

Hence $x \perp_{UHH-I} y$. *On the other hand, we have*

$$\int_0^1 \|(1-t)x - ty\|^2 dt = \int_0^{\frac{2}{5}} (2 - \frac{5}{2}t)^2 dt + \int_{\frac{2}{5}}^1 dt = \frac{23}{15}.$$

While,

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^{\frac{6}{7}} (2 - \frac{3}{2}t)^2 dt + \int_{\frac{6}{7}}^1 (1-2t)^2 dt = \frac{265}{147},$$

which implies that $x \not\perp_{HH-I} y$ *and* $x \not\perp_{HH-P} y$. *Therefore* $\perp_{UHH-I} \not\subseteq \perp_{HH-I}$, *and* $\perp_{UHH-I} \not\subseteq \perp_{HH-P}$ *in X.*

Corollary 2.11. *Let $(X, \|\cdot\|)$ be a normed linear space, whose dimension is at least three. Then the following statements are equivalent:*

(1) $\perp_{UHH-I} \subseteq \perp_{HH-I}$.

(2) X is an inner product space.

Proof. The implication (2)⇒(1) is clear. To prove (1)⇒(2), suppose that $x, y \in \mathbb{S}_X$ and $x \perp_{HH-I} y$. Then $x \perp_{UHH-I} y$, and so $x \perp_{UHH-I} \lambda y$ for all $\lambda > 0$. It follows from (1) that $x \perp_{HH-I} \lambda y$ for all $\lambda > 0$. Hence we have proved that for each $x, y \in \mathbb{S}_X$,

$$x \perp_{HH-I} y \Rightarrow x \perp_{HH-I} \lambda y \quad (\forall \lambda > 0),$$

for symmetric orthogonality relation, HH-I-orthogonality. Therefore X is an inner product space, by Theorem 2.6. □

Corollary 2.12. *Let $(X, \|\cdot\|)$ be a normed linear space, whose dimension is at least three. Then the following statements are equivalent:*

(1) $\perp_{UHH-I} \subseteq \perp_{HH-P}$.

(2) X is an inner product space.

Proof. Assume that $x, y \in \mathbb{S}_X$ and $x \perp_{HH-I} y$. Then $x \perp_{UHH-I} \lambda y$ for all $\lambda > 0$, and so $x \perp_{HH-P} \lambda y$ for all $\lambda > 0$. It means

$$\int_0^1 \|(1-t)x + \lambda t y\|^2 dt = \frac{1}{3}(1+\lambda^2) \quad (\forall \lambda > 0). \tag{3}$$

Also, we have $x \perp_{UHH-I} -y$, which follows that $x \perp_{HH-P} -\lambda y$ for all $\lambda > 0$. Then

$$\int_0^1 \|(1-t)x - \lambda t y\|^2 dt = \frac{1}{3}(1+\lambda^2) \quad (\forall \lambda > 0). \tag{4}$$

Combining (3) and (4) imply that $x \perp_{HH-I} \lambda y$ for all $\lambda > 0$. Therefore Theorem 2.6, yields that X is an inner product space. □

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