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On Elements whose (b, c)-Inverse is Idempotent in a Monoid

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Abstract. In this paper, we investigate the elements whose (b, c)-inverse is idempotent in a monoid. Let S be a monoid and $a, b, c \in S$. Firstly, we give several characterizations for the idempotency of $a^{\|(b,c)}$ as follows: $a^{\|(b,c)}$ exists and is idempotent if and only if $cab = cb$, $cS = cbS$, $Sb = Scb$ if and only if both $a^{\|(b,c)}$ and $1^{\|(b,c)}$ exist and $a^{\|(b,c)} = 1^{\|(b,c)}$, which establish the relationship between $a^{\|(b,c)}$ and $1^{\|(b,c)}$. They imply that $a^{\|(b,c)}$ merely depends on b, c but is independent of a when $a^{\|(b,c)}$ exists and is idempotent. Particularly, when $b = c$, more characterizations which ensure the idempotency of $a^{\|b}$ by inner and outer inverses are given. Finally, the relationship between $a^{\|b}$ and $a^{\|b^n}$ for any $n \in \mathbb{N}^+$ is revealed.

1. Introduction

Recall that an involution $*$: $a \mapsto a^*$ in a monoid S is an anti-isomorphism of degree 2, i.e. $(a^*)^* = a$, $(ab)^* = b^*a^*$, for arbitrary $a, b \in S$. Throughout the paper, unless otherwise stated, S denotes a monoid and $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices. For any $A \in \mathbb{C}^{m\times n}$, the rank of A is denoted by $\mathrm{rk}(A)$. We use $\mathbb{N}$ to denote the set of all nonnegative integers and $\mathbb{N}^+$ to denote the set of all positive integers.

Let S be a monoid with an involution. An element $a \in S$ is called Moore-Penrose invertible [9, 12, 15] if there exists $x \in S$ satisfying the following four equations:

$$(1)\ axa = a,\ (2)\ xax = x,\ (3)\ (ax)^* = ax,\ (4)\ (xa)^* = xa.$$

Such x is unique if it exists, so that is called the Moore-Penrose inverse of a and denoted by $a^\dagger$. The symbol $S^\dagger$ denotes the set of all Moore-Penrose invertible elements in S.

We call that $a \in S$ is regular if there exists $x \in S$ such that the equation (1) holds, in which case $x = a^-$ is called an inner inverse of a. If x satisfies the equation (2), then x is called an outer inverse of a.

And $a \in S$ is called group invertible if there exists $x \in S$ satisfying

$$(1)\ axa = a,\ (2)\ xax = x,\ (5)\ ax = xa.$$

Such x is unique if it exists, so that is called the group inverse of a and denoted by $a^\#$. The symbol $S^\#$ denotes the set of all group invertible elements in S.

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The concept of the (b, c)-inverse was first introduced by Drazin [6] in 2012. Rakić [16] gave another equivalent definition of the (b, c)-inverse as follows. Let $a, b, c \in S$. Then a is said to be (b, c)-invertible if there exists $y \in S$ such that

$$y \in bS \cap Sc, \ yab = b, \ cay = c.$$

Such y is unique if it exists, so that is called the (b, c)-inverse of a and denoted by $a^{\|(b,c)}$. Obviously, $a^{\|(b,c)}$ is an outer inverse of a.

In particular, when $b = c$, the (b, c)-inverse reduces to the (b, b)-inverse, which is also called the inverse along an element b [14]. Let $a, b \in S$. Then a is said to be (b, b)-invertible if there exists $y \in S$ such that

$$y \in bS \cap Sb, \ yab = b = bay.$$

Such y is unique if it exists, so that is called the (b, b)-inverse of a and denoted by $a^{\|b}$.

Actually, the (b, c)-inverse can be regarded as a generalization of many generalized inverses, such as the Moore-Penrose inverse (i.e. (a^*, a^*)-inverse) [14], the Drazin inverse (i.e. (a^j, a^j)-inverse, for some $j \in \mathbb{N}$) [14], the core inverse (i.e. (a, a^*)-inverse) [17] and so on.

In [4, Fact 8.7.6], Bernstein proved that $A^\dagger$ is idempotent if and only if $A^2 = AA^*A$ for $A \in \mathbb{C}^{n\times n}$. In [2], Baksalary and Trenkler investigated matrices whose Moore-Penrose inverse is idempotent. They gave more characterizations for the idempotency of $A^\dagger$, as well as both A and $A^\dagger$ being idempotent. Recently, the authors investigated elements whose Moore-Penrose inverse is idempotent in a $*$-ring and generalized above results from complex matrices to $*$-rings. More equivalent conditions which ensure the idempotecny of $a^\dagger$ (as well as a) were shown in [19].

Motivated by the above work, we investigate the elements whose (b, c)-inverse is idempotent in a monoid. The paper is organized as follows. Let $a, b, c \in S$. In section 2, we first give several concise characterizations for the idempotency of $a^{\|(b,c)}$: $a^{\|(b,c)}$ exists and is idempotent if and only if $cab = cb$, $cS = cbS$, $Sb = Scb$ if and only if both $a^{\|(b,c)}$ and $1^{\|(b,c)}$ exist and $a^{\|(b,c)} = 1^{\|(b,c)}$, which connect $a^{\|(b,c)}$ and $1^{\|(b,c)}$ to some extend (Theorem 2.7). They imply that $A^{\|(B,C)}$ exists and is idempotent if and only if $CAB = CB$, $\mathrm{rk}(C) = \mathrm{rk}(CB) = \mathrm{rk}(B)$ for any $A, B, C \in \mathbb{C}^{n\times n}$ (Corollary 2.9), and that $a^{\|(b,c)}$ merely depends on b, c but is independent of a when $a^{\|(b,c)}$ exists and is idempotent (Corollary 2.10). In section 3, we focus on the case when $b = c$. A characterization for $a^{\|b}$ being idempotent is given: $a^{\|b}$ exists and is idempotent if and only if $a^{\|b}$ exists and $bab = b^2$ if and only if $b \in S^\#$ and $bab = b^2$, which connects (b, b)-invertibility and group invertibility (Theorem 3.1). Then, we present several characterizations for $a^{\|b}$ being idempotent by inner and outer inverses (Theorem 3.8). Furthermore, the equivalent condition under which both b and $a^{\|b}$ are idempotent is provided (Proposition 3.11). Finally, the relationship between (b, b)-inverses and (b^n, b^n)-inverses for any $n \in \mathbb{N}^+$ is revealed (Proposition 3.13).

2. Characterizations for the idempotency of (b, c)-inverses

In this section, we investigate the elements whose (b, c)-inverse is idempotent and give several equivalent characterizations for the idempotency of (b, c)-inverses in a monoid. Firstly, let us recall some auxiliary lemmas.

Lemma 2.1. [10] *Let $a \in S$. Then $a \in S^\#$ if and only if $a \in a^2S \cap Sa^2$. Moreover, if $a = a^2x = ya^2$ for some $x, y \in S$, then $a^\# = yax$.*

Lemma 2.2. [6] *Let $a, b, c \in S$. Then a is (b, c)-invertible if and only if $b \in Scab$ and $c \in cabS$.*

Definition 2.3. [7] *Let $a, b, c \in S$. Then a is said to be left (resp. right) (b, c)-invertible if $b \in Scab$ (resp. $c \in cabS$), in which case any $x \in Sc$ (resp. $x \in bS$) satisfying $xab = b$ (resp. $cax = c$) is called a left (resp. right) (b, c)-inverse of a, and denoted by $a_l^{\|(b,c)}$ (resp. $a_r^{\|(b,c)}$).*

Therefore, a is (b, c)-invertible if and only if a is both left (b, c)-invertible and right (b, c)-invertible by Lemma 2.2. And in this case, $a^{\|(b,c)} = a_l^{\|(b,c)} = a_r^{\|(b,c)}$ [7].

Lemma 2.4. [14] *Let $a, b \in S$. Then the following statements are equivalent:*

(i) *a is (b, b)-invertible;*

(ii) *$ab \in S^{\#}$ and $Sb = Sab$;*

(ii) *$ba \in S^{\#}$ and $bS = baS$.*

In this case, $a^{\|b} = b(ab)^{\#} = (ba)^{\#}b$.

In [11, Theorem 2.7], Ke et al. proved that for any $a, b, c \in S$, if $a^{\|(b,c)}$ exists, then $(a^{\|(b,c)})^2 = a^{\|(b,c)}$ if and only if $a^{\|(b,c)}b = b$. Based on their results, we first give a lemma to characterize the idempotency of (b, c)-inverses.

Lemma 2.5. *Let $a, b, c \in S$ and a be (b, c)-invertible. Set $x = a^{\|(b,c)}$. Then the following statements are equivalent:*

(i) *x is idempotent;*

(ii) *$cx = c$;*

(iii) *$xb = b$.*

Proof. According to the definition of the (b, c)-inverse, we have $xab = b, cax = c, x \in Sc \cap bS$.

(i) ⇒ (ii). Since $x^2 = x$, we get $cx = caxx = cax = c$.

(ii) ⇒ (iii). Since $x \in Sc$, there exists $y_1 \in S$ such that $x = y_1c$. Then $b = xab = y_1cab = y_1cxab = y_1cb = xb$.

(iii) ⇒ (i). Since $x \in bS$, there exists $y_2 \in S$ such that $x = by_2$. Then $x = by_2 = xby_2 = xx = x^2$. □

Remark 2.6. *When a is merely left (resp. right) (b, c)-invertible, set $y = a_l^{\|(b,c)}$ (resp. $y = a_r^{\|(b,c)}$). We find that y being idempotent can imply that $yb = b$ (resp. $cy = c$), but it does not hold conversely.*

For example, let $S = \mathbb{C}^{2\times2}$, $a = c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = 0$. Then $y = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is the left (b, c)-inverse of a and satisfies $yb = 0 = b$, but y is not idempotent. Similarly, let $a = b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $c = 0$. Then $y = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is the right (b, c)-inverse of a and satisfies $cy = 0 = c$, but y is not idempotent.

In [19, Theorem 2.8], the authors gave a concise characterization for the idempotency of $a^{\dagger}$ in a $*$-ring R: $a \in R^{\dagger}$ and $a^{\dagger}$ is idempotent if and only if $a \in R^{\#}$ and $a^2 = aa^*a$, which connects Moore-Penrose invertibility and group invertibility. Inspired by previous work, we generalize the results to (b, c)-inverses in monoids.

Theorem 2.7. *Let $a, b, c \in S$. Then the following statements are equivalent:*

(i) *$a^{\|(b,c)}$ exists and is idempotent;*

(ii) *$cab = cb, cS = cbS$ and $Sb = Scb$;*

(iii) *$cab = cb$ and $1^{\|(b,c)}$ exists;*

(iv) *Both $a^{\|(b,c)}$ and $1^{\|(b,c)}$ exist and $a^{\|(b,c)} = 1^{\|(b,c)}$;*

(v) *There exist a right (b, c)-inverse of a and a left (b, c)-inverse of 1 such that $a_r^{\|(b,c)} = 1_l^{\|(b,c)}$;*

(vi) *There exist a left (b, c)-inverse of a and a right (b, c)-inverse of 1 such that $a_l^{(b,c)} = 1_r^{\|(b,c)}$;*

(vii) *$1^{\|(b,c)}$ exists and a is an inner inverse of $1^{\|(b,c)}$.*

Proof. (i) ⇒ (ii). Let $x = a^{\|(b,c)}$. Then according to Lemma 2.2, we have $b \in Scab$, $c \in cabS$. Since $x^2 = x$, $cab = ca(xab) = (cax)(xab) = cb$. And $c \in cabS = cbS$, $b \in Scab = Scb$. Therefore, $cS = cbS$ and $Sb = Scb$.

(ii) ⇒ (i). Since $cS = cbS = cabS, Sb = Scb = Scab$, we have $b \in Scab$ and $c \in cabS$. Thus, a is (b,c)-invertible by Lemma 2.2 and we denote $x = a^{\|(b,c)}$. According to the definition of the (b,c)-inverse, $x \in bS \cap Sc$, so there exist $y_1, y_2 \in S$ such that $x = by_1 = y_2c$. Then $x^2 = y_2cby_1 = y_2caby_1 = xax = x$.

(ii) ⇔ (iii). Since $cS = cbS, Sb = Scb$ is equivalent to $1^{\|(b,c)}$ existing, we obtain (ii) ⇔ (iii).

(i) ⇒ (iv). Let $x = a^{\|(b,c)}$. Since $x^2 = x$, according to Lemma 2.5, we have $xb = b$, $cx = c$, $x \in bS \cap Sc$. Then, $1^{\|(b,c)}$ exists and $1^{\|(b,c)} = x = a^{\|(b,c)}$.

(iv) ⇒ (v), (vi), (vii). According to the definition and property of the (b,c)-inverse, they are obvious.

(v) ⇒ (i). Suppose that there exists $y \in S$ such that $y = a_r^{\|(b,c)} = 1_l^{\|(b,c)}$, we have $y \in bS \cap Sc$, $cay = c$, $yb = b$. Thus, $cb = cayb = cab$ and there exists $w \in S$ such that $y = bw$, then $cy = caybw = cabw = cay = c$. Therefore, $1_r^{\|(b,c)}$ exists and is also equal to y. Therefore, by Lemma 2.2 and Definition 2.3, $1^{\|(b,c)}$ exists and $cab = cb$. According to the equivalence of (i) and (iii), the proof is completed.

(vi) ⇒ (i). The proof is similar to that of (v) ⇒ (i).

(vii) ⇒ (i). Let $x = 1^{\|(b,c)}$. Then we have $x \in bS \cap Sc$, $cx = c$, $xb = b$. Since $xax = x$, $xab = xa(xb) = xb = b$, $cax = (cx)ax = cx = c$. Therefore, $a^{\|(b,c)}$ exists and $a^{\|(b,c)} = x$. According to Lemma 2.5, x is idempotent. □

Corollary 2.8. *Let R be a ring with identity and $a, b, c \in R$. Then $a^{\|(b,c)}$ exists and is idempotent if and only if $1^{\|(b,c)}$ exists and $a \in T = 1^{\|(b,c)} + (1 - 1^{\|(b,c)})R + R(1 - 1^{\|(b,c)})$.*

Proof. According to [1, Lemma 3], for $r \in R$ with an inner inverse r_0, the set of all inner inverses of r can be represented by $r_0 + (1-r_0r)R + R(1-rr_0)$. By the equivalence between (i) and (iii) in Theorem 2.7, it is clear that $1^{\|(b,c)}$ is idempotent if it exists. Since $1^{\|(b,c)}$ is idempotent and is an inner inverse of itself, take $r = r_0 = 1^{\|(b,c)}$ and we can get that the set of inner inverses of $1^{\|(b,c)}$ is equal to $T = 1^{\|(b,c)} + (1 - 1^{\|(b,c)})R + R(1 - 1^{\|(b,c)})$. Then, by the equivalence between (i) and (vii) in Theorem 2.7, the proof is completed. □

Particularly, according to the equivalence between (i) and (ii) in Theorem 2.7, we can get a concise characterization for the idempotency of (B,C)-inverse in the case of complex matrices.

Corollary 2.9. *Let $A, B, C \in \mathbb{C}^{n\times n}$. Then $A^{\|(B,C)}$ exists and is idempotent if and only if $CAB = CB$, $\mathrm{rk}(C) = \mathrm{rk}(CB) = \mathrm{rk}(B)$.*

In view of the equivalence between (i) and (iv) in Theorem 2.7, we can obtain the following corollary, which shows that under the condition that (b,c)-inverse of an element is idempotent, the (b,c)-inverse merely depends on b, c and has nothing to do with the element itself.

Corollary 2.10. *Let $a_1, a_2, b, c \in S$. If a_1, a_2 are (b,c)-invertible and their (b,c)-inverses are idempotent, then $a_1^{\|(b,c)} = a_2^{\|(b,c)} = 1^{\|(b,c)}$.*

In [19, Theorem 2.8], the authors proved that in a $*$-ring R, when $a \in R^{\dagger}$ and $a^{\dagger}$ is idempotent, a is also group invertible. However, for (b,c)-inverses, even though $a^{\|(b,c)}$ exists and is idempotent, it can not imply that b or c is group invertible.

Example 2.11. *Let $S = \mathbb{C}^{2\times 2}$, $a = I$, $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. By computation, we have $cab = cb = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathrm{rk}(c) = \mathrm{rk}(cb) = \mathrm{rk}(b) = 1$. Thus, according to Corollary 2.9, $a^{\|(b,c)}$ exists and is idempotent. However, b and c are not group invertible by Lemma 2.1.*

3. Characterizations for the idempotency of (b, b)-inverses

In this section, for any given $a, b \in S$, we consider the case of the (b,b)-inverse by using the results in section 2. Furthermore, several characterizations for the idempotency of $a^{\|b}$ are as follows.

Theorem 3.1. *Let $a, b \in S$. Then the following statements are equivalent:*

(i) *$a^{\|b}$ exists and is idempotent;*

(ii) *$a^{\|b}$ exists and $bab = b^2$;*

(iii) *$b \in S^{\#}$ and $bab = b^2$.*

In this case, $a^{\|b} = bb^{\#}$ and $b^{\#} = a^{\|b}b^{-}a^{\|b}$, where b^{-} is an inner inverse of b.

Proof. (i) ⇒ (ii). Let $x = a^{\|b}$. Then $xab = b = bax$. Thus, $bab = ba(xab) = (bax)(xab) = bb = b^2$.

(ii) ⇒ (iii). Let $x = a^{\|b}$. Then $x \in bS \cap Sb$. Thus, there exist $t, s \in S$ such that $x = bt = sb$. Then we have $b = bax = babt = b^2t \in b^2S$ and $b = xab = sbab = sb^2 \in Sb^2$. Thus, according to Lemma 2.1, $b \in S^{\#}$ and $b^{\#} = sbt = sbb^{-}bt = xb^{-}x$, where b^{-} is an inner inverse of b.

(iii) ⇒ (i). Set $x = bb^{\#}$. Then $x \in bS \cap Sb, x^2 = x$ and $xab = bb^{\#}ab = b^{\#}b^2 = b, bax = babb^{\#} = b^2b^{\#} = b$. Thus, $a^{\|b}$ exists and $a^{\|b} = x$ is idempotent. □

Example 3.2. *When $a^{\|b}$ exists and is idempotent, if $b^{\|a}$ exists, it may not imply that $b^{\|a}$ is idempotent. Let $S = \mathbb{C}^{2\times 2}$, $a = I$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By computation, we have $bab = b^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,* $\mathrm{rk}(b) = \mathrm{rk}(b^2) = 2$, *$aba \neq a^2$ and* $\mathrm{rk}(a) = \mathrm{rk}(aba) = 2$. *Therefore, according to Lemma 2.2 and Corollary 2.9, $a^{\|b}$ and $b^{\|a}$ exist and $a^{\|b}$ is idempotent, but $b^{\|a}$ is not idempotent.*

Example 3.3. *When $a^{\|b}$ exists and is idempotent, even though $a^2 = aba$ holds, it may not imply that $b^{\|a}$ exists as well. Let $S = \mathbb{C}^{2\times 2}$, $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = 0$. By computation, we have that $b^2 = bab = 0$, $a^2 = aba = 0$ and b is clearly group invertible. But* $\mathrm{rk}(a) \neq \mathrm{rk}(aba)$, *that is to say, $a \notin Saba \cap abaS$. Therefore, according to Lemma 2.2 and Theorem 3.1, $a^{\|b}$ exists and is idempotent, $a^2 = aba$ holds but $b^{\|a}$ does not exist.*

The above Theorem 3.1 generalizes [19, Theorem 2.8] from Moore-Penrose inverses to inverses along an element (i.e. (b, b)-inverses). As a special case, we give the corresponding results on weighted Moore-Penrose inverses.

Firstly, recall the definition of weighted Moore-Penrose inverses [5, 8, 13]. Let S be a monoid with an involution and $a, e, f \in S$, where e, f are invertible and Hermitian. Then a is called weighted Moore-Penrose invertible with weights e, f if there exists $x \in S$ satisfying the following equations:

$$(1')\ axa = a,\ (2')\ xax = x,\ (3')\ (eax)^* = eax,\ (4')\ (fxa)^* = fxa.$$

Such x is unique if it exists, so that is called the weighted Moore-Penrose inverse of a with weights e, f and denoted by $a^{\dagger}_{e,f}$.

Corollary 3.4. *Let S be a monoid with an involution and $a, e, f \in S$, where e, f are invertible and Hermitian. Then the following statements are equivalent:*

(i) *$a^{\dagger}_{e,f}$ exists and is idempotent;*

(ii) *$a^{\dagger}_{e,f}$ exists and $(eaf^{-1})^2 = (eaf^{-1})a^*(eaf^{-1})$;*

(iii) *$(eaf^{-1})^{\#}$ exists and $(eaf^{-1})^2 = (eaf^{-1})a^*(eaf^{-1})$.*

In this case, $a^{\dagger}_{e,f} = [(eaf^{-1})^{\#}(eaf^{-1})]^*$ and $(eaf^{-1})^{\#} = (a^{\dagger}_{e,f})^*(fa^{\dagger}_{e,f}e^{-1})(a^{\dagger}_{e,f})^*$.

Proof. By [3, Theorem 3.2], we have $a^{\dagger}_{e,f} = a^{\|f^{-1}a^*e}$. In Theorem 3.1, take $b = f^{-1}a^*e$, then the above results can be easily verified. □

In the following, we discuss several results about the equalities $b^2 = bab$ and $a^2 = aba$, which generalize [19, Proposition 2.12] and are useful in the subsequent proof.

Proposition 3.5. *Let $a, b \in S$ satisfying $b^2 = bab$, $a^2 = aba$ and $n \geq 2$. Then $b^n a = ba^n$, $b^n = ba^{n-1}b$ and $a^n = ab^{n-1}a$. For any positive integer $k_1, k_2, l_1, l_2 \in \mathbb{N}^+$, if $k_1 + k_2 = l_1 + l_2$, then $b^{k_1}a^{k_2} = b^{l_1}a^{l_2}$ and $a^{k_1}b^{k_2} = a^{l_1}b^{l_2}$.*

Proof. Obviously, when $n = 1, ba = ba$. Suppose that when $n = k$ $(k \geq 1)$, $b^k a = ba^k$ holds. Then according to induction hypothesis, for $n = k + 1$, we have $b^{k+1}a = bb^k a = bba^k = baba^k = babaa^{k-1} = ba^2a^{k-1} = ba^{k+1}$. Thus, $b^n a = ba^n$ for any $n \geq 1$ holds.

It is clear that when $n = 2$, $b^2 = bab$. Suppose that when $n = k$ $(k \geq 2)$, $b^k = ba^{k-1}b$ holds. Then according to induction hypothesis, for $n = k + 1$, we obtain $b^{k+1} = bb^k = bba^{k-1}b = bb^{k-1}ab = b^k ab = ba^k b$. Thus, $b^n = ba^{n-1}b$ for any $n \geq 2$ holds.

Assume that $k_1 \geq l_1 \geq 1$, since $k_1+k_2 = l_1+l_2$, we have $b^{k_1}a^{k_2} = b^{l_1-1}b^{k_1-l_1+1}aa^{k_2-1} = b^{l_1-1}ba^{k_1-l_1+1}a^{k_2-1} = b^{l_1}a^{l_2}$.

Due to the symmetry between a and b, we can immediately obtain that the following two equalities $a^n = ab^{n-1}a$ $(n \geq 2)$ and $a^{k_1}b^{k_2} = a^{l_1}b^{l_2}$ hold as well. □

Combining Theorem 3.1 and Proposition 3.5, we can get the following two corollaries.

Corollary 3.6. *Let $a, b \in S$ and $k_1, k_2, l_1, l_2 \in \mathbb{N}^+$ satisfying $k_1 + k_2 = l_1 + l_2$. If both $a^{\parallel b}$ and $b^{\parallel a}$ exist and are idempotent, then $b^{k_1}a^{k_2} = b^{l_1}a^{l_2}$ and $a^{k_1}b^{k_2} = a^{l_1}b^{l_2}$.*

Proposition 3.7. *Let $a, b \in S$ and $m_1, m_2, n_1, n_2 \in \mathbb{N}$ satisfying $m_1 + n_1 \neq 0$, $m_2 + n_2 \neq 0$. If both $a^{\parallel b}$ and $b^{\parallel a}$ exist and are idempotent, then a is $(a^{m_1}b^{n_1}, b^{n_2}a^{m_2})$-invertible, $(a^{m_1}b^{n_1}, a^{m_2}b^{n_2})$-invertible, $(b^{n_1}a^{m_1}, b^{n_2}a^{m_2})$-invertible and $(b^{n_1}a^{m_1}, a^{m_2}b^{n_2})$-invertible.*

Proof. Here, we only prove that a is $(a^{m_1}b^{n_1}, b^{n_2}a^{m_2})$-invertible, the rest can be verified similarly. Since both $a^{\parallel b}$ and $b^{\parallel a}$ exist and are idempotent, by Theorem 3.1, $b^2 = bab$, $a^2 = aba$ and a, b are group invertible. We prove in three cases:

Case 1. $m_1, m_2, n_1, n_2 \in \mathbb{N}^+$.

By Corollary 3.6, we have $(b^{n_2}a^{m_2})a(a^{m_1}b^{n_1}) = b^{n_2}a^{m_2+1+m_1}b^{n_1} = b^{m_1+m_2+n_1+n_2}$. Thus, $a^{m_1}b^{n_1} \in Sb^{n_1} = Sb^{m_1+m_2+n_1+n_2} = S(b^{n_2}a^{m_2})a(a^{m_1}b^{n_1})$ and $b^{n_2}a^{m_2} \in b^{n_2}S = b^{m_1+m_2+n_1+n_2}S = (b^{n_2}a^{m_2})a(a^{m_1}b^{n_1})S$. Then, according to Lemma 2.2, a is $(a^{m_1}b^{n_1}, b^{n_2}a^{m_2})$-invertible.

Case 2. Only one of m_1, m_2, n_1, n_2 is equal to 0.

(i). If $m_1 = 0, m_2, n_1, n_2 \in \mathbb{N}^+$, then we need to prove that a is $(b^{n_1}, b^{n_2}a^{m_2})$-invertible. By Proposition 3.5, we have $b^{n_2}a^{m_2+1}b^{n_1} = b^{n_1+n_2+m_2}$. Thus, $b^{n_1} \in Sb^{n_1+n_2+m_2} = Sb^{n_2}a^{m_2+1}b^{n_1}$. Since $b^{n_2}a^{m_2} \in b^{n_2}S = b^{n_1+n_2+m_2}S = b^{n_2}a^{m_2+1}b^{n_1}S$, according to Lemma 2.2, a is $(b^{n_1}, b^{n_2}a^{m_2})$-invertible.

(ii). If $n_1 = 0, m_1, m_2, n_2 \in \mathbb{N}^+$, then we need to prove that a is $(a^{m_1}, b^{n_2}a^{m_2})$-invertible. By Corollary 3.6, since a is group invertible, we have $a^{m_1} = (a^\#)^{m_2+n_2+1}a^{m_1+m_2+n_2+1} = (a^\#)^{m_2+n_2+1}abaa^{m_1+m_2+n_2-1} = (a^\#)^{m_2+n_2+1}ab^{n_2}a^{m_1+m_2+1} \in Sb^{n_2}a^{m_1+m_2+1}$, and $b^{n_2}a^{m_2} = b^{n_2}a^{m_1+m_2+1}(a^\#)^{m_1+1} \in b^{n_2}a^{m_1+m_2+1}S$. According to Lemma 2.2, a is $(a^{m_1}, b^{n_2}a^{m_2})$-invertible.

(iii). If $m_2 = 0, m_1, n_1, n_2 \in \mathbb{N}^+$, then we need to prove that a is $(a^{m_1}b^{n_1}, b^{n_2})$-invertible. Since $b^{n_2}a^{m_1+1}b^{n_1} = b^{m_1+n_1+n_2}$ and b is group invertible, we have $a^{m_1}b^{n_1} \in Sb^{m_1+n_1+n_2} = Sb^{n_2}a^{m_1+1}b^{n_1}$, and $b^{n_2} \in b^{m_1+n_1+n_2}S = b^{n_2}a^{m_1+1}b^{n_1}S$. According to Lemma 2.2, a is $(a^{m_1}b^{n_1}, b^{n_2})$-invertible.

(iv). If $n_2 = 0, m_1, n_1, m_2 \in \mathbb{N}^+$, then we need to prove that a is $(a^{m_1}b^{n_1}, a^{m_2})$-invertible. Since a is group invertible, we have $a^{m_1}b^{n_1} = (a^\#)^{m_2+1}a^{m_1+m_2+1}b^{n_1} \in Sa^{m_1+m_2+1}b^{n_1}$. By Corollary 3.6, $a^{m_2} = a^{m_1+m_2+n_1+1}(a^\#)^{m_1+n_1+1} = a^{m_1+m_2+n_1-1}aba(a^\#)^{m_1+n_1+1} = a^{m_1+m_2+1}b^{n_1}a(a^\#)^{m_1+n_1+1} \in a^{m_1+m_2+1}b^{n_1}S$. According to Lemma 2.2, a is $(a^{m_1}b^{n_1}, a^{m_2})$-invertible.

Case 3. Two of m_1, m_2, n_1, n_2 are equal to 0.

(i). If $m_1 = m_2 = 0, n_1, n_2 \in \mathbb{N}^+$, then we need to prove that a is (b^{n_1}, b^{n_2})-invertible. By Proposition 3.5, $b^{n_2}ab^{n_1} = b^{n_2-1}babb^{n_1-1} = b^{n_1+n_2}$. Since b is group invertible, we have $b^{n_1} \in Sb^{n_1+n_2} = Sb^{n_2}ab^{n_1}$ and $b^{n_2} \in b^{n_1+n_2}S = b^{n_2}ab^{n_1}S$. According to Lemma 2.2, a is (b^{n_1}, b^{n_2})-invertible.

(ii). If $m_1 = n_2 = 0, n_1, m_2 \in \mathbb{N}^+$, then we need to prove that a is (b^{n_1}, a^{m_2})-invertible. Since a and b are group invertible, by Corollary 3.6, we have $b^{n_1} = (b^\#)^{m_2+1}b^{n_1+m_2+1} = (b^\#)^{m_2+1}babb^{n_1+m_2-1} = (b^\#)^{m_2+1}ba^{m_2+1}b^{n_1} \in$

$Sa^{m_2+1}b^{n_1}$, and $a^{m_2} = a^{m_2+n_1+1}(a^\#)^{n_1+1} = a^{m_2+n_1-1}aba(a^\#)^{n_1+1} = a^{m_2+1}b^{n_1}a(a^\#)^{n_1+1} \in a^{m_2+1}b^{n_1}S$. According to Lemma 2.2, a is (b^{n_1}, a^{m_2})-invertible.

(iii). If $m_2 = n_1 = 0, m_1, n_2 \in \mathbb{N}^+$, then we need to prove that a is (a^{m_1}, b^{n_2})-invertible. Since a and b are group invertible, by Corollary 3.6, we have $a^{m_1} = (a^\#)^{n_2+1}a^{m_1+n_2+1} = (a^\#)^{n_2+1}abaa^{m_1+n_2-1} = (a^\#)^{n_2+1}ab^{n_2}a^{m_1+1} \in Sb^{n_2}a^{m_1+1}$, and $b^{n_2} = b^{n_2+m_1+1}(b^\#)^{m_1+1} = b^{n_2+m_1-1}bab(b^\#)^{m_1+1} = b^{n_2}a^{m_1+1}b(b^\#)^{m_1+1} \in b^{n_2}a^{m_1+1}S$. According to Lemma 2.2, a is (a^{m_1}, b^{n_2})-invertible.

(iv). If $n_1 = n_2 = 0, m_1, m_2 \in \mathbb{N}^+$, then we need to prove that a is (a^{m_1}, a^{m_2})-invertible. Since a is group invertible, we have $a^{m_1} \in Sa^{m_2}aa^{m_1}$ and $a^{m_2} \in a^{m_2}aa^{m_1}S$. According to Lemma 2.2, a is (a^{m_1}, a^{m_2})-invertible. □

Particularly, let S be a monoid with an involution. Taking $b = a^*, m_1 = n_2 = 1, m_2 = n_1 = 0$ in Proposition 3.7, we can have that $a \in R^\dagger$ and $a^\dagger$ being idempotent can imply that a is core invertible, which is first proved in [19, Proposition 2.11].

Theorem 3.8. *Let $a, b \in S$ and a be (b, b)-invertible. Denote $x = a^{\|b}$, then the following statements are equivalent:*

(i) *x is idempotent;*

(ii) *$bx = b$;*

(iii) *$xb = b$;*

(iv) *$(ab)^\#$ is an inner inverse of b;*

(v) *$(ab)^\#$ is an outer inverse of b;*

(vi) *$(ba)^\#$ is an inner inverse of b;*

(vii) *$(ba)^\#$ is an outer inverse of b.*

Proof. (i) ⇔ (ii) ⇔ (iii). Follows from Lemma 2.5.

(iii) ⇒ (iv). By Lemma 2.4, $x = b(ab)^\# = (ba)^\#b$. Then $b(ab)^\#b = xb = b$.

(iv) ⇒ (i). It is clear that $x^2 = b(ab)^\#b(ab)^\# = b(ab)^\# = x$.

(ii) ⇒ (v). Since $bx = b$ and $x = b(ab)^\#$, we have $(ab)^\#b(ab)^\# = (ab)^\#x = [(ab)^\#]^2abx = [(ab)^\#]^2ab = (ab)^\#$.

(v) ⇒ (i). Since $(ab)^\#b(ab)^\# = (ab)^\#$ and $x = b(ab)^\#$, we have $x = b(ab)^\# = b(ab)^\#b(ab)^\# = x^2$.

Similarly, by Lemma 2.4, we can prove (ii) ⇒ (vi) ⇒ (i) and (iii) ⇒ (vii) ⇒ (i). □

Lemma 3.9. *Let $a, b, c \in S$. If a is both (b, c)-invertible and (c, b)-invertible, then $a^{\|bac}$ and $a^{\|(cab)}$ exist, and $a^{\|bac} = a^{\|(b,c)}$, $a^{\|cab} = a^{\|(c,b)}$.*

Proof. When a is both (b, c)-invertible and (c, b)-invertible, according to [18, Theorem 2.6], we have that $abac$, $acab$, $baca$, $caba$ are group invertible and $a^{\|(b,c)} = bac(abac)^\# = (baca)^\#bac$, $a^{\|(c,b)} = cab(acab)^\# = (caba)^\#cab$. Thus, according to the definition of the (b, c)-inverse, we have $b = (baca)^\#bacab$. Then, $bac = (baca)^\#bacabac \in Sabac$. Thus, $Sbac = Sabac$. By Lemma 2.4, a is (bac, bac)-invertible, and in this case $a^{\|bac} = bac(abac)^\# = (baca)^\#bac = a^{\|(b,c)}$. Similarly, we can prove $a^{\|cab} = cab(acab)^\# = (caba)^\#cab = a^{\|(c,b)}$. □

Combining Theorem 3.8 and Lemma 3.9, we can have the following corollary, which further characterizes $a^{\|(b,c)}$ being idempotent under the condition that a is both (b, c)-invertible and (c, b)-invertible, in which case $a^{\|(b,c)} = a^{\|bac}$.

Corollary 3.10. *Let $a, b, c \in S$. If a is both (b, c)-invertible and (c, b)-invertible and denote $x = a^{\|(b,c)}$, then the following statements are equivalent:*

(i) *x is idempotent;*

(ii) *$bacx = bac$;*

(iii) *$xbac = bac$;*

(iv) $(abac)^\#$ *is an inner inverse of bac;*

(v) $(abac)^\#$ *is an outer inverse of bac;*

(vi) $(baca)^\#$ *is an inner inverse of bac;*

(vii) $(baca)^\#$ *is an outer inverse of bac.*

Similarly, under the same condition that a is both (b,c)-invertible and (c,b)-invertible, we can obtain the corresponding characterizations for $a^{\|(c,b)}$ being idempotent, which are omitted here.

In [19, Theorem 3.1], the authors gave some equivalent conditions of a and $a^\dagger$ being idempotent simultaneously in a $*$-ring. Next, we generalize the results to (b,b)-inverses in a monoid.

Proposition 3.11. *Let $a, b \in S$ and $n \in \mathbb{N}^+$. Suppose that a is (b,b)-invertible and $x = a^{\|b}$ is idempotent. Then $b^{n+1} = b$ if and only if $b^n = x$. Particularly, b is idempotent if and only if $b = x$.*

Proof. ($\Rightarrow$). Since x is idempotent, according to Theorem 3.1 and Theorem 3.8, we have $bab = b^2$ and $bx = b$. Then, by Lemma 2.4, we have $x = b(ab)^\# = bab[(ab)^\#]^2 = b^2[(ab)^\#]^2 = \cdots = b^{n+1}[(ab)^\#]^{n+1} = b[(ab)^\#]^{n+1}$. Premultiplying b^n on both sides, we obtain $b^n x = b^{n+1}[(ab)^\#]^{n+1}$, i.e. $b^n = b[(ab)^\#]^{n+1}$. Thus, $b^n = x$.

($\Leftarrow$). Since x is idempotent and $b^n = x$, according to Theorem 3.8, $b = bx = bb^n = b^{n+1}$. □

Example 3.12. *Generally, when $a^{\|b}$ exists, $a^{\|b^n}$ may not exist for any $n \in \mathbb{N}^+$. Let $S = \mathbb{C}^{4\times 4}$, $a = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}$ and $b = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}$. By computation, we have $b = bab$ and $\mathrm{rk}(b^2ab^2) = \mathrm{rk}(b^3) \neq \mathrm{rk}(b^2)$, that is to say, $b \in Sbab \cap babS$ but $b^2 \notin Sb^2ab^2 \cap b^2ab^2S$. Therefore, according to Lemma 2.2, $a^{\|b}$ exists but $a^{\|b^2}$ does not exist.*

However, the next proposition reveals the connection between (b,b)-inverses and (b^n,b^n)-inverses for any $n \in \mathbb{N}^+$ under the condition that $a^{\|b}$ exists and is idempotent.

Proposition 3.13. *Let $a, b \in S$ and a be (b,b)-invertible. Denote $x = a^{\|b}$, if $x^2 = x$, then both $1^{\|b^n}$ and $a^{\|b^n}$ exist and $1^{\|b^n} = a^{\|b^n} = x$, for any $n \in \mathbb{N}^+$.*

Proof. Since $x = a^{\|b}$ exists, $xab = b = bax$, $x \in bS \cap Sb$. Since x is idempotent, by Theorem 3.8, $bx = b = xb$. Therefore, there exist $u, v \in S$ such that $x = bu = bxu = b(bu)u = b^2u^2 = \cdots = b^nu^n \in b^nS$ and $x = vb = vxb = v(vb)b = v^2b^2 = \cdots = v^nb^n \in Sb^n$ for any $n \in \mathbb{N}^+$ hold. And we have $b^nx = b^n = xb^n$. Since $xab = b = bax$ holds, we have $xab^n = b^n = b^nax$. Therefore, for any $n \in \mathbb{N}^+$, we can obtain that both $1^{\|b^n}$ and $a^{\|b^n}$ exist and $1^{\|b^n} = a^{\|b^n} = x$. □

Conversely, even though $a^{\|b^n}$ exists and is idempotent for any $n \geq 2$, it may not imply that a is (b,b)-invertible. Here, we give an example.

Example 3.14. *Let $S = \mathbb{C}^{2\times 2}$, $a = 0$ and $b = \begin{pmatrix} 0&1\\ 0&0 \end{pmatrix}$. It is obvious that for any $n \geq 2$, $b^n = b^{2n} = b^nab^n = 0$, which implies that $a^{\|b^n}$ exists and is idempotent by Lemma 2.2 and Theorem 3.1. However, $b \notin Sbab \cap babS$, that is to say, $a^{\|b}$ does not exist.*

Besides, it is worth noting that even though $a^{\|b^n}$ exists for any $n \in \mathbb{N}^+$, it may not imply that $a^{\|b}$ is idempotent. The counterexample is as follows.

Example 3.15. *Let* $S = \mathbb{C}^{2\times 2}$, $a = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ *and* $b = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$. *By computation, we have* $bab \neq b^2$. *Thus,* $\mathrm{rk}(b^n) = \mathrm{rk}(b^n a b^n) = 2$, *that is to say,* $b^n \in Sb^n ab^n \cap b^n ab^n S$ *for any* $n \in \mathbb{N}^+$ *holds. Then, according to Lemma 2.2 and Theorem 3.1, for any* $n \in \mathbb{N}^+$, $a^{\| b^n}$ *exists but* $a^{\| b}$ *is not idempotent.*

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