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Characterization of Weighted (b,c) Inverse of an Element in a Ring

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Abstract. The notion of the weighted (b, c)-inverse of an element in rings were introduced very recently. In this paper, we further elaborate on this theory by establishing a few characterizations of this inverse and their relationships with other (v, w)-weighted (b, c)-inverses. We discuss a few necessary and sufficient conditions for the existence of the hybrid (v, w)-weighted (b, c)-inverse and the annihilator (v, w)-weighted (b, c)-inverse of an element in a ring. In addition, we explore a few sufficient conditions for the reverse-order law of the annihilator (v, w)-weighted (b, c)-inverses.

1. Introduction

1.1. Background and motivation

The theory of generalized inverses has generated tremendous interest in many research areas in mathematics [1, 11, 17, 20, 22–24, 26]. Several types of generalized inverses are available in the literature, such as Moore-Penrose inverse [15], group inverse [12], Drazin inverse [6], and core inverse [23]. It is worth mentioning that Drazin in [7] introduced (b, c)-inverse in the setting of a semigroup, which is a generalization of Moore–Penrose inverse. Further, the notion of (b, c)-inverse [2, 3, 13, 14] extended to rings along with various characterizations and representations [29]. The concepts of annihilator (b, c)-inverses and hybrid (b, c)-inverses were established as generalizations of (b, c)-inverses in [7]. Further, several characterizations of hybrid and annihilator (b, c)-inverse have been discussed in [27, 28]. Mary proposed the inverse along an element (see [18] Definition 4), as a new type of generalized inverse. Many researchers [8, 9] explored numerous properties of these inverses and interconnections with other generalized inverses. Among the extensive work of generalized inverses, there has been a growing interest in "weighted" generalized inverses [5, 19, 25] for encompassing the above-mentioned generalized inverses.

In connection with the theory of (b, c)-inverses (see [7], Definition 1.3 and [21]) and the Bott-Duffin inverse [4], Drazin explored the Bott-Duffin (e, f)-inverse (see [7], Definition 3.2) in a semigroup. Further, "(v, w)-weighted version" of (b, c)-inverses are introduced in [10], e.g., annihilator (v, w)-weighted (b, c)-inverses (see Definition 4.1) and hybrid (v, w)-weighted (b, c)-inverses (see Definition 4.2). The vast work on the hybrid and annihilator (b, c)-inverse along with the above weighted (b, c)-inverse, motivate us to study a few characterizations and representations for hybrid and annihilator (v, w)-weighted (b, c)-inverse.

More precisely, the main contributions of this paper are as follows:

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- A few necessary and sufficient conditions for the existence of the (v, w)-weighted (b, c) inverses of elements in rings are introduced.
- Some characterizations of the (v, w)-weighted hybrid (b, c)-inverse and annihilator (v, w)-weighted (b, c)-inverses are investigated.
- The construction of the (v, w)-weighted hybrid (b, c)-inverse via group inverse is presented.

1.2. Outline

Our presentation is organized as follows. We present some notations and definitions in Section 2. In Section 3, we have discussed a few characterizations for the (v, w)-weighted (b, c)-inverse. Various equivalent properties of the hybrid (v, w)-weighted (b, c)-inverse are presented in Section 4. In Section 5, we study the representation of the annihilator (v, w)-weighted (b, c)-inverse. The contribution of our work is summarized in Section 6.

2. Preliminaries

Throughout this paper, $\mathcal{R}$ is an associative ring with unity 1. The sets of all left annihilators and right annihilators of a are respectively defined by

$$lann(a) = \{x \in \mathcal{R} : xa = 0\} \text{ and } rann(a) = \{z \in \mathcal{R} : az = 0\}.$$

We denote the left and right ideals by $a\mathcal{R} = \{ar : r \in \mathcal{R}\}$ and $\mathcal{R}a = \{za : z \in \mathcal{R}\}$. An element $y \in \mathcal{R}$ is called generalized or inner inverse of $a \in \mathcal{R}$ if $aya = a$. If such y exist, we say a is regular. The set of inner inverses of a is denoted by $a\{1\}$ and an inner inverse of a is represented by a^-. The following result proved in [16], gives the relation between ideals and annihilators.

Proposition 2.1. *If a is idempotent then $rann(a) = (1 - a)\mathcal{R}$ and $lann(a) = \mathcal{R}(1 - a)$.*

Next, we recall the definition of group inverse [6] of an element in $\mathcal{R}$. An element y is called group inverse of $a \in \mathcal{R}$ if $aya = a$, $yay = y$, and $ay = ya$. The group inverse of a is denoted by $a^\#$. The necessary and sufficient condition for the existence of group inverse is stated in the next result.

Lemma 2.2. *[12, Theorem 1] Let $a \in \mathcal{R}$. Then a is group invertible if and only if $a \in a^2\mathcal{R} \cap \mathcal{R}a^2$.*

We now recall the "(v, w)-weighted" version of (b, c) inverse.

Definition 2.3. *[10, Theorem 2.1 (i)] Let $a, b, c, v, w \in \mathcal{R}$. An element $y \in \mathcal{R}$ satisfying*

$$y \in b\mathcal{R}wy \cap yv\mathcal{R}c,\ yvawb = b \text{ and } cvawy = c,$$

is called the (v, w)-weighted (b, c)-inverse of a and denoted by $a^{v,w}_{b,c}$.

In [10], Drazin proved that [see Theorem 2.4, [10] for a proof] $a^{v,w}_{b,c}$ is unique if exists.

An equivalent characterization of the (v, w)-weighted (b, c)-inverse is presented below.

Lemma 2.4. *[10, Theorem 2.1 and 2.8] Let $a, b, c, v, w \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *a has a (v, w)-weighted (b, c)-inverse.*

(ii) *$c \in cvawb\mathcal{R}$ and $b \in \mathcal{R}cvawb$.*

(iii) *there exists $y \in \mathcal{R}$ such that $yvawy = y$, $yv\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}wy = \mathcal{R}c$.*

Following the definition (see [18], Definition 4) of the inverse along an element of $\mathcal{R}$, we next, define (v, w)-weighted inverse of a along $d \in \mathcal{R}$.

Definition 2.5. *Let $a, d, v, w \in \mathcal{R}$. An element $y \in \mathcal{R}$ satisfying*

$$yvawd = d = dvawy, \mathcal{R}wy \subseteq Rd \text{ and } yv\mathcal{R} \subseteq d\mathcal{R},$$

is called the (v, w)-weighted inverse of a along $d \in \mathcal{R}$ and denoted by $a_{\|d}^{v,w}$.

Here is an example illustrating the above definition.

Example 2.6. Let $\mathcal{R} = M_2(\mathbb{R})$, with $a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $d = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Since the matrix $y = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ satisfies

$$yvawd = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = d,$$

$$dvawy = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = d,$$

$wy = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = r_1 d$ and $yv = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = dr_2$ for some $r_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $r_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, it follows that $a_{\|d}^{v,w} = y$.

In view of right [resp. left] hybrid (v, w)-weighted (b, c)-inverse (see [10], Definition 4.2) and annihilator (v, w)-weighted (b, c)-inverse (see [10], Definition 4.1) of $a \in \mathcal{R}$, we next present the definition of the hybrid (v, w)-weighted (b, c)-inverse and annihilator (v, w)-weighted (b, c)-inverse of $a \in \mathcal{R}$.

Definition 2.7. *[10, Definition 4.2] Let $a, b, c, v, w \in \mathcal{R}$. An element $y \in \mathcal{R}$ satisfying*

$$yvawy = y,\ yv\mathcal{R} = b\mathcal{R}, \text{ and } rann(c) = rann(wy),$$

is called the right hybrid (or hybrid) (v, w)-weighted (b, c)-inverse of a and denoted by $a_{b,c}^{h,v,w}$.

In section 4, we will discuss some results on right hybrid inverse (v, w)-weighted (b, c)-inverse, which can be similarly proved for left hybrid (v, w)-weighted (b, c)-inverse. So from here onward, we call the right hybrid (v, w)-weighted (b, c)-inverse as hybrid (v, w)-weighted (b, c)-inverse.

The existence of hybrid (v, w)-weighted (b, c)-inverse over a semigroup, as proved in [10], is restated for a ring $\mathcal{R}$, below.

Lemma 2.8. *Let $a, b, c, v, w \in \mathcal{R}$. Then $a_{b,c}^{h,v,w}$ exists if and only if $rann(cvawb) \subseteq rann(b)$ and $c \in cvawb\mathcal{R}$.*

Definition 2.9. *[10, Definition 4.2] Let $a, b, c, v, w \in \mathcal{R}$. An element $y \in \mathcal{R}$ satisfying*

$$yvawy = y,\ lann(yv) = lann(b), \text{ and } rann(c) = rann(wy),$$

is called the annihilator (v, w)-weighted (b, c)-inverse of a and denoted by $a_{b,c}^{a,v,w}$.

In [10], it is proved that both $a_{b,c}^{h,v,w}$ and $a_{b,c}^{a,v,w}$ are unique. In view of Bott-Duffin inverse [7], we next introduce the (v, w)-weighted Bott-Duffin (e, f)-inverse.

Definition 2.10. *Let $a, v, w, e, f \in \mathcal{R}$ with $e^2 = e$ and $f^2 = f$. An element $z \in \mathcal{R}$ is called (v, w)-weighted Bott-Duffin (e, f)-inverse of a if it satisfies*

$$z = ewz = zvf,\ zvawe = e,\ fvawz = f.$$

The (v, w)-weighted Bott-Duffin (e, f)-inverse of the element a is denoted as $a_{e,f}^{b,v,w}$.

Example 2.11. Let $\mathcal{R} = M_2(\mathbb{R})$ with $a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $e = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, and $f = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to verify that the matrix $z = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ satisfies

$$zvawe = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = e,$$

$fvawz = f$, and $ewz = z = zvf$. Hence $a_{e,f}^{b,v,w} = z$.

3. Further results on (v, w)-weighted (b, c)-inverse

In this section, we derive a few useful representations and properties of (v, w)-weighted (b, c)-inverse.

Proposition 3.1. *Let $v, w, d \in \mathcal{R}$. Then the following hold:*

(i) *If $\mathcal{R}wy = \mathcal{R}d$ ($\mathcal{R}wy \subseteq \mathcal{R}d$) then $rann(wy) = rann(d)$ ($rann(d) \subseteq rann(wy)$).*

(ii) *If $yv\mathcal{R} = d\mathcal{R}$ ($yv\mathcal{R} \subseteq d\mathcal{R}$) then $lann(yv) = lann(d)$ ($lann(d) \subseteq lann(yv)$).*

(iii) *If $rann(d) \subseteq rann(wy)$ and d^- exists, then $\mathcal{R}wy \subseteq \mathcal{R}d$.*

(iv) *If $lann(d) \subseteq lann(yv)$ and d^- exists, then $yv\mathcal{R} \subseteq d\mathcal{R}$.*

Proof. (i) Let $x \in rann(wy)$. Then $wyx = 0$. From $\mathcal{R}wy = \mathcal{R}d$, we obtain $d = twy$ for some $t \in \mathcal{R}$. Now $dx = twyx = 0$. Hence $rann(wy) \subseteq rann(d)$. Again from $\mathcal{R}wy = \mathcal{R}d$, we have $wy = sd$ for some $s \in \mathcal{R}$. If $z \in rann(d)$ then $dz = 0$ and hence $wyz = sdz = 0$. Thus $rann(d) \subseteq rann(wy)$.
(ii) A similar argument as (i).
(iii) Let $x \in d\{1\}$. Then $(1 - xd) \in rann(d) \subseteq rann(wy)$, which implies $wy = (wyx)d$. Therefore, $\mathcal{R}wy \subseteq \mathcal{R}d$.
(iv) Is similar to part (iii). □

Proposition 3.2. *Let $a, b, c, v, w \in \mathcal{R}$. If a has (v, w)-weighted (b, c)-inverse, then both b and c are regular.*

Proof. Let y be the (v, w)-weighted (b, c)-inverse of a. Then by Definition 2.3, $yvawb = b$, $cvawy = c$ and $y \in bRwy \cap yvRc$. From the ideals, we further obtain $y = bswy$ and $y = yvtc$ for some $s, t \in \mathcal{R}$. Now $b = yvawb = bswyvawb = bswb$. Thus b is regular. Similarly, we have $c = cvawy = cvawyvtc = cvtc$ and completes the proof. □

An equivalent characterization of the (v, w)-weighted (b, c)-inverse is presented in the next result.

Theorem 3.3. *Let $a, b, c, v, w \in \mathcal{R}$. Then the following statements are equivalent:*

(i) *a has (v, w)-weighted (b, c)-inverse.*

(ii) *b is regular, $\mathcal{R} = \mathcal{R}cvaw \oplus lann(b)$ and $lann(vaw) \cap \mathcal{R}c = \{0\}$.*

(iii) *$\mathcal{R} = \mathcal{R}cvaw \oplus lann(b)$, $lann(vaw) \cap \mathcal{R}c = \{0\}$ and $cvawb$ is regular.*

(iv) *c is regular, $\mathcal{R} = vawb\mathcal{R} \oplus rann(c)$ and $rann(vaw) \cap b\mathcal{R} = \{0\}$.*

(v) *$\mathcal{R} = vawb\mathcal{R} \oplus rann(c)$, $rann(vaw) \cap b\mathcal{R} = \{0\}$ and $cvawb$ is regular.*

Proof. (i)⇒(ii) Assume that a has a (v, w)-weighted (b, c)-inverse. By Proposition 3.2, we have b is regular. From Lemma 2.4, there exist $p, q \in R$ such that $b = pcvawb$ and $c = cvawbq$. Let $r = 1 - pcvaw$. Then $r \in lann(b)$. For any $t \in \mathcal{R}$,

$$t = t \cdot 1 = t(pcvaw + r) = tpcvaw + tr \in \mathcal{R}cvaw + lann(b).$$

Therefore, $\mathcal{R} = \mathcal{R}cvaw + lann(b)$. If $u \in \mathcal{R}cvaw \cap lann(b)$ then $ub = 0$ and $u = xcvaw$ for some $x \in \mathcal{R}$. Now $xc = x(cvawbq) = (ub)q = 0$ and $u = xcvaw = 0$. Thus $\mathcal{R}cvaw \cap lann(b) = \{0\}$.

If $m \in lann(vaw) \cap \mathcal{R}c$ then $mvaw = 0$ and $m = sc$, for some $s \in \mathcal{R}$. Therefore, $m = sc = scvawbq = mvawbq = 0$ and hence $lann(vaw) \cap Rc = \{0\}$.
(ii)⇒(iii) Let $\mathcal{R} = \mathcal{R}cvaw \oplus lann(b)$. Then $1 = gcvaw + h$ for some $g \in \mathcal{R}$ and $h \in lann(b)$. Therefore, $b = gcvawb \in \mathcal{R}cvawb$, which implies $\mathcal{R}b \subseteq \mathcal{R}cvawb$. Since $\mathcal{R}cvawb \subseteq \mathcal{R}b$ is trivial, it follows that $\mathcal{R}b = \mathcal{R}cvawb$. From $\mathcal{R}b = \mathcal{R}cvawb$, we have $b = scvawb$ and $cvawb = tb$ for some $s, t \in \mathcal{R}$. Now

$$cvawb = tb = tbb^-b = cvawbb^-scvawb, \text{ where } b^- \in b\{1\}.$$

Hence, $cvawb$ is regular.
(iii)⇒(i) Let $\mathcal{R} = \mathcal{R}cvaw \oplus lann(b)$. Then $1 = gcvaw + h$ for some $g \in \mathcal{R}$ and $h \in lann(b)$. Therefore, $b = gcvawb \in \mathcal{R}cvawb$. Now, we will prove $lann(c) = lann(cvawb)$. Obviously, $lann(c) \subseteq lann(cvawb)$. For $x \in lann(cvawb)$, we have $xcvaw \in lann(b) \cap \mathcal{R}cvaw = \{0\}$, i.e. $xcvaw = 0$. This implies that $xc \in lann(vaw) \cap Rc = \{0\}$. Thus $x \in lann(c)$, i.e. $lann(c) = lann(cvawb)$. Now, let $t \in (cvawb)\{1\}$. Since $(1 - cvawbt)cvawb = 0$, we have $1 - cvawbt \in lann(cvawb) = lann(c)$. Thus, $c = cvawbtc \in cvawb\mathcal{R}$. By Lemma 2.4, a has (v, w)-weighted (b, c)-inverse.
The proof of (i)⇒(iv)⇒(v)⇒(i) is similar to (i)⇒(ii)⇒(iii)⇒(i). □

Theorem 3.4. *Let $a, b, c, w, v \in \mathcal{R}$. Then the following statements are equivalent:*

(i) $y = a^{v,w}_{b,c}$.

(ii) $yvawy = y$, $yv\mathcal{R} = b\mathcal{R}$ *and* $\mathcal{R}wy = \mathcal{R}c$.

(iii) $yvawy = y$, $lann(yv) = lann(b)$, $\mathcal{R}wy = \mathcal{R}c$, *and b is regular.*

(iv) $yvawy = y$, $yv\mathcal{R} = b\mathcal{R}$, $rann(wy) = rann(c)$, *and c is regular.*

(v) $yvawy = y$, $lann(yv) = lann(b)$ *and* $rann(wy) = rann(c)$, *and both b, c are regular.*

(vi) *both b, c are regular,* $y = bb^-y$, $bb^- = yvawbb^-$, $yc^-c = y$, *and* $c^-c = c^-cvawy$.

(vii) *both b, c are regular,* $bb^- \in \mathcal{R}(c^-cvawbb^-)$, *and* $c^-c \in (c^-cvawbb^-)\mathcal{R}$.

(viii) *both b, c are regular, and there exists $s, t \in \mathcal{R}$ such that* $bb^- = tc^-cvawbb^-$, $c^-c = c^-cvawbb^-s$.

Proof. (i)⇔(ii) The proof of this equivalence follows from Lemma 2.4.
(ii)⇒(iii) The regularity of b is follows from the equivalence of (ii)⇔(i) and Proposition 3.2. For any $z \in lann(yv)$, we have $zyv = 0$. Now $zb = zyvt = 0$. Thus $z \in lann(b)$ and subsequently $lann(yv) \subseteq lann(b)$. The reverse inclusion $lann(b) \subseteq lann(yv)$ can be shown similarly. Therefore, $lann(yv) = lann(b)$.
(iii)⇒(iv) Let $\mathcal{R}wy = \mathcal{R}c$. Then $wy = sc$ and $c = twy$ for some $s, t \in \mathcal{R}$. Now

$$c = twy = tw(yvawy) = cva(wy) = c(vas)c.$$

Thus c is regular. From $yvawy = y$, we have $(yvaw - 1) \in lann(y) \subseteq lann(yv) = lann(b)$. Further, $yvawb = b$. Thus $b\mathcal{R} \subseteq yv\mathcal{R}$. The reverse inclusion $yv\mathcal{R} \subseteq b\mathcal{R}$ can be shown using Proposition 3.1(iv). Hence $yv\mathcal{R} = b\mathcal{R}$. For any $z \in rann(wy)$, we have $wyz = 0$. Now $cz = twyz = 0$. This implies $z \in rann(c)$. Hence $rann(wy) \subseteq rann(c)$. On the other hand, if $x \in rann(c)$ then $cx = 0$. Now $wyx = scx = 0$. Therefore, $rann(wy) = rann(c)$.
(iv)⇒(v) It is enough to show b is regular and $lann(yv) = lann(b)$. The regularity of b and $lann(yv) = lann(b)$ can be proved in the similar way as (ii)⇒(iii).
(v)⇒(ii) Follows from Proposition 3.1(iii) and (iv).
(i)⇒(vi) Let $y = a^{v,w}_{b,c}$. Then there exist $s, t \in \mathcal{R}$ such that $y = bswy$ and $y = yvtc$. Now

$$bb^-y = bb^-bswy = bswy = y \text{ and } yvawbb^- = bb^-.$$

Similarly we can show, $yc^-c = y$ and $c^-c = c^-cvawy$. Further, the regularity of b and c follows by Proposition 3.2.
(vi)⇒(vii) If (vi) holds, then $bb^- = yvawbb^- = yc^-cvawbb^- \in \mathcal{R}(c^-cvawbb^-)$. Similarly we can show $c^-c \in (c^-cvawbb^-)\mathcal{R}$.
(vii)⇒(viii) It is obvious.
(viii)⇒(i) Let $bb^- = tc^-cvawbb^-$. Post-multiplying by b, we obtain $b = bb^-b = tc^-cvawb \in \mathcal{R}cvawb$. Similarly, pre-multiplying c to $c^-c = c^-cvawbb^-s$, we obtain $c = cvawbb^-s \in cvawb\mathcal{R}$. Hence by Lemma 2.4, we obtain $a_{b,c}^{v,w} = y$. □

The relation between group inverse and (v, w)-weighted (b, c)-inverse is presented in the next result.

Theorem 3.5. *Let $a, b, c, v, w \in \mathcal{R}$ and $a_{b,c}^{v,w}$ exist. If there exist an element $s \in \mathcal{R}$ such that $s\mathcal{R} = b\mathcal{R}$ and $rann(s) = rann(c)$, then $vaws, svaw \in \mathcal{R}^\#$ and $a_{b,c}^{v,w} = s(vaws)^\# = (svaw)^\# s$.*

Proof. First, we will show that $vaws \in \mathcal{R}^\#$ and $a_{b,c}^{v,w} = s(vaws)^\#$. Let $g \in rann(vaws)$. Then $vawsg = 0$, which implies $sg \in rann(vaw) \cap s\mathcal{R} = rann(vaw) \cap b\mathcal{R} = \{0\}$ by Theorem 3.3. It follows that $sg = 0$ and $g \in rann(s)$. Thus, $rann(vaws) \subseteq rann(s)$ and consequently $rann(vaws) = rann(s) = rann(c)$. Since $s\mathcal{R} = b\mathcal{R}$, we have $vaws\mathcal{R} = vawb\mathcal{R}$. Using Theorem 3.3, we get

$$\mathcal{R} = vawb\mathcal{R} \oplus rann(c) = vaws\mathcal{R} \oplus rann(vaws).$$

Thus $1 = vawsu + t$ for some $u \in \mathcal{R}$ and $t \in rann(vaws)$. Now $vaws = vawsvawsu$. This yields $vaws(vaws - vawsuvaws) = 0$. Hence $(vaws - vawsuvaws) \in rann(vaws) \cap vaws\mathcal{R} = \{0\}$ and subsequently,

$$vaws = vawsuvaws = vawsvawsu. \tag{1}$$

Clearly, $vawsu$ is idempotent. Using Proposition 2.1, we obtain $rann(vawsu) = (1 - vawsu)\mathcal{R}$. Using equation (1), we obtain $rann(vawsu) \subseteq rann(vaws)$. For $h \in rann(vaws)$, we have $vawsh = 0$. By equation (1), $vawsvawsuh = vawsh = 0$. Thus $vawsuh \in rann(vaws) \cap vaws\mathcal{R} = \{0\}$ and hence $rann(vaws) \subseteq rann(vawsu)$. Again, $vaws - uvawsvaws \in rann(vaws) = rann(vawsu)$. Thus $vawsu(vaws - uvawsvaws) = 0$. From equation (1), we get

$$vaws = vawsuvaws = (vawsu^2)vawsvaws. \tag{2}$$

From equations (1) and (2), we have $vaws \in \mathcal{R}(vaws)^2 \cap (vaws)^2\mathcal{R}$. Hence by Lemma 2.2, $vaws$ is group invertible.

Next we will show that $s(vaws)^\#$ is the (v, w)-weighted (b, c)-inverse of a. Let $t = s(vaws)^\#$. Then

$$tvawt = s(vaws)^\# vaws(vaws)^\# = s(vaws)^\# = t.$$

Clearly, $tv\mathcal{R} = (s(vaws)^\#)v\mathcal{R} \subseteq sv\mathcal{R} \subseteq s\mathcal{R} = b\mathcal{R}$. Since $vaws((vaws)^\# vaws - 1) = 0$ and $rann(vaws) = rann(s)$, it follows that $s(vaws)^\# vaws = s$. Hence,

$$bR = s\mathcal{R} = (s(vaws)^\# vaws)\mathcal{R} = tvaws\mathcal{R} \subseteq tv\mathcal{R}.$$

Similarly, we have

$$rann(wt) = rann(ws(vaws)^\#) \subseteq rann(vaws(vaws)^\#) = rann(vaws) = rann(s) = rann(c),$$

and

$$\begin{aligned} rann(c) &= rann(s) = rann(vaws) = rann(vaws(vaws)^\#) \subseteq rann(s(vaws)^\# vaws(vaws)^\#) \\ &= rann(s(vaws)^\#) = rann(t) \subseteq rann(wt). \end{aligned}$$

Hence by Proposition 3.2 and Theorem 3.4(iv), we obtain $a_{b,c}^{v,w} = t = s(vaws)^\#$. Similarly, it can be shown that $svaw \in \mathcal{R}^\#$ and $a_{b,c}^{v,w} = (vaws)^\# s$. □

Theorem 3.6. *Let $a, v, w \in \mathcal{R}$. If $e, f \in \mathcal{R}$ with $e^2 = e$ and $f^2 = f$, then the following are equivalent:*

(i) *$e \in e\mathcal{R}fvawe$ and $f \in fvawe\mathcal{R}f$.*

(ii) *there exist $m, n \in \mathcal{R}$ such that $p = mfvawe + 1 - e$ is invertible and $fvawep^{-1}n = f$.*

(iii) *there exist $m, n \in \mathcal{R}$ such that $q = fvawen + 1 - f$ is invertible and $mq^{-1}fvawe = e$.*

(iv) *there exist $m, n \in \mathcal{R}$ such that $p = mfvawe + 1 - e$ and $q = fvawen + 1 - f$ are invertible.*

Proof. (i)⇒ (ii),(iii) Let $e \in \mathcal{R}fvawe$ and $f \in fvawe\mathcal{R}$. Then there exist $m, n \in \mathcal{R}$ such that $e = mfvawe$ and $f = fvawen$. Take $p = mfvawe + 1 - e$ and $q = fvawen + 1 - f$. Then $p = q = 1$, $f = fvawen = fvawep^{-1}n$ and $e = mq^{-1}fvawe$.
(ii)⇒(i) From $f = fvawep^{-1}n$ and $pe = mfvawe$, we have $e = p^{-1}mfvawe$. Post-multiplying $f = fvawep^{-1}n$ by f and pre-multiplying $e = p^{-1}mfvawe$ by e, we obtain $e \in e\mathcal{R}fvawe$ and $f \in fvawe\mathcal{R}f$.
(iii)⇒ (i) Is similar to (ii)⇒(i).
(ii)⇒(iv) Using (ii), we have $f = fvawep^{-1}n$ and $p = mfvawe + 1 - e$ is invertible. Let $n_1 = p^{-1}n$. Then $fvawen_1 + 1 - f = f + 1 - f = 1$. Thus $q = fvawen_1 + 1 - f$ is invertible.
(iv)⇒(i) Let $p = mfvawe + 1 - e$. Then $pe = mfvawe$ and subsequently, $e = p^{-1}mfvawe$. Now $e = e^2 = ep^{-1}mfvawe \in e\mathcal{R}fvawe$. Similarly, we can show $f \in fvawe\mathcal{R}f$. □

Following the Definition 2.10, we present the following characterizations for (v, w)-weighted Bott-Duffin (e, f)-inverse.

Proposition 3.7. *Let $a, v, w, e, f \in \mathcal{R}$ with $e^2 = e$ and $f^2 = f$. If $a^{b,v,w}_{e,f}$ exist then $e \in e\mathcal{R}fvawe$ and $f \in fvawe\mathcal{R}f$.*

Proof. Let $z = a^{b,v,w}_{e,f}$. Then by Definition 2.10, $e = zvawe = ewzvawe = e(wzv)fvawe \in e\mathcal{R}fvawe$ and $f = fvawz = fvawzvf = fvawe(wzv)f \in fvawe\mathcal{R}f$. □

Theorem 3.8. *Let $a, e, f, v, w \in \mathcal{R}$ such that $e = e^* = e^2$ and $f = f^* = f^2$. If $a^{b,v,w}_{e,f}$ exist then the following hold:*

(i) $e \in \mathcal{R}(fvawe)^*fvawe$ and $f \in fvawe(fvawe)^*\mathcal{R}$.

(ii) $p = (fvawe)^*fvawe + 1 - e$ is invertible and $fvawep^{-1}(fvawe)^* = f$.

(iii) $q = fvawe(fvawe)^* + 1 - f$ is invertible and $(fvawe)^*q^{-1}fvawe = e$.

Proof. (i) Let $a^{b,v,w}_{e,f}$ exists. Then by Theorem 3.6 and Proposition 3.7, we get $r = gfvawe + 1 - e$ is invertible and $fvawer^{-1}h = f$ for some $g, h \in \mathcal{R}$. Using this, we have $e = r^{-1}gfvawe$. Now

$$e^* = (r^{-1}gfvawe)^* = (fvawe)^*(r^{-1}g)^* = (fvawe)^*f(r^{-1}g)^* = (fvawe)^*fvawer^{-1}h(r^{-1}g)^*.$$

Thus $e = (r^{-1}g)(r^{-1}h)^*(fvawe)^*fvawe \in \mathcal{R}(fvawe)^*fvawe$. Similarly, we can show that $f = fvawe(fvawe)^*(r^{-1}g)^*r^{-1}h$ and $f \in fvawe(fvawe)^*\mathcal{R}$.
(ii) From part (i), we have $e = r^{-1}gfvawe$ and $fvawer^{-1}h = f$. So $r^{-1}gf = er^{-1}h$.
Let $\beta = (r^{-1}g)(r^{-1}h)^*$. Then

$$\beta e = \beta e^* = (r^{-1}g)(r^{-1}h)^*(fvawe)^*(r^{-1}g)^* = r^{-1}gf(r^{-1}g)^* = er^{-1}h(r^{-1}g)^* = e\beta^*$$

Thus $(\beta e+1-e)((fvawe)^*fvawe + 1 - e) = 1 = ((fvawe)^*fvawe + 1 - e)(\beta e+1-e)$. Hence $p = (fvawe)^*fvawe+1-e$ is invertible and $p^{-1} = (\beta e + 1 - e)$. Further,

$$\begin{aligned} fvawep^{-1}(fvawe)^* &= fvawe(\beta e + 1 - e)(fvawe)^* = fvawe\beta^*(fvawe)^* = fvawer^{-1}g(r^{-1}h)^*(fvawe)^* \\ &= fvawer^{-1}gf = fvawer^{-1}h = f. \end{aligned}$$

(iii) Analogous to (ii). □

4. Hybrid (v,w)-weighted (b,c)-inverse

First we discuss an equivalent definition of the hybrid (v, w)-weighted (b, c)-inverse, which will help us to prove more characterizations of this inverse.

Theorem 4.1. *Let $a, b, c, v, w, y \in \mathcal{R}$ with either v or w invertible. Then the followings are equivalent.*

(i) $yvawy = y$, $yv\mathcal{R} = b\mathcal{R}$ *and* $rann(wy) = rann(c)$.

(ii) $yvawb = b$, $cvawy = c$, $yv\mathcal{R} \subseteq b\mathcal{R}$ *and* $rann(c) \subseteq rann(wy)$.

Proof. (i)⇒(ii) Let $yv\mathcal{R} = b\mathcal{R}$. Then there exist a $t \in \mathcal{R}$ such that $b = yvt$. Now $b = yvt = yvawyvt = yvawb$. From $yvawy = y$, we obtain $1 - vawy \in rann(y) \subseteq rann(wy) = rann(c)$. Thus $c = cvawy$.
(ii)⇒(i) Let $yv\mathcal{R} \subseteq b\mathcal{R}$. Then $yv = br$ for some $r \in \mathcal{R}$. Multiplying $yvawb = b$ by rv^{-1} on the right gives $yvawy = y$. If w is invertible then $yvawy = y$ is similarly follows from $cvawy = c$. Using $yvawb = b$, we get $b\mathcal{R} \subseteq yvR$, and hence $yv\mathcal{R} = b\mathcal{R}$. Now, let $s \in rann(wy)$. Then $wys = 0$. Further, $s \in rann(c)$ since $cs = cvawys = 0$. Hence $rann(wy) = rann(c)$. □

In view of Lemma 2.8, we explore a necessary condition for the hybrid (v, w)-weighted (b, c)-inverse in the below result.

Theorem 4.2. *Let $a, b, c, v, w \in \mathcal{R}$ with either v or w invertible. If the hybrid (v, w)-weighted (b, c)-inverse of a exists, then there exist a $t \in \mathcal{R}$ such that bt is the hybrid (v, w)-weighted (b, c)-inverse of a satisfying $c = cvawbt$.*

Proof. Let a has a hybrid (v, w)-weighted (b, c)-inverse. Then by Lemma 2.8,

$$c = cvawbt \text{ for some } t \in \mathcal{R}.$$

Let $y = bt$. Now we will claim that y is the hybrid (v, w)-weighted (b, c)-inverse of a. Clearly, $yv\mathcal{R} = btv\mathcal{R} \subseteq b\mathcal{R}$. Using $c = cvawbt$, we obtain $cvawb = cvawbtvawb$. Thus $(1 - tvawb) \in rann(cvawb) \subseteq rann(b)$ by Lemma 2.8. Hence

$$b = btvawb = yvawb.$$

For $x \in rann(c)$, we have $cvawbtx = cx = 0$, which yields $tx \in rann(cvawb)$. Further, by Lemma 2.8, $tx \in rann(b)$ and consequently $wyx = wbtx = 0$. Therefore,

$$rann(c) \subseteq rann(wy).$$

By Theorem 4.1, we get $y = bt$ is the hybrid (v, w)-weighted (b, c)-inverse of a. □

A necessary and sufficient condition for the existence of hybrid (v, w)-weighted (b, c) is presented below.

Theorem 4.3. *Let $a, b, c, v, w \in \mathcal{R}$ with either v or w invertible. Then $a^{h,v,w}_{b,c}$ exists if and only if $\mathcal{R} = vawb\mathcal{R} \oplus rann(c)$ and $rann(vaw) \cap b\mathcal{R} = \{0\}$.*

Proof. Let a has a hybrid (v, w)-weighted (b, c)-inverse. Then by Theorem 4.2, bt is the hybrid (v, w)-weighted (b, c)-inverse of a satisfying $c = cvawbt$, where $t \in \mathcal{R}$. Subsequently $z := (1 - vawbt) \in rann(c)$. For any $x \in \mathcal{R}$, we can write

$$x = 1 \cdot x = (z + vawbt)x = zx + vawbtx \in rann(c) + vawb\mathcal{R}.$$

Thus $\mathcal{R} = rann(c) + vawb\mathcal{R}$ since the reverse inclusion is trivial. If $r \in rann(c) \cap vawb\mathcal{R}$, then $cr = 0$ and $r = vawbu$ for some $u \in \mathcal{R}$. From Theorem 4.1, taking $y = bt$, we have

$$rann(wbt) = rann(c) \text{ and } btvawb = b, \tag{3}$$

which yields $wbtvawbu = wbtr = 0$ and $r = vaw(b)u = va(wbtvawbu) = 0$. Hence $R = vawb\mathcal{R} \oplus rann(c)$. Next we will show that $rann(vaw) \cap b\mathcal{R} = \{0\}$. Let $h \in rann(vaw) \cap b\mathcal{R}$, then $vawh = 0$ and $h = bk$ for some $k \in \mathcal{R}$, which implies $vawbk = 0$. Using second part of equation (3), we have $h = bk = bt(vawbk) = 0$. Thus

$rann(vaw) \cap b\mathcal{R} = \{0\}$.
Conversely, let $\mathcal{R} = vawbR \oplus rann(c)$. Then $1 = vawbm + n$ for some $m \in \mathcal{R}$ and $n \in rann(c)$. Further,

$$c = cvawbm + cn = cvawbm \in cvawb\mathcal{R} \text{ since } n \in rann(c). \tag{4}$$

If $x \in rann(cvawb)$, then $cvawbx = 0$ and hence $vawbx \in rann(c) \cap vawb\mathcal{R} = \{0\}$. Thus $bx \in rann(vaw) \oplus b\mathcal{R} = \{0\}$ since $rann(vaw) \cap b\mathcal{R} = \{0\}$. Hence $x \in rann(b)$ and subsequently, we obtain

$$rann(cvawb) \subseteq rann(b). \tag{5}$$

In view of equations (4), (5) and Lemma 2.8, a has a hybrid (v, w)-weighted (b, c)-inverse. □

Lemma 4.4. *Let $a, b, c, v, w \in \mathcal{R}$ with either v or w invertible. Assume that $a^{h,v,w}_{b,c}$ exists. Then $a^{v,w}_{b,c}$ exists if and only if any one of the following holds.*

(i) *$cvawb$ is regular.*

(ii) *c is regular.*

Proof. (i) Let $a^{v,w}_{b,c}$ exists. Then $b \in \mathcal{R}cvawb$ and $c \in cvawb\mathcal{R}$. Furtehr, $b = scvawb$ and $c = cvawbt$ for some $s, t \in \mathcal{R}$. Now

$$b = scvawb = scvawbtvawb = btvawb,\ c = cvawbt = cvawscvawbt = c(vaws)c, \text{ and}$$

$$cvawb = cvawbtvawscvawb = cvawb(tvaws)cvawb.$$

Hence $cvawb$ is regular.

Conversely, let $cvawb$ be regular. Then there exist an element $z \in \mathcal{R}$ such that $cvawb = cvawbzcvawb$. Since a has a hybrid (v, w)-weighted (b, c)-inverse, by Lemma 2.8, we have $c \in cvawb\mathcal{R}$ and $b = bzcvawb \in \mathcal{R}cvawb$ due to the fact that $1 - zcvawb \in rann(cvawb) \subseteq rann(b)$. Hence by Lemma 2.4, a has a (v, w)-weighted (b, c)-inverse.
(ii) The regularity of c is follows from Theorem 3.4.

Conversely, let c be regular and the hybrid (v, w)-weighted (b, c)-inverse of a exist. Then by Theorem 4.3, $\mathcal{R} = vawb\mathcal{R} \oplus rann(c)$ and subsequently $1 = vawbs + t$ for some $s \in \mathcal{R}$ and $t \in rann(c)$. Therefore, $c = cvawbs + ct = cvawbs$. Since c is regular, there exist an element $x \in \mathcal{R}$ such that $c = cxc$. Now

$$cvawb = (c)vawb = (c)xcvawb = cvawbsxcvawb = cvawb(sx)cvawb.$$

Thus $cvawb$ is regular. Hence by part (i), a has a (v, w)-weighted (b, c)-inverse. □

We next present the following characterizations of hybrid (v, w)-weighted (b, c) through annihilators.

Theorem 4.5. *Let $a, b, c, v, w \in \mathcal{R}$ with either v or w invertible. If a has a hybrid (v, w)-weighted (b, c)-inverse then the following statements hold:*

(i) *$rann(vawb) = rann(b)$.*

(ii) *If $rann(b) = rann(c)$ then $rann(vawbs) = rann(vawb)$, where $s \in \mathcal{R}$ satisfies $vawb = (vawb)^2 s$.*

Proof. (i) It is trivial that $rann(b) \subseteq rann(vawb)$. Let a has a hybrid (v, w)-weighted (b, c)-inverse. Then by Theorem 4.3, $rann(vaw) \cap b\mathcal{R} = \{0\}$. For $r \in rann(vawb)$, we have $br \in rann(vaw)$ and $br \in b\mathcal{R}$. Thus $br \in rann(vaw) \cap b\mathcal{R} = \{0\}$ and hence $r \in rann(b)$. Therefore, $rann(vawb) \subseteq rann(b)$.
(ii) Using the condition $\mathcal{R} = vawb\mathcal{R} \oplus rann(c)$ of Theorem 4.3, we have $1 = vawbs + t$ for some $s \in \mathcal{R}$ and $t \in rann(c) = rann(b)$. Thus $b = bvawbs$ and $vawb = vawbvawbs = (vawb)^2 s$. Let $x \in rann(vawb)$. Then $(vawb)^2 sx = vawbx = 0$. Now

$$vawbsx \in rann(vawb) \cap vawb\mathcal{R} = rann(b) \cap vawb\mathcal{R} = rann(c) \cap vawb\mathcal{R} = \{0\}.$$

Therefore $x \in rann(vawbs)$ and consequently, $rann(vawb) \subseteq rann(vawbs)$.

Conversely, let $z \in rann(vawbs)$. Then $vawbz = (vawb)^2sz = 0$, which implies $z \in rann(vawb)$. Thus, $rann(vawbs) \subseteq rann(vawb)$ and hence $rann(vawbs) = rann(vawb)$, where s satisfies $vawb = (vawb)^2s$. □

The following result represent a necessary and sufficient condition for hybrid (v,w)-weighted (b,c) inverse through group inverse.

Theorem 4.6. *Let $a,b,c,v,w \in \mathcal{R}$ with either v or w invertible. Assume that $rann(vawb) = rann(b) = rann(c)$. Then the hybrid (v,w)-weighted (b,c)-inverse of a exists if and only of $vawb$ is group invertible.*

Proof. Let a have a hybrid (v,w)-weighted (b,c)-inverse. Then by Theorem 4.3, we have $1 = vawbs + t$ for some $s \in \mathcal{R}$ and $t \in rann(c) = rann(vawb)$, which implies

$$vawb = (vawb)^2s \in (vawb)^2\mathcal{R} \text{ and } (vawb)^2 = (vawb)^2svawb. \tag{6}$$

Using the second part of equation (6) and Theorem 4.3, we obtain

$$vawb - vawbsvawb \in rann(vawb) \cap vawb\mathcal{R} = rann(c) \cap vawb\mathcal{R} = \{0\}.$$

Thus

$$vawb = vawbsvawb \text{ and } (vawb)^2 = vawbs(vawb)^2. \tag{7}$$

Applying equation (7) and Theorem 4.5, we have

$$vawb - s(vawb)^2 \in rann(vawb) = rann(vawbs).$$

Further, $vawbs^2(vawb)^2 = vawbsvawb = vawb$ and $vawb \in \mathcal{R}(vawb)^2$. Hence by Lemma 2.2, $vawb$ is group invertible since $vawb \in (vawb)^2\mathcal{R} \cap \mathcal{R}(vawb)^2$.

Conversely, let $y = b(vawb)^\#$. From $vawb = (vawb)^2(vawb)^\#$ and $rann(vawb) = rann(c)$, we have $c(1 - vawb(vawb)^\#) = 0$ and

$$c = cvawb(vawb)^\# = cvawy.$$

Similarly by applying $rann(vawb) = rann(b)$, we obtain

$$b = b(vawb)^\#vawb = yvawb.$$

The condition $yv\mathcal{R} \subseteq y\mathcal{R} \subseteq bR$ follows from $y = b(vawb)^\#$. Next we will show that $rann(c) \subseteq rann(wy)$. Let $x \in rann(c) = rann(b)$. Then $bx = 0$.
Now $wyx = wb(vawb)^\#x = wb(vawb)^\#(vawb)^\#vawbx = 0$. Thus $x \in rann(wy)$ and hence $rann(c) \subseteq rann(wy)$. By Theorem 4.1, $y = b(vawb)^\#$ is the hybrid (v,w)-weighted (b,c)-inverse of a. □

Remark 4.7. *Let $a,b,c,v,w \in \mathcal{R}$ with either v or w invertible. If $rann(vawb) = rann(b) = rann(c)$ and $vawb$ is group invertible, then $b(vawb)^\#$ is the hybrid (v,w)-weighted (b,c)-inverse of a.*

Corollary 4.8. *Let $a,b,c,v,w \in \mathcal{R}$ with either v or w be invertible and $y = a_{b,c}^{h,v,w}$. Then $rann(b) = rann(vawb) = rann(c)$ if and only of $vawb$ is group invertible with $y = b(vawb)^\#$.*

Proof. Let $vawb$ be group invertible with $y = b(vawb)^\#$. From $y = a_{b,c}^{h,v,w}$, we have $rann(c) = rann(wy)$. If $x \in rann(vawb)$, then by Theorem 4.3, $bx \in rann(vaw) \cap b\mathcal{R} = \{0\}$ and hence $x \in rann(b)$. Thus $rann(vawb) = rann(b)$ since the reverse inclusion $rann(b) \subseteq rann(vawb)$ is obvious. Next we will show that $rann(b) = rann(wy)$. Let $z \in rann(b)$. Then $bz = 0$ and consequently

$$wyz = wb(vawb)^\#z = wb(vawb)^\#(vawb)^\#vawbz = 0.$$

Therefore, $z \in rann(wy)$ and $rann(b) \subseteq rann(wy)$. If $x \in rann(wy)$, then $wyx = 0$ and

$$vawbx = (vawb)^2(vawb)^\#x = vawbvawyx = 0.$$

Further, by Theorem 4.3, we obtain $bx \in rann(vaw) \cap b\mathcal{R} = \{0\}$. Thus $x \in rann(b)$. Hence $rann(b) = rann(wy) = rann(c)$.

The converse part follows from Theorem 4.6. □

The following result presents hybrid (v, w)-weighted (b, c) inverse in the relationships with annihilators and (v, w)-weighted inverse of a along $d \in \mathcal{R}$.

Theorem 4.9. *Let $a, d, v, w, y \in \mathcal{R}$ with either v or w invertible. Then the following statements are equivalent:*

(i) *y is the (v, w)-weighted inverse of a along d.*

(ii) *$yvawd = d = dvawy$, $\mathcal{R}wy \subseteq \mathcal{R}d$, and $lann(d) \subseteq lann(yv)$.*

(iii) *$yvawy = y$, $\mathcal{R}wy = \mathcal{R}d$, and $lann(yv) = lann(d)$.*

(iv) *$yvawd = d = dvawy$, $yv\mathcal{R} \subseteq d\mathcal{R}$, and $rann(wy) = rann(d)$.*

(v) *$yvawy = y$, $yv\mathcal{R} = d\mathcal{R}$, and $rann(wy) = rann(d)$.*

(vi) *y is the hybrid (v, w)-weighted (d, d)-inverse of a.*

(vii) *y is the (v, w)-weighted (d, d)-inverse of a.*

Proof. (i)⇒(ii) The proof follows from the Definition 2.5 and Proposition 3.1 (ii).
(ii)⇒(iii) Let $\mathcal{R}wy \subseteq \mathcal{R}d$. Then $wy = sd$ for some $s \in \mathcal{R}$. Pre-multiplying $d = dvawy$ by $w^{-1}s$, we obtain

$$y = w^{-1}sd = w^{-1}sdvawy = yvawy.$$

From $d = dvawy$, we have $\mathcal{R}d \subseteq \mathcal{R}wy$ and hence $\mathcal{R}wy = \mathcal{R}d$. Next we will show that $lann(yv) \subseteq lann(d)$. If $z \in lann(yv)$, then by applying $yvawd = d$, we obtain

$$zd = z(yvawd) = (zyv)awd = 0.$$

Thus $lann(yv) \subseteq lann(d)$ and consequently $lann(d) = lann(yv)$.
(iii)⇒(iv) Let $y = yvawy$. Then $(1 - yvaw) \in lann(y) \subseteq lann(yv) = lann(d)$. Thus $d = yvawd$. From $\mathcal{R}d = \mathcal{R}wy$, we have

$$d = swy \text{ and } wy = td \text{ for some } s, t \in \mathcal{R}. \tag{8}$$

Pre-multylying $yvawy = y$ by sw, we obtain $d = sw(y) = (swy)vawy = dvawy$. The condition $rann(wy) = rann(d)$ follows from Proposition 3.1 (i). Using the second part of equation (8), we get d is regular since

$$d = dvawy = d(vat)d.$$

Hence by Proposition 3.1 (iv), $yv\mathcal{R} \subseteq d\mathcal{R}$.
(iv)⇒(v) The proof is similar to (ii)⇒(iii).
(v)⇔(vi) This part is trivial and follows from the definition.
(v)⇒(vii) To establish the result, it is sufficient to show

$$yvawd = d = dvawy \text{ and } y \in yv\mathcal{R}d \cap d\mathcal{R}wy.$$

Let y be the hybrid (v, w)-weighted (d, d)-inverse of a. Then $yvawy = y$, $yvR = dR$ and $rann(wy) = rann(d)$. From $y = yvawy$, we obtain $(1 - vawy) \in rann(y) \subseteq rann(wy) = rann(d)$. Thus $d = dvawy$. From $yv\mathcal{R} = d\mathcal{R}$, we have $d = yvs$ and $yv = dt$ for some $s, t \in \mathcal{R}$. Therefore,

$$y = yvawy = d(ta)wy \in d\mathcal{R}wy,\ d = yvs = yvawyvs = yvawd \text{ and } d = d(taw)d.$$

Hence d is regular and by Proposition 3.1 (iii), we obtain $\mathcal{R}wy \subseteq \mathcal{R}d$, which implies $wy = zd$ for some $z \in \mathcal{R}$ and $y = yvawy = yv(az)d \in yv\mathcal{R}d$. Hence y is the (v, w)-weighted (d, d)-inverse of a.
(vii)⇒(i) Let y be the (v, w)-weighted (d, d)-inverse of a. Then $yvawd = d = dvawy$, and $y = dswy = yvtd$ for some $s, t \in \mathcal{R}$. To establish the result, it is enough to show $yv\mathcal{R} \subseteq d\mathcal{R}$ and $\mathcal{R}wy \subseteq \mathcal{R}d$. Since $yv = d(swyv) = ds_1$ and $wy = wyvtd = t_1d$ for some $s_1 = swyv \in \mathcal{R}$ and $t_1 = wyvt \in \mathcal{R}$, it follows that $yv\mathcal{R} \subseteq d\mathcal{R}$ and $\mathcal{R}wy \subseteq \mathcal{R}d$. Hence by Definition 2.5, y is the (v, w)-weighted inverse of a along d. □

In view of Theorem 4.9, and taking $b = c = d$ in Theorem 3.3, we obtain the following result as a corollary.

Corollary 4.10. *Let $a, d, v, w \in \mathcal{R}$ with either v or w invertible. Then the following statements are equivalent:*

(i) *a has a (v, w)-weighted inverse along d.*

(ii) *d is regular, $\mathcal{R} = \mathcal{R}dvaw \oplus lann(d)$, and $lann(vaw) \cap \mathcal{R}d = \{0\}$.*

(iii) *$\mathcal{R} = \mathcal{R}dvaw \oplus lann(d)$, $lann(vaw) \cap \mathcal{R}d = \{0\}$ and $dvawd$ is regular.*

(iv) *d is regular, $\mathcal{R} = vawd\mathcal{R} \oplus rann(d)$, and $rann(vaw) \cap d\mathcal{R} = \{0\}$.*

(v) *$\mathcal{R} = vawd\mathcal{R} \oplus rann(d)$, $rann(vaw) \cap d\mathcal{R} = \{0\}$ and $dvawd$ is regular.*

The relation between (v, w)-weighted inverse along $d \in \mathcal{R}$ and the group inverse of an element is discussed in the next result.

Corollary 4.11. *Let $a, d, v, w \in \mathcal{R}$. Then $a^{v,w}_{\|d}$ exists if and only if $vawd$ is group invertible and $rann(vawd) = rann(d)$.*

Proof. Let y be the (v, w)-weighted inverse of a along along d. Then by Theorem 4.9, y is the hybrid (v, w)-weighted (d, d)-inverse of a and $yvawd = d$. From the condition $yvawd = d$, we have $\mathcal{R}d \subseteq \mathcal{R}vawd$. Then $rann(vawd) \subseteq rann(d)$ follows directly from Proposition 3.1 (i). Hence $rann(vawd) = rann(d)$ since the reverse inclusion $rann(d) \subseteq rann(vawd)$ is trivial. Replacing b and c by d in Corollary 4.8, we get $vawd$ is group invertible. The converse part follows from Theorem 4.6 □

5. Annihilator (v,w)-weighted (b,c)-inverse

This section is devoted to the characterizations of annihilator (v,w)-weighted (b,c)-inverse. The first result is represent an equivalent definition of annihilator (v, w)-weighted (b, c)-inverse, which will be used in the subsequent results.

Theorem 5.1. *Let $a, b, c, v, w, y \in \mathcal{R}$ with either v or w invertible. Then the following statements are equivalent:*

(i) *$yvawy = y$, $rann(wy) = rann(c)$ and $lann(yv) = lann(b)$.*

(ii) *$yvawb = b$, $cvawy = c$, $rann(c) \subseteq rann(wy)$ and $lann(b) \subseteq lann(yv)$.*

Proof. (i)⇒(ii) Let $yvawy = y$. Then $(yvaw - 1) \in lann(yv) = lann(b)$. This yields $yvawb = b$. Similarly $cvawy = c$ follows from

$$(vawy - 1) \in rann(wy) = rann(c).$$

Hence completes the proof.
(ii)⇒(i) Let $rann(c) \subseteq rann(wy)$ and $cvawy = c$. Then $(vawy - 1) \in rann(c) \subseteq rann(wy)$. Thus $wyvawy = wy$. Similarly, from $lann(b) \subseteq lann(yv)$ and $yvawb = b$, we can obtain $yvawyv = yv$. If either v or w is invertible then $yvawy = y$. Next we will claim that $rann(wy) \subseteq rann(c)$ and $lann(yv) \subseteq lann(b)$. For $x \in rann(wy)$, we have $wyx = 0$. Now $cx = cva(wyx) = 0$. Thus $rann(wy) \subseteq rann(c)$. If $z \in lann(yv)$, then $zyv = 0$. Further, $zb = (zyv)awb = 0$. Hence $lann(yv) \subseteq lann(b)$. □

With the help of Theorem 5.1 (i), we present the following property of annihilator (v, w)-weighted (b, c)-inverse.

Proposition 5.2. *For $i = 1, 2$, let $a_i, b_i, c_i, v, w, y_i \in \mathcal{R}$ with both v, w invertible and $y_i = {a_i}_{b,c}^{a,v,w}$. If $rc_1 = c_2 r$ $rva_1 w = va_2 wr$ and $rb_1 = b_2 r$ for any $r \in \mathcal{R}$, then $ry_1 = y_2 r$.*

Proof. Let $y_i = {a_i}_{b,c}^{a,v,w}$. Then by Theorem 5.1, we obtain $y_2 va_2 wb_2 = b_2$ and $lann(b_1) \subseteq lann(y_1 v)$. Thus

$$rb_1 = b_2 r = y_2 va_2 wb_2 r = y_2 (va_2 wr) b_1 = y_2 (rva_1 w) b_1.$$

Further, $(r - y_2 rva_1 w) \in lann(b_1) \subseteq lann(y_1 v)$, which implies

$$ry_1 v = y_2 rva_1 wy_1 v \tag{9}$$

Similarly, we have $c_2 r = rc_1 = rc_1 va_1 wy_1 = c_2 rva_1 wy_1$ and $(r - rva_1 wy_1) \in rann(c_2) \subseteq rann(wy_2)$. Thus

$$wy_2 r = wy_2 rva_1 wy_1. \tag{10}$$

Using the invetibility of v and w in equation (9) and (10), we get $ry_1 = y_2 r$. □

In the similar manner, we have the following result for the (v, w)-weighted (b, c)-inverse.

Corollary 5.3. *For $i = 1, 2$, let $a_i, b_i, c_i, v, w, y_i \in \mathcal{R}$ and y_i be the (v, w)-weighted (b_i, c_i)-inverse of a_i. If $rc_1 = c_2 r$, $rva_1 w = va_2 wr$ and $rb_1 = b_2 r$ for any $r \in \mathcal{R}$, then $ry_1 = y_2 r$.*

Proof. We first note that $(rva_1 w)b_1 = (va_2 wr)b_1 = va_2 w(rb_1)$ and similarly $rva_1 wb_1 = va_2 wb_2 r$, $rc_1 va_1 w = c_2 va_2 wr$. Since y_i is the (v, w)-weighted (b_i, c_i)-inverse of a_i, we have $c_1 va_1 wy_1 = c_1$ and $y_2 va_2 wb_2 = b_2$. Also we can write $y_1 = b_1 ewy_1$ and $y_2 = y_2 vfc_2$ for some $e, f \in \mathcal{R}$. Now we find

$$\begin{aligned} ry_1 &= r(b_1 ewy_1) = (rb_1)ewy_1 = (b_2 r)ewy_1 = (y_2 va_2 wb_2)rewy_1 = y_2(va_2 wb_2 r)ewy_1 \\ &= y_2(rva_1 wb_1)ewy_1 = y_2 rva_1 w(b_1 ewy_1) = y_2 rva_1 wy_1, \end{aligned}$$

$$\begin{aligned} y_2 r &= (y_2 vfc_2)r = y_2 vf(c_2 r) = y_2 vf(rc_1) = y_2 vfr(c_1 va_1 wy_1) = y_2 vf(rc_1 va_1 w)y_1 = y_2 vf(c_2 va_2 wr)y_1 \\ &= (y_2 vfc_2)va_2 wry_1 = y_2(va_2 wr)y_1 = y_2(rva_1 w)y_1. \end{aligned}$$

Hence $ry_1 = y_2 r$. □

The next result concerning on the reverse order law for the annihilator (v, w)-weighted (b, c)-inverse.

Theorem 5.4. *Let $s, t, b, c, v, w \in \mathcal{R}$ with both v and w invertible. Assume that both $s_{b,c}^{a,v,w}$ and $t_{b,c}^{a,v,w}$ exists. If $bvtw = vtwb$ and $cvsw = vswc$ then $(swvt)_{b,c}^{a,v,w} = t_{b,c}^{a,v,w} s_{b,c}^{a,v,w}$.*

Proof. Let $y = t_{b,c}^{a,v,w} s_{b,c}^{a,v,w}$. Then we have

$$yv(swvt)wb = t_{b,c}^{a,v,w} s_{b,c}^{a,v,w} vswvtwb = t_{b,c}^{a,v,w} s_{b,c}^{a,v,w} vswbvtw = t_{b,c}^{a,v,w} bvtw = t_{b,c}^{a,v,w} vtwb = b.$$

Similalry, we can show $cv(swvt)wy = cvswvtwt_{b,c}^{a,v,w} s_{b,c}^{a,v,w} = c$. From Definition 2.9, we have $lann(b) = lann(t_{b,c}^{a,v,w} v)$ and $rann(c) = rann(ws_{b,c}^{a,v,w})$. Now for any $z \in lann(b)$, we obtain $zt_{b,c}^{a,v,w} v = 0$ and

$$zyv = zt_{b,c}^{a,v,w} s_{b,c}^{a,v,w} v = zt_{b,c}^{a,v,w} vtwt_{b,c}^{a,v,w} s_{b,c}^{a,v,w} v = 0.$$

Hence $lann(b) \subseteq lann(yv)$. Let $z \in rann(c) = rann(ws_{b,c}^{a,v,w})$. Then $ws_{b,c}^{a,v,w} z = 0$. Now

$$wyz = wt_{b,c}^{a,v,w} s_{b,c}^{a,v,w} z = wt_{b,c}^{a,v,w} s_{b,c}^{a,v,w} vswsw_{b,c}^{a,v,w} z = 0.$$

Thus $rann(c) \subseteq rann(wy)$. Hence by Theorem 5.1 (ii), we obtain $(swvt)_{b,c}^{a,v,w} = y = t_{b,c}^{a,v,w} s_{b,c}^{a,v,w}$. □

Corollary 5.5. *Let $s, t, b, c, v, w \in \mathcal{R}$ and both $s_{b,c}^{v,w}$, $t_{b,c}^{v,w}$ exists. If $cvsw = vswc$ and $bvtw = vtwb$ then $(swvt)_{b,c}^{v,w} = t_{b,c}^{v,w} s_{b,c}^{v,w}$.*

Proof. Let $y = t_{b,c}^{v,w} s_{b,c}^{v,w}$. Then

$$yvswvtwb = t_{b,c}^{v,w}s_{b,c}^{v,w}vswvtwb = t_{b,c}^{v,w}s_{b,c}^{v,w}vswbvtw = t_{b,c}^{v,w}bvtw = t_{b,c}^{v,w}vtwb = b.$$

Similalry,

$$cvswvtwy = cvswvtwt_{b,c}^{v,w}s_{b,c}^{v,w} = vswcvtwt_{b,c}^{v,w}s_{b,c}^{v,w} = vswcs_{b,c}^{v,w} = cvsws_{b,c}^{v,w} = c.$$

Since $t_{b,c}^{v,w} \in b\mathcal{R}wt_{b,c}^{v,w}$, we have

$$y = t_{b,c}^{v,w}s_{b,c}^{v,w} \in b\mathcal{R}wt_{b,c}^{v,w}s_{b,c}^{v,w} = b\mathcal{R}wy.$$

From $s_{b,c}^{v,w} \in s_{b,c}^{v,w}v\mathcal{R}c$, we obtain

$$y = t_{b,c}^{v,w}s_{b,c}^{v,w} \in t_{b,c}^{v,w}s_{b,c}^{v,w}v\mathcal{R}c = yv\mathcal{R}c.$$

Hence $(swvt)_{b,c}^{v,w} = t_{b,c}^{v,w}s_{b,c}^{v,w}$. $\square$

The following result present annihilator (v, w)-weighted (b, c) inverse in the relationships with hybrid (v, w)-weighted (b, c) inverse and (v, w)-weighted inverse of a along $d \in \mathcal{R}$.

Proposition 5.6. *Let $a, v, w, y, e \in \mathcal{R}$ with either v or w invertible and let e be regular. Then the following conditions are equivalent:*

(i) $y = a_{\|e}^{v,w}$.

(ii) $yvawe = e = evawy$, $\mathcal{R}wy \subseteq \mathcal{R}e$ *and* $lann(e) \subseteq lann(yv)$.

(iii) $yvawy = y$, $\mathcal{R}wy \subseteq \mathcal{R}e$ *and* $lann(yv) = lann(e)$.

(iv) $yvawe = e = evawy$, $yv\mathcal{R} \subseteq e\mathcal{R}$ *and* $rann(e) \subseteq rann(wy)$.

(v) $yvawy = y$, $yv\mathcal{R} \subseteq e\mathcal{R}$ *and* $rann(wy) = rann(e)$.

(vi) $y = a_{e,e}^{h,v,w}$.

(vii) $y = a_{e,e}^{v,w}$.

(viii) $y = a_{e,e}^{a,v,w}$.

Proof. The equivalence of (i)⇔(vii) follows from Theorem 4.9. Next we will show that (vi)⇔(viii). Since $a_{e,e}^{h,v,w}$ is a special case of $a_{e,e}^{a,v,w}$, it is enough show (viii)⇒(vi). Let $y = a_{e,e}^{a,v,w}$. Then

$$yvawy = y,\ rann(wy) = rann(e), \text{ and } lann(yv) = lann(e).$$

Clearly both e and y are regular. So by Proposition 3.1, $lann(yv) = lann(e)$ gives $yv\mathcal{R} = e\mathcal{R}$ and $rann(wy) = rann(e)$ gives $\mathcal{R}wy = \mathcal{R}e$. Hence $y = a_{e,e}^{v,w}$. $\square$

Lemma 5.7. *For $i = 1, 2$, let $a_i, b_i, c_i, v, w, y_i \in \mathcal{R}$ with v, w both invertible and $y_i = {a_i}_{b_i,c_i}^{a,v,w}$. If $b_1 = b_2$ then $y_1va_1wy_2 = y_2$ and $y_2va_2wy_1 = y_1$. Mutually, if $c_1 = c_2$ then $y_1va_2wy_2 = y_1$ and $y_2va_1wy_1 = y_2$.*

Proof. Let $y_i = {a_i}_{b_i,c_i}^{a,v,w}$. Then by Theorem 5.1, $y_1va_1wb_1 = b_1$ and consequently,

$$(y_1va_1w - 1) \in lann(b_1) = lann(b_2) \subseteq lann(y_2v).$$

Thus $y_1va_1wy_2v = y_2v$. Post-multiplying by v^{-1} we get $y_1va_1wy_2 = y_2$. From $y_2va_2wb_2 = b_2$, we have

$$(y_2va_2w - 1) \in lann(b_2) = lann(b_1) \subseteq lann(y_1v).$$

Therefore, $y_2va_2wy_1v = y_1v$. Again post-multiplying by v^{-1}, we obtain $y_2va_2wy_1 = y_1$. In the similar manner, we can show if $c_1 = c_2$ then $y_1va_2wy_2 = y_1$ and $y_2va_1wy_1 = y_2$. $\square$

Theorem 5.8. *Let $a_1, a_2, v, w, b, c \in \mathcal{R}$ with both v and w invertible. If $y_1 = {a_1}_{b,c}^{a,v,w}$ and $y_2 = {a_2}_{b,c}^{a,v,w}$ then $y_1 + y_2 = y_1 v(a_1 + a_2)wy_2 = y_2 v(a_1 + a_2)wy_1$.*

Proof. Let $y_1 = {a_1}_{b,c}^{a,v,w}$ and $y_2 = {a_2}_{b,c}^{a,v,w}$. Then by Lemma 5.7, we have $y_1 va_1 wy_2 = y_2$, $y_2 va_2 wy_1 = y_1$, $y_2 va_1 wy_1 = y_2$, and $y_1 va_2 wy_2 = y_1$. Now

$$y_1 v(a_1 + a_2)wy_2 = y_1 va_1 wy_2 + y_1 va_2 wy_2 = y_2 + y_1 \text{ and}$$

$y_2 v(a_1 + a_2)wy_1 = y_2 va_1 wy_1 + y_2 va_2 wy_1 = y_2 + y_1$. □

6. Conclusion

We have discussed a few necessary and sufficient conditions for the existence of the (v, w)-weighted (b, c) inverse of an elements in a ring. Derived representations are used in generating corresponding representations of the (v, w)-weighted hybrid (b, c)-inverse and annihilator (v, w)-weighted (b, c)-inverse. We have also explored a few results related to the reverse order law for annihilator (v, w)-weighted (b, c)-inverses. In addition, the notion of (v, w)-weighted Bott-Duffin (e, f)-inverse was introduced along with a few characterizations of this inverse.

Conflicts of interest
No potential conflict of interest was reported by the authors.

References

[1] A. Ben-Israel, T. N. E. Greville, Generalized inverses: theory and applications. John Wiley & Sons, New York-London-Sydney, 1974.
[2] E. Boasso, G. K. Montiel, The (b, c)-inverse in rings and in the Banach context, Mediterranean Journal of Mathematics, 14(3) (2017).
[3] X. Chen, J. Chen, The (b, c)-inverse in semigroups and rings with involution, Front. Math. China, 15(6) (2020) 1089—1104
[4] R. Bott, R. J. Duffin, On the algebra of networks, Trans. Amer. Math. Soc., 74 (1953) 99—109.
[5] S. Das, J. K. Sahoo, R. Behera, Further results on weighted core inverse in a ring, arXiv preprint arXiv:2005.01130, 2020.
[6] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly, 65 (1958) 506-–514.
[7] M. P. Drazin, A class of outer generalized inverses. Linear Algebra Appl., 436 (7) (2012) 1909-–1923.
[8] M. P. Drazin, Commuting properties of generalized inverses, Linear Multilinear Algebra, 61(12) (2013) 1675-–1681.
[9] M. P. Drazin, Left and right generalized inverses, Linear Algebra Appl., 510 (2016) 64-–78.
[10] M. P. Drazin, Weighted (b, c)-inverses in categories and semigroups, Comm. Algebra, 48(4) (2020) 1423-–1438.
[11] T. N. E. Greville ,Note on the generalized inverse of a matrix product, SIAM Rev. 8 (1966) 518–521.
[12] R. E. Hartwig, J. Luh, A note on the group structure of unit regular ring elements, Pacific J. Math. 71(2) (1977) 449—461.
[13] Y. Ke, S. D. Cvetković-Ilić, J. Chen, J. Višnjić, New results on (b, c)-inverses, Linear Multilinear Algebra, 66 (3) (2018) 447–458.
[14] Y. Ke, Y. Gao, J. Chen, Representations of the (b, c)-inverses in rings with involution, Filomat, 31:9 (2017) 2867–2875.
[15] J. J. Koliha, D. Djordjević, D. Cvetković, Moore-Penrose inverse in rings with involution. Linear Algebra Appl. 426(2-3) (2007) 371—381.
[16] J. J. Koliha, V. Rakočević, Invertibility of the difference of idempotents, Linear Multilinear Algebra, 51(1) (2003) 97-–110.
[17] W. Long, N. C. Gonzalez, J. Chen, Characterizations of outer generalized inverses, Canad. Math. Bull. 60(4) (2017) 861–871.
[18] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl., 434(8) (2011) 1836—1844.
[19] D. Mosić, C. Deng, H. Ma, On a weighted core inverse in a ring with involution, Comm. Algebra, 46(6) (2018) 2332–2345.
[20] D. Mosić, D. Djordjević, Weighted outer inverse, Monatsh. Math. 188(2) (2019) 297–307.
[21] D. Mosić, H. Zou, J. Chen, On the (b, c)-inverse in rings, Filomat 32(4) (2018) 1221–1231.
[22] D. Mosić, P. S. Stanimirović, J. K. Sahoo, R. Behera, V. N. Katsikis, One-sided weighted outer inverses of tensor,J. Comput. Appl. Math. 388, Paper No. 113293, 22 (2021)
[23] D. S. Rakić, N. C. Dinčić, D. S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115–133.
[24] J. K. Sahoo, R. Behera, P. S. Stanimirović, V. N. Katsikis, H. Ma, Core and core-EP inverses of tensors. Comput. Appl. Math. 39(1) (2020), 10.1007/s40314-019-0983-5.
[25] P. S. Stanimirović, V. N. Katsikis, H. Ma, Representations and properties of the W-weighted Drazin inverse, Linear Multilinear Algebra, 65(6) (2017) 1080-–1096.
[26] P. S. Stanimirović, D. Mosić, H. Ma, New classes of more general weighted outer inverses, Linear Multilinear Algebra 70 (1) (2022) 122–147.
[27] L. Wang, Further results on hybrid (b, c)-inverses in rings, Filomat, 33:15 (2019) 4943–4950.
[28] H. Zhu, Further results on several types of generalized inverses, Comm. Algebra, 46(8) (2018) 3388—3396.

[29] Bhaskara Rao, K. P. S. The theory of generalized inverses over commutative rings. With a foreword by Adi Ben-Israel. Algebra, Logic and Applications, 17. Taylor & Francis Group, London, 2002. xii+167 pp. ISBN: 0-415-27248-3 MR1919967