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Results on Impulsive Fractional Integro-Differential Equations Involving Atangana-Baleanu Derivative

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Abstract. In this paper, we consider the impulsive fractional integro-differential equations involving Atangana-Baleanu fractional derivative. The main tools consist a fractional integral operator contains generalized Mittag-Leffler function, Gronwall-Bellman inequality with continuous functions and the Krasnoselskii's fixed point theorem.

1. Introduction

Fractional differential equations play a key role to describe various problems in different areas of science. Fractional models are more useful than the classical models. Fractional differential equations are used in economics, image processing, physics, and so on. For detailed information on fractional differential equations and their applications, see [2, 4, 7, 9, 11, 21, 22, 33].

Nonsingular Caputo and Riemann-Liouville version of fractional differential operator with Mittag-Leffler function as its kernel is introduced in [5]. Bonyah et al. [8] constituted a mathematical model involving AB-fractional derivative for co-infection of cancer and hepatitis diseases. They analyzed stability analysis, existence and uniqueness, and reproductive number. The fractional-order tumor-immune-vitamin model with AB-fractional derivative was presented for existence, uniqueness, and Hyers-Ulam stability in [3]. Researchers [20] prepared a chaotic and comparative work of tumor and effector cells through the fractional tumor-immune dynamical mode with AB-fractional derivative. In [12], the numerical solution of the fractional immunogenetic tumor model was studied by utilizing the fractional AB derivative.

It was given that a work on transmission dynamics of COVID-19 mathematical model under ABC-fractional-order derivative [29]. A mathematical model with AB-fractional derivative was investigated [23] for spreading of COVID-19 infection in the world. Moreover, Logeswari et al. created a framework that generates numerical outcomes to predict the outcome of the infection spreading all over India. For other important works on this topic, see [1, 6, 13–15, 32].

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In [30], Liang et al. discussed the impulsive fractional differential equations with boundary value problems of the form

$$
\begin{aligned}
&{}^{C}D_t^{\alpha}x(t) = f(t, x(t)), t \in J' : J\ t_1, t_2, ..., t_m,\ J = [0, T],\\
&\Delta x(t_k) = u(t_k^+ - t_k^-) = I_k(t_k^-),\ \ k = 0, 1, 2, ..., m,\\
&ax(0) + bx(T) = c,
\end{aligned}
$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ with the lower limit zero, $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous and t_k satisfies $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T, x(t_k^+) = \lim \epsilon \to 0 + x(t_k + \epsilon)$ and $x(t_k^-) = \lim \epsilon \to 0 - x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k.I_k \in C(\mathbb{R}, \mathbb{R})$, and a, b, c are real constants with $a + b \neq 0$.

Yukunthorn et al. [34] studied the impulsive Hadamard fractional differential equations with boundary value problems of the form:

$$
\begin{aligned}
{}^{C}D_{t_k}^{p_k}x(t) &= f(t, x(t)),\ \ t \in J_k \subset [t_0, T],\ \ t = t_k,\\
\Delta x(t_k) &= \varphi_k(x(t_k)),\ \ k = 1, 2, ..., m,\\
\alpha x(t_0) + \beta x(T) &= \sum_{i=0}^{m} \gamma_i J_{t_i}^{q_i} x(t_{i+1}),
\end{aligned}
$$

where ${}^{C}D_{t_k}^{p_k}$ is the Hadamard fractional derivative of order $0 < p_k \leq 1$ on intervals $J_k = (t_k, t_{k+1}], k = 1, 2, ..., m$, with $J_0 = [t_0, t_1], 0 < t_1 < t_2 < t_3 < \ < t_k < \ < t_m < t_{m+1} = T$ are the impulse points, $J := [t_0, T], f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi_k \in C(\mathbb{R}, \mathbb{R}), J_{t_i}^{q_i}$ is the Hadamard fractional integral of order $q_i > 0, i = 1, 2, ..., m$. The jump conditions are defined by $\Delta x(t_k) = x(t_k^+) - x(t_k), x(t_k^+) = \lim \epsilon \to 0^+ x(t_k + \epsilon), k = 1, 2, 3, ..., m$.

Inspired by the works of [19, 26, 31], on the line of [18, 24], we take into consideration multi-derivative nonlinear impulsive FDEs involving Riemann-Liouville version of AB-fractional derivative (ABR derivative) of the from:

$$
{}_0^{*}D_{\tau}^{\alpha}\omega(\tau) = f(\tau, \omega(\tau), B\omega(\tau)), t \in J, \tag{1}
$$

$$
\omega(t_k^+) = \omega(t_k^-) + y_k,\ \ y_k \in \mathbb{R}, \tag{2}
$$

$$
\omega(0) = \omega_0 \in \mathbb{R} \tag{3}
$$

where $J = [0, T]$, $T > 0$, $0 < \alpha < 1$, D_{τ}^{α} denotes the ABR-fractional differential operator of order α and $f \in C(J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ is a nonlinear function,

$$
B\omega(\tau) = \int_0^{\tau} k(\tau, s, \omega(s))ds,\ \ k : \Delta \times [0, T] \to \mathbb{R},\ \ \Delta = \{(\tau, s) : 0 \leq s \leq \tau \leq T\},
$$

$0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_m = 1$, $\Delta\omega|_{\tau=\tau_k} = \omega(\tau_k^+) - \omega(\tau_k^-)$,

$$
\omega(\tau_k^+) = \lim_{h \to 0^+} \omega(\tau_k + h)\ \text{ and }\ \omega(\tau_k^-) = \lim_{h \to 0^-} \omega(\tau_k + h)
$$

represent the right and left hand limits of $\tau(t)$ at $\tau = \tau_k$.

We provide an equivalent fractional integral equation to ABR-FDEs (1)-(2) analytically. Using the properties of fractional integral operator $\varepsilon_{\rho,\mu,\omega;a+}^{\gamma}$, we obtain some results. The existence of solution is established by using Krasnoselskii's fixed point theorem. We get uniqueness of solution via Gronwall-Bellman inequality as well as using the properties of fractional integral operator $\varepsilon_{\rho,\mu,\omega;a+}^{\gamma}$.

The paper is structured as follows: In section 2, we introduce the required background for the development of the paper. The existence and uniqueness results of impulsive fractional integro differential equations are discussed in section 3.

2. Preliminaries

This section includes some definitions and facts on AB-fractional derivative and the generalized Mittag-Leffler function.

Definition 2.1. ([16]) *Let $p \in [1, \infty)$ and ω be an open subset of $\mathbb{R}$. The Sobolev space $H^p(\omega)$ is defined by*

$$H^p(\omega) = \{f \in L^2(\omega) : D^\beta f \in L^2(\omega) \text{ for all } |\beta| \le \omega\}.$$

Definition 2.2. ([5]) *Let $x \in H^1(0,1)$ and $0 < \alpha < 1$. The left Atangana-Baleanu fractional derivative of ω of order α in Riemann-Liouville sense (ABR derivative) is defined by*

$$D_\tau^\alpha = \frac{B(\alpha)}{(1-\alpha)} \frac{d}{d\sigma} \int_0^\tau \mathbb{E}_\alpha \left(\frac{-\alpha}{(1-\alpha)} (\tau - \sigma)^\alpha \right) \omega(\sigma) d\sigma,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$ and $\mathbb{E}$ is one parameter Mittag-Leffler function.

Definition 2.3. ([5]) *Let $\omega \in H^1(0,1)$ and $0 < \alpha < 1$. The left Atangana-Baleanu fractional derivative of x of order α in Caputo sense is defined by*

$$D_t^\alpha = \frac{B(\alpha)}{(1-\alpha)} \int_0^t \mathbb{E}_\alpha \left(\frac{-\alpha}{(1-\alpha)} (\tau - \sigma)^\alpha \right) \omega'(\sigma) d\sigma.$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$ and $\mathbb{E}$ is one parameter Mittag-Leffler function.

Definition 2.4. ([10, 17]) *The generalized Mittag-Leffler function $\mathbb{E}^\gamma_{\alpha,\beta}(z)$ for the complex α, β with $Re(\alpha) > 0$ is defined by*

$$\mathbb{E}^\gamma_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}$$

where γ_k is the Pochhammer symbol given by

$$\gamma_0 = 1, \ \gamma_k = \gamma(\gamma + 1 \dots (\gamma + k - 1)), \ k = 1, 2, 3, \dots$$

Note that

$$\mathbb{E}^1_{\alpha,\beta}(z) = \mathbb{E}_{\alpha,\beta}(z), \ \mathbb{E}^1_{\alpha,1}(z) = \mathbb{E}_\alpha(z).$$

We need the following results related with Laplace transformation.

Lemma 2.5. ([5]) *If $L\{f(\tau); p\} = \bar{F}(p)$, then $D_t^\alpha\{f(\tau); p\} = \frac{B(\alpha)}{1-\alpha} \frac{p^\alpha \bar{F}(p)}{p^\alpha + \frac{\alpha}{1-\alpha}}$.*

Lemma 2.6. ([27]) $L\left[t^{k\alpha+\beta-1} \mathbb{E}^{(k)}_{\alpha,\beta}(\pm a t^\alpha); p\right] = \frac{k! p^{\alpha-\beta}}{(p^\alpha \pm a)^{k+1}}$, $\mathbb{E}^{(k)} t = \frac{d^k}{dt^k} t$.

Definition 2.7. ([17, 28]) *Let $\rho, \mu, \omega, \gamma \in \mathbb{C}(Re(\rho), Re(\mu) > 0)$, $b > a$. The fractional integral operator $\mathcal{E}^\gamma_{\rho,\mu,\omega;a+}$ on a class $L(a, b)$ is defined by*

$$(\mathcal{E}^\gamma_{\rho,\mu,\omega;a+} \phi)\tau = \int_a^t (\tau - \sigma)^{\gamma - 1} \mathbb{E}^\gamma_{\rho,\mu}[\omega(\tau - \sigma)^\rho] \phi(\sigma) d\sigma, \ \tau \in [a, b].$$

Lemma 2.8. ([17, 28]) *Let $\rho, \mu, \omega, \gamma \in \mathbb{C}(Re(\rho), Re(\mu) > 0)$, $b > a$. The operator $\mathcal{E}^\gamma_{\rho,\mu,\omega;a+}$ is bounded on $C[a, b]$ such that*

$$\|(\mathcal{E}^\gamma_{\rho,\mu,\omega;a+} \phi)(\tau)\| \le Q \|\phi\|$$

where

$$Q = (b - a)^{Re(u)} \sum_{k=0}^{\infty} \frac{|(\gamma)_k|}{|\Gamma(\rho k + \mu)| [(Re(\rho) k + \rho(\mu))]} \frac{|\omega(b - a)^{Re(\rho)}|^k}{k!}.$$

Lemma 2.9. ([17, 28]) *Let $\rho, \mu, \omega, \gamma \in \mathbb{C}(Re(\rho), Re(\mu) > 0)$. The operator $\mathcal{E}^{\gamma}_{\rho,\mu,\omega;a+}$ is invertible in the space $\mathcal{L}(a, b)$ and for $f \in \mathcal{L}(a, b)$ its left inversion is given by the relation*

$$\left([\mathcal{E}^{\gamma}_{\rho,\mu,\omega;a+}]^{-1} f\right)\tau = (D^{\mu+\nu}_{a+}\mathcal{E}^{-\gamma}_{\rho,\mu,\omega;a+}f)\tau, \;\; a < \tau \leq b,$$

where $\nu \in C$, $(Re(\nu) > 0)$ and $\mathcal{D}^{\mu+\nu}_{a+}$ is the Riemann-Liouville fractional differential operator of order $\mu + \nu$ with lower terminal a.

Lemma 2.10. ([17, 28]) *Let $\rho, \mu, \omega, \gamma \in \mathbb{C}(Re(\rho), Re(\mu) > 0)$. If the integral equation*

$$\int_a^t (t - \sigma)^{\gamma-1}\mathbb{E}^{\gamma}_{\rho,\mu}[x(t - \sigma)^{\rho}]\phi(\sigma)d\sigma = f(t), \; a < t \leq b,$$

is solvable in the space $L(a, b)$, then its unique solution $\phi(\tau)$ is given by

$$\phi(\tau) = (D^{\mu+\nu}_{a+}\mathcal{E}^{-\gamma}_{\rho,\mu,\omega;a+}f)\tau, \;\; a < \tau \leq b,$$

where $\nu \in C$, $(Re(\nu) > 0)$ and $D^{\mu+\nu}_{a+}$ is the Riemann-Liouville fractional differential operator of order $\mu + \nu$ with lower terminal a.

Lemma 2.11. ([2]) *(Krasnoselskii's fixed point theorem) Let ω be a Banach space. Let $\mathcal{S}$ be a bounded, closed, convex subset of ω and $\mathcal{F}_1$,$\mathcal{F}_2$ be maps of $\mathcal{S}$ into ω such that $\mathcal{F}_1\omega + \mathcal{F}_2\eta \in \mathcal{S}$ for every pair $\omega, \eta \in \mathcal{S}$. If $\mathcal{F}_1$ is contraction and $\mathcal{F}_2$ is completely continuous, then the equation*

$$\mathcal{F}_1\omega + \mathcal{F}_2\omega = \omega$$

has a solution on $\mathcal{S}$.

Lemma 2.12. ([25]) *(Gronwall-Bellman inequality) Let u and f be continuous and nonnegative functions defined on $J = [\alpha, \beta]$, and c be a nonnegative constant. Then the inequality*

$$u(\tau) \leq C + \int_\alpha^\tau f(\sigma)u(\sigma)d(\sigma), \;\; \tau \in J$$

implies that

$$u(\tau) \leq Cexp(\int_\alpha^\tau f(\sigma)d(\sigma)), \;\; \tau \in J.$$

Lemma 2.13. *For any function $h \in C(J)$, the function $\omega \in C(J)$ is a solution of ABR-FDEs*

$$ {}^*_0D^{\alpha}_{\tau}\omega(\tau) = h(\tau), \tau \in \mathbb{J}, \tag{4}$$
$$\omega(\tau_k^+) = \omega(\tau_k^-) + y_k, \;\; y_k \in \mathbb{R}, \tag{5}$$
$$\omega(0) = \omega_0 \in \mathbb{R}, \tag{6}$$

if and only if x is a solution of fractional integral equation

$$\omega(\tau) = \begin{cases} \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma, \; for\ \tau \in [0, \tau_1), \\ y_1 + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma, \; for\ \tau \in (\tau_1, \tau_2), \\ y_1 + y_2 + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma, \; for\ \tau \in (\tau_2, \tau_3), \\ . \\ . \\ . \\ \sum_{i=1}^m y_i + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma, \;\; for\ \tau \in (\tau_m, T]. \end{cases} \tag{7}$$

Proof. (⇒) Assume that ω satisfies (4)-(6). If $\tau \in [0,\tau_1)$, then we obtain the followings:

$$ {}_0^*D_\tau^\alpha \omega(\tau) = h(\tau), t \in \mathbb{J}, \tag{8}$$
$$ \omega(0) = \omega_0 \in \mathbb{R}, \tag{9}$$

$$\omega(\tau) = \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma.$$

If $\tau \in (\tau_1,\tau_2)$, then we have

$$ {}_0^*D_\tau^\alpha \omega(\tau) = h(\tau), \tau \in \mathbb{J},$$
$$ \omega(\tau_k^+) = \omega(\tau_k^-) + y_k, \;\; y_k \in \mathbb{R},$$

and so

$$\begin{aligned}\omega(\tau) &= \omega(\tau_1^+) - \int_0^{\tau_1} h(\sigma)d\sigma + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma\\ &= \omega(\tau_1^+) + y_1 - \int_0^{\tau_1} h(\sigma)d\sigma + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma\\ &= y_1 + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma.\end{aligned}$$

If $\tau \in (\tau_2,\tau_3)$, then we find

$$\begin{aligned}\omega(\tau) &= \omega(\tau_2^+) - \int_0^{\tau_2} h(\sigma)d\sigma + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma\\ &= \omega(\tau_2^+) + y_2 - \int_0^{\tau_1} h(\sigma)d\sigma + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma\\ &= y_1 + y_2 + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau_1} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma.\end{aligned}$$

Let consider the case $\tau \in (\tau_m, T]$. Then we conclude

$$\omega(\tau) = \sum_{i=1}^m y_i + \omega_0 + \frac{B(\alpha)}{1-\alpha}\int_0^{\tau} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega'(\sigma)d\sigma + \int_0^\tau h(\sigma)d\sigma. \tag{10}$$

(⇐) Conversely, assume that ω satisfies the impulsive equations (7). Using the definition of fractional integral operator $\mathcal{E}^\gamma_{\rho,\mu,\omega;a+}$, the equivalent fractional integral equation (7) to the ABR-FDEs (4)-(6) is given by

$$\omega(\tau) = \sum_{i=1}^m y_i + \omega_0 - \frac{B(\alpha)}{1-\alpha}\left(\mathbb{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega\right)(\tau) + \int_0^\tau h(\sigma)d\sigma, \;\; \tau \in J.$$

□

Theorem 2.14. *For any $f \in C(J \times \mathcal{R}, \mathcal{R})$, the function $\omega \in C(J)$ is a solution of ABR-FDEs (1)–(2) if and only if ω is a solution of fractional integral equation*

$$\omega(\tau) = \sum_{i=1}^m y_i + \omega_0 - \frac{B(\alpha)}{1-\alpha}\int_0^{\tau} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau-\sigma)^\alpha\right)\omega(\sigma)d\sigma + \int_0^t f(\sigma,\omega(\sigma))d\sigma, \;\; t \in J. \tag{11}$$

Proof. Proof follows by taking $h(\tau) = f(\tau, \omega(\tau)),\ \tau \in J$ in the Lemma 2.8. □

The proof of following theorem is based on the properties of fractional integral operator $\mathcal{E}^{\gamma}_{\rho,\mu,\omega;a+}$ studied in [46, 47].

Theorem 2.15. *Let* $0 < \alpha < 1$. *Define the function* $\mathcal{F}$ *on* $C(J)$ *by*

$$(\mathcal{F}\omega)(\tau) = \frac{B(\alpha)}{1-\alpha}\left(\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega\right)(\tau), \omega \in C(J),\ \ \tau \in J. \tag{12}$$

Then we have the followings:

1. $\mathcal{F}$ *is bounded linear operator on* $C(J)$.
2. $\mathcal{F}$ *satisfies Lipschitz condition.*
3. $\mathcal{F}(S)$ *is equicontinuous, where* S *is any bounded subset of* $C(J)$.
4. $\mathcal{F}$ *is invertible and for any* $f \in C(J)$, *the operator equation* $\mathcal{F}\omega = f$ *has unique solution in* $C(J)$.

Proof. (i) Since, by definition and Lemma 2.3, the integral operator $\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}$ is bounded and linear operator on $C(J)$, such that

$$\|\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+\omega}\| \leq Q\|\omega\|,\ \ \tau \in J,$$

where we find

$$Q = \sum_{k=0}^{\infty}\frac{(1)_k}{\|\Gamma(\alpha k+1)(\alpha k+1)\|}\frac{|\frac{-\alpha}{1-\alpha}T^{\alpha^k}}{k!} = \sum_{k=0}^{\infty}\frac{\frac{\alpha}{1-\alpha}T^{\alpha^k}}{\Gamma(\alpha k+2)} = T\mathbb{E}_{\alpha,2}\left(\frac{\alpha}{1-\alpha}T^{\alpha}\right).$$

Since

$$\mathcal{F}\omega = |\frac{B(\alpha)}{1-\alpha}|\|\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+\omega}\| \leq Q\frac{B(\alpha)}{1-\alpha}\|\omega\|,\ \ \text{for all}\omega \in C(J),$$

$\mathcal{F}$ is bounded linear operator on $C(J)$.
(ii) Let $\omega, \eta \in C(J)$. Then using the linearity of $\mathcal{F}$ and boundedness of operator $\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}$, we find

$$\begin{aligned}|\mathcal{F}\omega(\tau) - \mathcal{F}\eta(\tau)| = |(\mathcal{F}\omega - \mathcal{F}\eta)(\tau)| = \frac{B(\alpha)}{1-\alpha}|\left(\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega - \eta\right)(\tau)| &\leq \frac{B(\alpha)}{1-\alpha}\|\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega - \eta\| \\ &\leq Q\frac{B(\alpha)}{1-\alpha}\|\omega - \eta\|\end{aligned}$$

for any $\tau \in J$. This gives

$$\|\mathcal{F}\omega - \mathcal{F}\eta\| \leq Q\frac{B(\alpha)}{1-\alpha}\|\omega - \eta\|,\ \ \omega, \eta \in C(J).$$

Thus the operator $\mathcal{F}$ satisfies Lipschitz condition with Lipschitz constant$\mathcal{T}\frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha,2}\left(\frac{\alpha}{1-\alpha}T^{\alpha}\right)$.
(iii) Let $\mathcal{S} = \{\omega \in C(J) : \|\omega\| \leq R\}$ be a closed and bounded subset of $C(J)$. Then for any $\omega \in \mathcal{S}$ and any

$\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$, we obtain

$$
\begin{aligned}
|\mathcal{F}\omega(\tau_1) - \mathcal{F}\eta(\tau_2)| &= |\frac{B(\alpha)}{1-\alpha}\left(\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega\right)(\tau_1) - \frac{B(\alpha)}{1-\alpha}\left(\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega\right)(\tau_2)| \\
&= \frac{B(\alpha)}{1-\alpha}|\int_0^\tau \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau_1-\sigma)^\alpha\right)\omega(\sigma)d\sigma - \int_0^\tau \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau_2-\sigma)^\alpha\right)\omega(\sigma)d\sigma| \\
&\le \frac{B(\alpha)}{1-\alpha}|\int_0^\tau \left\{\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau_1-\sigma)^\alpha\right) - \int_0^\tau \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau_2-\sigma)^\alpha\right)\right\}\omega(\sigma)d\sigma| \\
&+ \frac{B(\alpha)}{1-\alpha}|\int_{\tau_1}^{\tau_2} \mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}(\tau_2-\sigma)^\alpha\right)\omega(\sigma)d\sigma| \\
&\le \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^\infty |\left(\frac{-\alpha}{1-\alpha}\right)^k| \frac{1}{\Gamma(\alpha k+1)}\int_0^{\tau_1} |(\tau_1-\alpha)^{k\alpha} - (\tau_2-\alpha)^{k\alpha}|\omega(\sigma)d\sigma \\
&+ \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^\infty |\left(\frac{-\alpha}{1-\alpha}\right)^k| \frac{1}{\Gamma(\alpha k+1)}\int_{\tau_1}^{\tau_2} |(\tau_2-\alpha)^{k\alpha}|\omega(\sigma)d\sigma \\
&\le \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^\infty \left(\frac{\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(\alpha k+1)}\int_0^{\tau_1} \left\{(\tau_2-\alpha)^{k\alpha} - (\tau_1-\alpha)^{k\alpha}\right\}\omega(\sigma)d\sigma \\
&+ \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^\infty \left(\frac{\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(\alpha k+1)}\int_{\tau_1}^{\tau_2} (\tau_2-\alpha)^{k\alpha}\omega(\sigma)d\sigma \\
&\le \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^\infty \left(\frac{\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(\alpha k+2)}\left\{-(\tau_2-\tau_1)^{k\alpha+1} + (\tau_2)^{k\alpha+1} - (\tau_1)^{k\alpha+1} + (\tau_2-\tau_1)^{k\alpha+1}\right\} \\
&\le \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^\infty \left(\frac{\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(\alpha k+2)}\{(\tau_2)^{k\alpha+1} - (\tau_1)^{k\alpha+1}\}.
\end{aligned}
$$

From the above inequalities, it follows that if $|\tau_1 - \tau_2| \to 0$, then $|\mathcal{F}\omega(\tau_1) - \mathcal{F}\eta(\tau_2)| \to 0$. This proves that $\mathcal{F}(S)$ is equicontinious on J.
(iv) Using Lemma 2.4 and Lemma 2.5, for any $f \in C(J)$, we have

$$\left(\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}f\right)^{-1}(\tau) = \left(\mathcal{D}^{1+\beta}_{0+}\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}f\right)^{-1}(\tau), \tau \in (a,b), \tag{13}$$

where $\beta \in \mathbb{C}$, with $Re(\beta) > 0$.
Then from the definition of operator $\mathcal{F}$ and Eq. (13), we have

$$(\mathcal{F}^{-1}f)(\tau) = \left(\frac{B\alpha}{1-\alpha}\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}f\right)^{-1}(\tau) = \frac{1-\alpha}{B\alpha}\left(\mathcal{D}^{1+\beta}_{0+}\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}f\right)^{-1}(\tau), \quad \tau \in (a,b).$$

This proves that $\mathcal{F}$ is invertible on $C(J)$ and the operator equation

$$(\mathcal{F}\omega)(\tau) = f(\tau), \quad \tau \in J$$

has the unique solution

$$\omega(\tau) = \frac{1-\alpha}{B\alpha}\left(\mathcal{D}^{1+\beta}_{0+}\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}f\right)(\tau), \quad \tau \in (a,b).$$

□

We get the next existence theorem for the particular case of ABR-FDEs (1).

Theorem 2.16. *If the function $f \in C(J \times \mathcal{R}, \mathcal{R})$, then ABR-FDEs $\mathcal{D}^{\alpha}_{\tau} = f(\tau, \omega(\tau))$, $\tau \in J$ is solvable in $C(J)$ and has a solution in $C(J)$ given by*

$$\omega(\tau) = \frac{1-\alpha}{B\alpha}\left(\mathcal{D}^{1+\beta}_{0+}\mathcal{E}^{1}_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\bar{f}\right)(\tau), \quad \tau \in J,$$

where $\beta \in \mathbb{C}$ with $Re(\beta) > 0$ and $\int_0^{\tau} f(\sigma, \omega(\sigma))d(\sigma)$, $\tau \in J$.

3. Main results

Theorem 3.1. *(Existence Theorem) Let the function $f \in C(J \times \mathcal{R}, \mathcal{R})$, satisfies Lipschitz type condition*

$$|f(\tau, \omega, \kappa_1) - f(\tau, \eta, \kappa_2)| \leq p(\tau)[|\omega - \eta| + |\kappa_1 - \kappa_2|], \quad \omega, \eta, \kappa_1, \kappa_2 \in C(J),$$

where $p : J \to \mathbb{R}^+$, with $L = \sup p(\tau)$. If $0 < L < min\left\{1, \frac{1}{2T}\right\}$, then ABR-FDEs (1)–(2) has a solution in $C(J)$ provided

$$\frac{B(\alpha)T\mathbb{E}_{\alpha,2}\left(\frac{\alpha}{1-\alpha}\right)T^{\alpha}}{1-\alpha} < 1. \tag{14}$$

Proof. Define

$$R = \frac{|\omega_0| + M_f T + M^*}{1 - LT - \frac{B(\alpha)T\mathbb{E}_{\alpha,2}\left(\frac{\alpha}{1-\alpha}\right)T^{\alpha}}{1-\alpha}},$$

where $M_f = sup|f(\tau, 0, 0)$ and $M^* > 0$ is a constant such that $\sum_{i=1}^{m}\left|y_i\right| \leq M^*$. By the choice of L and condition (14), we have $R > 0$. Consider the following set

$$\mathcal{S} = \{\omega \in C(J) : \|\omega\| \leq R\}.$$

One can verify that $\mathcal{S}$ is closed, convex and bounded subset of Banach space ω . Consider the operators $\mathcal{F}_1 : S \to \omega$ and $\mathcal{F}_2 : S \to \omega$ defined by

$$(\mathcal{F}_1\omega)(\tau) = \sum_{i=1}^{m} y_i + \omega_0 + \int_0^{\tau} f(\sigma, \omega(\sigma), B\omega(\sigma))d\sigma, \quad \tau \in J,$$

$$(\mathcal{F}_2\omega)(\tau) = (\mathcal{F}\omega)(\tau), \quad \tau \in J$$

where we take $\mathcal{F}$ as defined in the Eq. (12). The equivalent fractional integral Eq. (11) to the ABR– FDEs (1)–(2) can be written as operator equation in the form $\omega = \mathcal{F}_1\omega + \mathcal{F}_2\omega$, $\omega \in C(J)$.

We prove that the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy conditions of Lemma 2.6. The proof is given in the following steps.

Step 1: $\mathcal{F}_1$ is contraction.

Using Lipschitz condition on f, for any $\omega, \eta \in C(J)$ and $\tau \in J$, we get

$$|\mathcal{F}(\tau, \omega(\tau), B\omega(\sigma)) - \mathcal{F}(\tau, \eta(\tau), B\eta(\tau))| \leq p(\tau)|\omega - \eta|.$$

This gives

$$\|\mathcal{F}_1\omega - \mathcal{F}_1\eta\| \leq LT\|\omega - \eta\|, \omega, \eta \in C(J).$$

Step 2: $\mathcal{F}_2$ is completely continuous.

By Ascoli-Arzela theorem and Theorem 2.10, it can be easily verified that the operator $\mathcal{F}_2 = -\mathcal{F}$ is completely continuous.

Step 3: $\mathcal{F}_1\omega + \mathcal{F}_2\eta \in \mathcal{S}$, for any $\omega, \eta \in \mathcal{S}$.

For any $\omega, \eta \in \mathcal{S}$, using Theorem 2.10, we find

$$|(\mathcal{F}_1\omega + \mathcal{F}_2\eta)(\tau)| \leq |(\mathcal{F}_1\omega)(\tau)| + |(\mathcal{F}_2\eta)(\tau)|$$

$$\leq |\omega_0| + \sum_{i=1}^{m} y_i + \int_0^\tau |f(\sigma,\omega(\sigma),B\omega(\sigma))| d\sigma + \frac{B(\alpha)}{1-\alpha} T\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}T^\alpha\right] \|\eta\| \tag{15}$$

$$\leq |\omega_0| + M^* + \int_0^\tau |f(\sigma,\omega(\sigma),B\omega(\sigma)) - f(\sigma,0,0)| d\sigma + \int_0^\tau |f(\sigma,0,0)| d\sigma + \frac{B(\alpha)}{1-\alpha} T\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}T^\alpha\right] R \tag{16}$$

$$\leq |\omega_0| + M^* + L\int_0^\tau |f(\omega(\sigma),B\omega(\sigma)) + M_f \int_0^\tau d\sigma + \frac{B(\alpha)}{1-\alpha} T\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}T^\alpha\right] R \tag{17}$$

$$\leq |\omega_0| + M^* + LR\tau + M_f\tau + \frac{B(\alpha)}{1-\alpha} T\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}T^\alpha\right] R \tag{18}$$

$$\leq |\omega_0| + M^* + LRT + M_fT + \frac{B(\alpha)}{1-\alpha} T\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}T^\alpha\right] R. \tag{19}$$

By definition of R, we get

$$|\omega_0| + M_fT + M^* = R1 - LT - \frac{B(\alpha)T\mathbb{E}_{\alpha,2}\left(\frac{\alpha}{1-\alpha}\right)T^\alpha}{1-\alpha}. \tag{20}$$

We write from inequalities (15) and (16)

$$|(\mathcal{F}_1\omega + \mathcal{F}_2\eta)(\tau)| \leq R, \quad \tau \in J.$$

This gives

$$\|(\mathcal{F}_1\omega + \mathcal{F}_2\eta)\| \leq R, \text{for all } \omega, \quad \eta \in \mathcal{S}.$$

This shows that $\mathcal{F}_1\omega + \mathcal{F}_2\eta \in \mathcal{S}$ for $\omega, \eta \in \mathcal{S}$. From steps 1 to 3, it follows that all the conditions of Lemma 2.6 are satisfied. Therefore by applying it, the operator equation

$$\omega = \mathcal{F}_1\omega + \mathcal{F}_2\omega$$

has a fixed point in S, which is a solution of ABR–FDEs (1)–(2). This completes the proof of the theorem. □

In the following theorem, we prove the uniqueness of solution to ABR-FDEs (1)-(2) in two different ways. Firstly we give the proof via properties of fractional integral operator $\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}$ and then by using the Gronwall-Bellman inequality.

Theorem 3.2. *(Uniqueness Result) Under the assumptions of Theorem 3.1, the ABR-FDEs (1)–(2) has unique solution in* $C(J)$.

Proof. There are two ways for proof.
Proof 1: The equivalent fractional integral equation to ABR-FDEs (1)–(2) can be stated in operator equation form as

$$\left(\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega\right)(\tau) = \tilde{f}(\tau), \quad \tau \in J \tag{21}$$

where

$$\tilde{f}(\tau) = \frac{1-\alpha}{B(\alpha)}\left(\omega_0 - \omega_\tau + \int_0^\tau f(\sigma,\omega(\sigma),B\omega(\sigma))d\sigma\right) + \sum_{i=1}^{m} y_i, \quad \tau \in J.$$

By Theorem 3.1, the operator Eq.(11) is solvable in $C(J)$. Therefore by applying Lemma 2.5, the operator equation Eq.(11) has unique solution in $C(J)$, which is the unique solution of ABR-FDEs (1)–(2).

Proof 2: Let ω, η be two solutions of ABR-FDEs (1)–(2). Using linearity of fractional integral operator, we get

$$\begin{aligned}
|\omega(\tau) - \eta(\tau)| = &\left|\left(\omega_0 - \frac{B(\alpha)}{1-\alpha}(\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\omega)(\tau) + \int_0^\tau f(\sigma, \omega(\sigma), B\omega(\sigma))d\sigma\right) - \right.\\
&\left.\left(\omega_0 - \frac{B(\alpha)}{1-\alpha}(\mathcal{E}^1_{\alpha,1,\frac{-\alpha}{1-\alpha};0+}\eta)(\tau) + \int_0^\tau f(\sigma, \eta(\sigma), B\eta(\sigma))d\sigma\right)\right|\\
&\le \frac{B(\alpha)}{1-\alpha}\int_0^\tau \mathbb{E}_\alpha\left(\left|\frac{-\alpha}{(1-\alpha)}(T-\sigma)^\alpha\right|\right)|\omega(\sigma) - \eta(\sigma)|d\sigma + \int_0^\tau pl_1(\sigma)|\omega(\sigma) - \eta(\sigma)|d\sigma\\
&\le \frac{B(\alpha)}{1-\alpha}\int_0^\tau \mathbb{E}_\alpha\left(\frac{\alpha}{(1-\alpha)}(T)^\alpha\right)|\omega(\sigma) - \eta(\sigma)|d\sigma + \int_0^\tau pl_1(\sigma)|\omega(\sigma) - \eta(\sigma)|d\sigma\\
&\le \int_0^\tau \left[\frac{B(\alpha)}{1-\alpha}\mathbb{E}_\alpha\left(\frac{\alpha}{(1-\alpha)}(T)^\alpha + pl_1(\sigma)\right)\right]|\omega(\sigma) - \eta(\sigma)|d\sigma
\end{aligned}$$

for any $\tau \in J$. Applying Lemma 2.7, we obtain

$$|\omega(\tau) - \eta(\tau)| \le 0, \quad \tau \in J$$

which shows that $\omega(\tau) = \eta(\tau)$ for all $\tau \in J$. This proves the uniqueness of solution of ABR-FDEs (1)–(2). □

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