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Pseudo-generalized Inverse II

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Abstract. The purpose of this paper is to continue the study initiated in [7] concerning the notion of pseudo-generalized inverse. In this work, we are interested in investigating the relationship between compactness, finite rank and pseudo-generalized invertibility for bounded operators. The preserving properties of products of pseudo-generalized invertible operators are also treated and related results are given. At the end of this paper, we discuss in some special cases the relationship between the sets of n-left pseudo-generalized inverses of a left pseudo-generalized invertible operator T.

1. Introduction

The present paper is a continuation of [7], where we have introduced the class of pseudo-generalized invertible operators needed in our study. Let $\mathscr{B}(\mathsf{X})$ be the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X and $\mathscr{K}(\mathsf{X})$ the space of compact operators in $\mathscr{B}(\mathsf{X})$. As we introduced in [7], for $n \in \mathbb{N}$, an operator $T \in \mathscr{B}(\mathsf{X})$ is called n-left (resp. n-right) pseudo-generalized invertible if $T \in \Omega_n^\ell$ (resp. Ω_n^r), where

$$\Omega_n^\ell = \Big\{ T \in \mathscr{B}(\mathsf{X}) \ : \exists\, S \in \mathscr{B}(\mathsf{X}) : \ T^n S T = T^n \Big\},$$

and

$$\Omega_n^r = \Big\{ T \in \mathscr{B}(\mathsf{X}) \ : \exists\, S \in \mathscr{B}(\mathsf{X}) : \ T S T^n = T^n \Big\}.$$

More generally, if $T \in \Omega^\ell$ (resp. $T \in \Omega^r$), then T is called left (resp. right) pseudo-generalized invertible, where

$$\Omega^\ell = \bigcup_{n=0}^{\infty} \Omega_n^\ell \ \text{ and } \ \Omega^r = \bigcup_{n=0}^{\infty} \Omega_n^r.$$

We simply call pseudo-generalized invertible operator any operator in $\Omega^\ell \cup \Omega^r$.

Recall also that a n-left (resp. n-right) pseudo-generalized inverse of T is an operator $S \in \mathscr{B}(\mathsf{X})$ satisfying the equation $T^n S T = T^n$ (resp. $T S T^n = T^n$). We denote by ${}^T\mathsf{S}_n^\ell$ (resp. ${}^T\mathsf{S}_n^r$) the subset of all n-left (resp.

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n-right) pseudo-generalized inverses of T and we simply call left (resp. right) pseudo-generalized inverse any operator in ${}^{T}\mathsf{S}^{\ell}_{n}$ (resp. ${}^{T}\mathsf{S}^{r}_{n}$), for any $n \in \mathbb{N}$. A pseudo-generalized inverse refers to a left or right pseudo-generalized inverse. Definitions and notations not explicitly given are taken from [7].

This paper is organized as follows. In Section 2 we discuss some properties of compact pseudo-generalized invertible operators, in particular we extend to pseudo-generalized invertible operators some classical findings obtained in [4]. Some decomposition results are also proved. Preserving properties of products of operators is an interesting topic in operator theory, so in this work we devote our attention to investigate some aspects of this topic. For example, in Section 3, we show that the product of two pseudo-generalized invertible operators is not necessarily in general pseudo-generalized invertible, even for commuting operators. In particular, this prompted us to give in certain cases some sufficient conditions preserving products of two commuting pseudo-generalized invertible operators. Some other product stability results are also obtained in Sections 4 and 5. Furthermore, as an application, we see that the notion of pseudo-generalized invertibility is useful to prove the g-invertibility in some particular cases. At the end of the paper, we discuss in some special cases the relationship between the sets of n-left pseudo-generalized inverses of a left pseudo-generalized invertible operator $T \in \mathscr{B}(\mathsf{X})$.

2. On Compact operators and pseudo-generalized invertibility

In this section, we prove some results related to both compact operators and pseudo-generalized invertible operators. First, from [10], recall that $T \in \mathscr{B}(\mathsf{X})$ is called a Fredholm operator if $\alpha(T) < +\infty$ and $\beta(T) < +\infty$, where $\alpha(T)$ denotes the dimension of $\mathsf{N}(T)$ and $\beta(T)$ is the codimension of $\mathsf{R}(T)$. Also, it is well-known that if $K \in \mathscr{K}(\mathsf{X})$, then $I + K$ is a Fredholm operator (see [8, Proposition 3.21]).

Now, we start with the following lemma.

Lemma 2.1. *Let $T \in \mathscr{K}(\mathsf{X})$.*

1) *If there exists $n \in \mathbb{N}\backslash\{0\}$, such that $T \in \Omega^{\ell}_{n}$, then T is not one-to-one.*

2) *If there exists $n \in \mathbb{N}\backslash\{0\}$, such that $T \in \Omega^{r}_{n}$, then T is not onto.*

Proof.
1) Suppose that T is one-to-one, then we know that $T \in \Omega^{\ell}_{0}$, and hence there exists $S \in \mathscr{B}(\mathsf{X})$ such that $I = ST \in \mathscr{K}(\mathsf{X})$, which is a contradiction because X is of infinite dimensional and therefore T is not one-to-one.
2) Can be proved in the same way as 1). □

Recall that $T \in \mathscr{K}(\mathsf{X})$ is g-invertible if and only if it is of finite rank [4]. In fact, this result can be extended for pseudo-generalized invertible operators, as we can see in the following theorem.

Theorem 2.2. *Let $T \in \mathscr{K}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. The following assertions are equivalent :*

(1) $T \in \Omega^{\ell}_{n}$,

(2) $T \in \Omega^{r}_{n}$,

(3) $\mathrm{rank}(T^{n}) < +\infty$.

Proof.
"(1) $\Longrightarrow$ (3)" Let $S \in {}^{T}\mathsf{S}^{\ell}_{n}$. As T is a compact operator, then $I - ST$ is a Fredholm operator and therefore $\beta(I - ST) < +\infty$. Now, since $\mathsf{R}(I - ST) \subseteq \mathsf{N}(T^{n})$, so $\mathrm{codim}\, \mathsf{N}(T^{n}) \leq \beta(I - ST) < +\infty$, and consequently,

$$\mathrm{rank}(T^{n}) = \mathrm{codim}\, \mathsf{N}(T^{n}) < +\infty.$$

"(2) $\Longrightarrow$ (3)" Let $S \in {}^{T}\mathsf{S}^{r}_{n}$. Since T is a compact operator, it follows that $I - TS$ is a Fredholm operator and therefore $\alpha(I - TS) < +\infty$. Now, as $\mathsf{R}(T^{n}) \subseteq \mathsf{N}(I - TS)$, so

$$\mathrm{rank}(T^{n}) \leq \alpha(I - TS) < +\infty.$$

"(3) $\Longrightarrow$ (1)" and "(3) $\Longrightarrow$ (2)" If $\operatorname{rank}(T^n) < +\infty$, we know that $T^n \in \Omega_1^\ell$ $(= \Omega_1^r)$ and hence $T \in \Omega_n^\ell \cap \Omega_n^r$. This completes the proof. □

Clearly, if $T \in \mathscr{K}(\mathsf{X})$ and n is the smallest non-negative integer such that T^n is of finite rank, then $T \in \left(\Omega^\ell \cap \Omega^r\right) \backslash (\Omega_{n-1}^\ell \cap \Omega_{n-1}^r\big)$. Also, if T is both compact and pseudo-generalized invertible, from [7, Example 2.2, Remark 2.3], we can deduced that the fact $\operatorname{rank}(T^n) < +\infty$, does not mean that T^{n-1} is also of finite rank. Now, from Theorem 2.2, we obtain the following remark.

Remark 2.3. Let $T \in \mathscr{K}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$. If T^n is of infinite rank, then $T \notin \Omega_n^\ell \cup \Omega_n^r$.

It follows that injective compact operators could not be pseudo-generalized invertible as can be seen in the following.

Corollary 2.4. *Let $T \in \mathscr{K}(\mathsf{X})$. If there exists $n \in \mathbb{N}\backslash\{0\}$, such that $T \in \Omega_n^\ell \cup \Omega_n^r$, then T is neither onto nor one-to-one.*

Proof.
By Theorem 2.2, we see that $T \in \Omega_n^\ell$ (resp. Ω_n^r) if and only if $T \in \Omega_n^\ell \cap \Omega_n^r$. Now, Lemma 2.1, allows us to conclude the stated result. □

The reasoning presented in Lemma 2.1, for compact pseudo-generalized invertible operators enables us to formulate the following result.

Lemma 2.5. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$.*

1) *If $T \in \Omega_n^\ell$ and $\mathscr{K}(\mathsf{X}) \cap {}^T\mathsf{S}_n^\ell \neq \emptyset$, then T is not one-to-one.*

2) *If $T \in \Omega_n^r$ and $\mathscr{K}(\mathsf{X}) \cap {}^T\mathsf{S}_n^r \neq \emptyset$, then T is not onto.*

As in "(1) $\Longrightarrow$ (3)" and "(2) $\Longrightarrow$ (3)" in Theorem 2.2, we can obtain the next proposition.

Proposition 2.6. *Let $T \in \mathscr{B}(\mathsf{X})$ and $n \in \mathbb{N}\backslash\{0\}$.*

1) *If $T \in \Omega_n^\ell$ and $\mathscr{K}(\mathsf{X}) \cap {}^T\mathsf{S}_n^\ell \neq \emptyset$ then* $\operatorname{rank}(T^n) < +\infty$.

2) *If $T \in \Omega_n^r$ and $\mathscr{K}(\mathsf{X}) \cap {}^T\mathsf{S}_n^r \neq \emptyset$ then* $\operatorname{rank}(T^n) < +\infty$.

Under the same assumptions as in Proposition 2.6, $\operatorname{rank}(T^n)$ is finite does not necessarily imply that $\operatorname{rank}(T^{n-1})$ is finite, as we can see in the following example.

Example 2.7. Let $s \in \mathbb{N}\backslash\{0, 1\}$ and $N \geq s - 1$. In a separable Hilbert space H with an orthonormal basis $(e_k)_{k \in \mathbb{N}\backslash\{0\}}$, let us consider the operator defined by

$$T_s(e_k) = \begin{cases} \frac{1}{k} e_{k+1} & \text{if } k \in \mathbb{N}\backslash\{s\mathbb{N} + s - 1\} \text{ and } k > N \\ e_k & \text{if } k \in s\mathbb{N} + s - 1 \text{ and } k \leq N \\ 0 & \text{if not.} \end{cases}$$

We see that $\operatorname{rank}(T_s^{s-1})$ is infinite and $\operatorname{rank}(T_s^s) < +\infty$, (see [7, Remark 2.3]). On the other hand, if S is a g_2-inverse of T_s^s, then $ST_s^{s-1} \in \mathscr{K}(\mathsf{X}) \cap {}^{T_s}\mathsf{S}_s^\ell$ and $T_s^{s-1}S \in \mathscr{K}(\mathsf{X}) \cap {}^{T_s}\mathsf{S}_s^r$. In fact, we know that $\operatorname{rank}(S) < +\infty$, so $\operatorname{rank}(ST_s^{s-1}) < +\infty$ and $\operatorname{rank}(T_s^{s-1}S) < +\infty$. Thus, both ST_s^{s-1} and $T_s^{s-1}S$ are compact operators. Finally, clearly we have $ST_s^{s-1} \in {}^{T_s}\mathsf{S}_s^\ell$ and $T_s^{s-1}S \in {}^{T_s}\mathsf{S}_s^r$. Hence the result.

In the following, we give some conditions to characterize some pseudo-generalized inverses of compact operators.

Proposition 2.8. *Let $T \in \mathscr{K}(\mathsf{X})$ and $S \in \mathscr{B}(\mathsf{X})$.*

1) *If there exist $L \in {}^{I-ST}\mathsf{S}_1^\ell$ and $n \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{R}(L) \subseteq \mathsf{N}(T^n - T^nST)$, then* $\mathrm{rank}(T^n) < +\infty$ *and $S \in {}^T\mathsf{S}_n^\ell$.*

2) *If there exist $L \in {}^{I-TS}\mathsf{S}_1^r$ and $n \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{R}(T^n - TST^n) \subseteq \mathsf{N}(L)$, then* $\mathrm{rank}(T^n) < +\infty$ *and $S \in {}^T\mathsf{S}_n^r$.*

Proof.
Since $T \in \mathscr{K}(\mathsf{X})$, then $I - ST$ and $I - TS$ are Fredholm operators. Therefore, $I - ST \in \Omega_1^\ell$ and $I - TS \in \Omega_1^r$.
1) As $L \in {}^{I-ST}\mathsf{S}_1^\ell$, then

$$(I - ST)L(I - ST) = I - ST.$$

Hence

$$I = ST + (I - ST)L(I - ST).$$

Consequently,

$$\begin{aligned} T^n &= T^nST + T^n(I - ST)L(I - ST) \\ &= T^nST + (T^n - T^nST)L(I - ST). \end{aligned}$$

In the other hand, from $\mathsf{R}(L) \subseteq \mathsf{N}(T^n - T^nST)$, we obtain $T^n = T^nST$. So, $T \in \Omega_n^\ell$ and $S \in {}^T\mathsf{S}_n^\ell$. Now the result can be deduced from Theorem 2.2.
2) This assertion can be proved in the same way as 1). □

Analogously, if some pseudo-generalized inverse is compact, then we have the following result.

Proposition 2.9. *Let $T \in \mathscr{B}(\mathsf{X})$ and $S \in \mathscr{K}(\mathsf{X})$.*

1) *If there exist $L \in {}^{I-ST}\mathsf{S}_1^\ell$ and $n \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{R}(L) \subseteq \mathsf{N}(T^n - T^nST)$, then* $\mathrm{rank}(T^n) < +\infty$ *and $S \in {}^T\mathsf{S}_n^\ell$.*

2) *If there exists $L \in {}^{I-TS}\mathsf{S}_1^r$ and $n \in \mathbb{N}\backslash\{0\}$, such that $\mathsf{R}(T^n - TST^n) \subseteq \mathsf{N}(L)$, then* $\mathrm{rank}(T^n) < +\infty$ *and $S \in {}^T\mathsf{S}_n^r$.*

Proof.
Since $S \in \mathscr{K}(\mathsf{X})$, then $I - ST$ and $I - TS$ are Fredholm operators. Therefore, as in the proof of Proposition 2.8, we deduce the stated result. □

We close this section with the following decomposition result.

Proposition 2.10. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{K}(\mathsf{X})$.*

1) *If $T \in \Omega_n^\ell$, then for all $S \in {}^T\mathsf{S}_n^\ell$, there exists $L \in {}^T\mathsf{S}_n^\ell$, such that*

$$T = TLT + TP,$$

where $P \in \mathscr{B}(\mathsf{X})$ is a projection of range $\mathsf{R}(I - ST)$.

2) *If $T \in \Omega_n^r$, then for all $S \in {}^T\mathsf{S}_n^r$, there exists $L \in {}^T\mathsf{S}_n^r$, such that*

$$T = TLT + PT,$$

where $P \in \mathscr{B}(\mathsf{X})$ is a projection of kernel $\mathsf{N}(I - TS)$.

Proof.

1) Let $S \in {}^T\mathsf{S}_n^\ell$, since T is a compact operator, then $I - ST$ is a Fredholm operator and so $I - ST \in \Omega_1^\ell$. Consequently, as in the proof of [7, Theorem 4.3], we can see that there exists $L \in {}^T\mathsf{S}_n^\ell$, such that $T = TLT + TP$, where P is a projection of range $\mathsf{R}(I - ST)$.

2) Let $S \in {}^T\mathsf{S}_n^r$. Since T is a compact operator, then $I - TS$ is a Fredholm operator and so $I - TS \in \Omega_1^r$. Now, as in the proof of [7, Theorem 4.3], we can see that there exists $L \in {}^T\mathsf{S}_n^r$, such that $T = TLT + PT$, where P is a projection of kernel $\mathsf{N}(I - TS)$. □

3. Product of commuting pseudo-generalized invertible operators

In this section, we give some cases where the product of two commuting pseudo-generalized invertible operators is preserved. First, from [1], recall that the product of two g-invertible operators is not necessary g-invertible, even for commuting operators. The same fact is proved for pseudo-generalized invertible operators, as we can see in the following examples inspired by those of Caradus [1] and Harte [4].

Example 3.1. Let H be a separable Hilbert space with orthonormal basis $(e_k)_{k\geq 1}$ and $M, C \in \mathscr{B}(\mathsf{H})$ defined by :

$$M(e_k) = \begin{cases} e_{k+1} & \text{if } k \in 3\mathbb{N}\backslash\{0\} \\ 0 & \text{if not,} \end{cases}$$

and

$$C(e_k) = \begin{cases} \frac{1}{k}e_{k-1} & \text{if } \in 3\mathbb{N}\backslash\{0\} + 1, \\ 0 & \text{if not.} \end{cases}$$

Let $\lambda \notin \sigma(C)$, where $\sigma(C)$ is the spectrum of C. We set $T = (\lambda I - C)M$, then

$$T(e_k) = \begin{cases} \lambda e_{k+1} - \frac{1}{k+1}e_k & \text{if } k \in 3\mathbb{N}\backslash\{0\} \\ 0 & \text{if not.} \end{cases}$$

It is clear that $\mathsf{R}(M)$ is closed, hence $M \in \Omega_1^\ell$ $(= \Omega_1^r)$. As $\lambda I - C$ is invertible, we get $T \in \Omega_1^\ell$ $(= \Omega_1^r)$ and therefore $T \in \Omega_n^\ell \cap \Omega_n^r$, for all $n \in \mathbb{N}\backslash\{0\}$. On the other hand, for $m \in \mathbb{N}\backslash\{0\}$, we see that

$$T^2(e_k) = \begin{cases} \frac{-\lambda}{k+1}e_{k+1} + \frac{1}{(k+1)^2}e_k & \text{if } k \in 3\mathbb{N}\backslash\{0\} \\ 0 & \text{if not} \end{cases}$$

and

$$T^{2m}(e_k) = \begin{cases} \frac{-\lambda}{(k+1)^{2m-1}}e_{k+1} + \frac{1}{(k+1)^{2m}}e_k & \text{if } k \in 3\mathbb{N}\backslash\{0\} \\ 0 & \text{if not.} \end{cases}$$

It is clear that T^{2m} is a compact operator of infinite rank, so by Remark 2.3, $T^2 \notin \Omega^\ell \cup \Omega^r$. Hence the result. □

Example 3.2. Let H be a seperable Hilbert space with an orthonormal basis $(e_k)_{k>0}$. Let $U, V \in \mathscr{B}(\mathsf{H})$, such that

$$U(e_k) = e_{k+1}$$

and

$$V(e_k) = \begin{cases} 0 & \text{if } k = 1, \\ e_{k-1} & \text{if not.} \end{cases}$$

We define $T, S \in \mathscr{B}(\mathsf{H} \times \mathsf{H})$ as follows

$$T(x, y) = (U(y), 0)$$

and

$$S(x, y) = (0, V(x)).$$

It is clear that

$$TST = T$$

and therefore $T \in (\Omega_1^\ell \cap \Omega_1^r) \subseteq (\Omega_n^\ell \cap \Omega_n^r)$, for all $n \in \mathbb{N}\backslash\{0\}$ and $S \in {}^T\mathsf{S}_1^\ell$ $(= {}^T\mathsf{S}_1^r)$. We consider the map $W \in \mathscr{K}(\mathsf{H})$, defined by $e_k \longmapsto \frac{1}{k+2}e_k$, for all $k \in \mathbb{N}\backslash\{0\}$. It is clear that $\|W\| < \frac{1}{2}$. Now, let $\lambda \in \mathbb{R}$, such that

$$2\|W\| < \lambda < 1.$$

We define the operator $L \in \mathscr{B}(\mathsf{H} \times \mathsf{H})$ as follows

$$L(x, y) = (\lambda x + W(y), \lambda y + W(x)),\ \forall (x, y) \in \mathsf{H} \times \mathsf{H}.$$

Since

$$(I-L)(x,y) = ((1-\lambda)x + W(y), (1-\lambda)y + W(x)),$$

we deduce that

$$\begin{aligned}\|(I-L)(x,y)\| &= \|((1-\lambda)x + W(y), (1-\lambda)y + W(x))\| \\ &= \|((1-\lambda)x + W(y)\| + \|(1-\lambda)y + W(x))\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|(I-L)(x,y)\| &\leq \|(1-\lambda)x\| + \|W(y)\| + \|(1-\lambda)y\| + \|W(x)\| \\ &\leq (1-\lambda)\|x\| + \|W\|\|y\| + (1-\lambda)\|y\| + \|W\|\|x\|.\end{aligned}$$

Now, since $2\|W\| < \lambda < 1$, it follows that

$$\|(I-L)(x,y)\| < (1-\lambda)\|x\| + \frac{\lambda}{2}\|y\| + (1-\lambda)\|y\| + \frac{\lambda}{2}\|x\|.$$

Hence,

$$\|(I-L)(x,y)\| < (1-\frac{\lambda}{2})\|(x,y)\|.$$

As a result, we obtain $\|I-L\| \leq (1-\frac{\lambda}{2}) < 1$ and L is therefore invertible. Now, since $T \in \Omega_1^\ell\ (= \Omega_1^r)$, it follows that $TL \in (\Omega_1^\ell \cap \Omega_1^r) \subseteq (\Omega_n^\ell \cap \Omega_n^r)$, for all $n \in \mathbb{N}\backslash\{0\}$ and

$$TL(x,y) = (\lambda U(y) + UW(x), 0).$$

Hence, for $n \in \mathbb{N}\backslash\{0\}$, we see that

$$(TL)^n(x,y) = \Big((UW)^{n-1}(\lambda U(y) + UW(x)), 0\Big).$$

Since W is a compact operator, then for any $n \in \mathbb{N}\backslash\{0, 1\}$, $(TL)^n$ is too. Now, let $k, s \in \mathbb{N}\backslash\{0\}$ and $n \in \mathbb{N}\backslash\{0, 1\}$, then we have

$$(TL)^n(e_k, e_s) = \Big(\lambda\Big(\prod_{r=3}^{n+1} s+r\Big)^{-1} e_{s+n} + \Big(\prod_{r=2}^{n+1} k+r\Big)^{-1} e_{k+n}, 0\Big).$$

It is clear that for all $n \in \mathbb{N}\backslash\{0,1\}$, $(TL)^n$ is of infinite rank and since $(TL)^n$ is a compact operator, then by Remark 2.3, $(TL)^n \notin \Omega^\ell \cup \Omega^r$. Hence the result. □

Consequently, the product stability of pseudo-generalized invertible operators is not preserved in general except for left-invertible or right-invertible operators. In the remainder of this section we focus on some specific cases in which the pseudo-generalized invertibility of the product of commuting pseudo-generalized invertible operators is preserved.

The following proposition serves as a starting point for our investigation.

Proposition 3.3. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T, A, B \in \mathscr{B}(\mathsf{X})$, such that $TA = BT$ and $TBT = BT^2$.*

1) *If $T \in \Omega_n^\ell$ and $B \in \Omega_0^\ell$, then $TA \in \Omega_n^\ell$.*

2) *If $T \in \Omega_n^r$ and $A \in \Omega_0^r$, then $TA \in \Omega_n^r$.*

3) *If $T, A \in \Omega_n^\ell$ and there exists $S \in {}^T\mathsf{S}_n^\ell$ such that $AS = SB$, then $TA \in \Omega_n^\ell$.*

4) *If $T, B \in \Omega_n^r$ and there exists $S \in {}^T\mathsf{S}_n^r$ such that $AS = SB$, then $TA \in \Omega_n^r$.*

Proof.
First, we will prove by induction that

$$T^kA = BT^k, \ \forall k \geq 1.$$

In fact, for $k = 1$ the result is obvious. Now, for $k \geq 2$, assume that $T^kA = BT^k$. Since $TBT = BT^2$, we see that

$$T^{k+1}A = TBT^k = (TBT)T^{k-1} = BT^{k+1},$$

hence the result.
Therefore, we are able to prove, by induction, that

$$(TA)^k = T^kA^k = B^kT^k = (BT)^k, \ \forall k \in \mathbb{N}.$$

For $k = 0$ and $k = 1$, the result is obvious. Now, for $k \geq 2$, assume that

$$(TA)^k = T^kA^k = B^kT^k = (BT)^k$$

and let us prove that

$$(TA)^{k+1} = T^{k+1}A^{k+1} = B^{k+1}T^{k+1} = (BT)^{k+1}.$$

We see that

$$(TA)^{k+1} = (TA)^kTA = B^kT^{k+1}A = B^{k+1}T^{k+1}.$$

Similarly,

$$(BT)^{k+1} = BT(BT)^k = BT^{k+1}A^k = T^{k+1}A^{k+1}.$$

1) Let $S \in {}^T\mathsf{S}_n^\ell$ and $L \in {}^B\mathsf{S}_0^\ell$, then

$$(TA)^nSLTA = (TA)^nSLBT = B^nT^nST = B^nT^n = (TA)^n.$$

2) Let $S \in {}^T\mathsf{S}_n^r$ and $L \in {}^A\mathsf{S}_0^r$, then

$$TALS(TA)^n = TST^nA^n = T^nA^n = (TA)^n.$$

3) Let $L \in {}^A\mathsf{S}_n^\ell$, then

$$(TA)^nLSTA = T^nA^nLSBT = T^nA^nLAST = T^nA^nST = B^nT^nST = (TA)^n.$$

4) Let $L \in {}^B\mathsf{S}_n^r$, then

$$TASL(TA)^n = TSBLB^nT^n = TSB^nT^n = TST^nA^n = T^nA^n = (TA)^n.$$

Therefore the proof is complete. □

Proposition 3.4. *Let $n \in \mathbb{N}\backslash\{0\}$, $T, A \in \mathscr{B}(\mathsf{X})$ such that $T^n = T^{n+1}$ and $TA = AT$.*

1) *If $A \in \Omega_n^\ell$, then $TA \in \Omega_n^\ell$.*

2) *If $A \in \Omega_n^r$, then $TA \in \Omega_n^r$.*

Proof.
1) Let $B \in {}^A\mathsf{S}_n^\ell$, then we see that

$$(TA)^nB(TA) = T^nA^nBAT = T^nA^nT = T^{n+1}A^n = T^nA^n = (TA)^n.$$

Hence the result.
2) This assertion can be proved in the same way as 1). □

As a consequence of Proposition 3.4, we obtain the following corollary.

Corollary 3.5. *Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \mathscr{B}(\mathsf{X})$ and $P \in \mathscr{B}(\mathsf{X})$ be a projection such that $PT = TP$.*

1) *If $T \in \Omega_n^\ell$, then $TP \in \Omega_n^\ell$.*

2) *If $T \in \Omega_n^r$, then $TP \in \Omega_n^r$.*

Now we state the following result.

Theorem 3.6. *Let $T, A \in \mathscr{B}(\mathsf{X})$ such that $T \in \Omega_1^\ell$, $A \in \Omega_n^\ell$, for some $n \in \mathbb{N}$ and let X_1 be a topological complement of $\mathsf{N}(T)$ in X. If*

(1) *$AT = TA$,*

(2) *$A(\mathsf{X}_1) \subseteq \mathsf{X}_1$,*

then $AT \in \Omega_n^\ell$.

Proof.
Let P be a projection of range $\mathsf{R}(T)$ and let us denote by i the canonical injection of X_1 into X. Define

$$\begin{array}{cccc} T_1: & \mathsf{X}_1 & \longrightarrow & \mathsf{R}(T) \\ & x & \longmapsto & Tx. \end{array}$$

We know that $S = iT_1^{-1}P$ is a g_2-inverse of T (see [1]). Since $A(\mathsf{X}_1) \subseteq \mathsf{X}_1$ and $AT = TA$, then

$$AT_1x = i_1T_1Ax,\ \forall x \in \mathsf{X}_1,$$

where i_1 is the canonical injection of X_1 into X. Now, let $y \in \mathsf{R}(T)$, then $T_1^{-1}y \in \mathsf{X}_1$. So

$$AT_1T_1^{-1}y = i_1T_1AT_1^{-1}y,\ \forall\, y \in \mathsf{R}(T),$$

and this implies that

$$Ay = i_1T_1AT_1^{-1}y,\ \forall y \in \mathsf{R}(T).$$

Since $AT = TA$, it follows that $A(\mathsf{R}(T)) \subseteq \mathsf{R}(T)$, and hence

$$iT_1^{-1}Ay = iT_1^{-1}T_1AT_1^{-1}y,\ \forall\, y \in \mathsf{R}(T).$$

Therefore,

$$iT_1^{-1}Ay = AT_1^{-1}y,\ \forall\, y \in \mathsf{R}(T). \tag{*}$$

Now, let $B \in {}^A\mathsf{S}_n^\ell$, since $TA = AT$ and $PT = T$, then

$$(AT)^nBS(AT) = T^nA^nBT_1^{-1}PAT = T^nA^nBT_1^{-1}AT.$$

So, by (*), we see that

$$(AT)^nBS(AT) = T^nA^nBAT_1^{-1}T.$$

As $A^nBA = A^n$, we get

$$(AT)^nBS(AT) = T^nA^nT_1^{-1}T = A^nT^nT_1^{-1}T = A^nT^nST = (AT)^n.$$

Hence the result. □

Consequently, under the same hypotheses as in Theorem 3.6, if T is g-invertible then the g-invertibility can be obtained for any power of T.

Corollary 3.7. *Let* $T \in \Omega_1^\ell$ *and* X_1 *be a topological complement of* $\mathsf{N}(T)$. *If* $\mathsf{R}(T) \subseteq \mathsf{X}_1$, *then*

$$T^n \in \Omega_1^\ell,\ \forall\, n \in \mathbb{N}\backslash\{0\}.$$

Proof.
Since $T \in \Omega_1^\ell$ and $T(\mathsf{X}_1) = \mathsf{R}(T)$, then $T(\mathsf{X}_1) \subseteq \mathsf{X}_1$. By Theorem 3.6, we deduce that $T^2 \in \Omega_1^\ell$. Now, suppose that for some $k \geq 2$, we have $T^k \in \Omega_1^\ell$. We will show that $T^{k+1} \in \Omega_1^\ell$. Notice that

$$T^k(\mathsf{X}_1) \subseteq T^{k-1}(\mathsf{X}_1) \subseteq \cdots \subseteq T(\mathsf{X}_1) \subseteq \mathsf{X}_1$$

and since $TT^k = T^kT$, then by Theorem 3.6 we deduce that $T^{k+1} \in \Omega_1^\ell$. Hence the result. □

We also have a result analogous to Theorem 3.6, for right pseudo-generalized invertible operators.

Theorem 3.8. *Let* $A, T \in \mathscr{B}(\mathsf{X})$ *such that* $A \in \Omega_n^r$, *where* $n \in \mathbb{N}$ *and* $T \in \Omega_1^r$. *Let* X_2 *be a topological complement of* $\mathsf{R}(T)$ *in* X. *If*

(1) $AT = TA$,

(2) $\mathsf{X}_2 \subseteq \mathsf{N}(A)$,

then $AT \in \Omega_n^r$.

Proof.
Let X_1 be a topological complement of $\mathsf{N}(T)$ in X, P the projection onto $\mathsf{R}(T)$ along X_2, i the canonical injection of X_1 into X and

$$\begin{array}{rccc} T_1: & \mathsf{X}_1 & \longrightarrow & \mathsf{R}(T) \\ & x & \longmapsto & Tx. \end{array}$$

We know that $S = iT_1^{-1}P$ is a g_2-inverse of T (see [1]). Let $x \in \mathsf{X}$, so there exists $(x_1, x_2) \in \mathsf{R}(T) \times \mathsf{X}_2$, such that $x = x_1 + x_2$. Since $\mathsf{X}_2 \subseteq \mathsf{N}(A)$, we obtain

$$APx = Ax_1 = Ax,$$

this implies that $AP = A$. Now, let $B \in \mathscr{B}(\mathsf{X})$, such that $ABA^n = A^n$. Then

$$(AT)SB(AT)^n = ATT_1^{-1}PBA^nT^n = APBA^nT^n = ABA^nT^n = (AT)^n.$$

Hence the result. □

As a consequence we obtain the following result.

Corollary 3.9. *Let* $T \in \Omega_1^r$ *and* X_2 *be a topological complement of* $\mathsf{R}(T)$. *If* $\mathsf{X}_2 \subseteq \mathsf{N}(T)$, *then*

$$T^n \in \Omega_1^r,\ \forall\, n \in \mathbb{N}\backslash\{0\}.$$

Proof. Since $T \in \Omega_1^r$ and $\mathsf{X}_2 \subseteq \mathsf{N}(T)$, by Theorem 3.8, we deduce that $T^2 \in \Omega_1^r$. Now, assume that for $k \geq 2$, we have $T^k \in \Omega_1^r$. We will show that $T^{k+1} \in \Omega_1^r$. First, we notice that

$$\mathsf{X}_2 \subseteq \mathsf{N}(T) \subseteq \mathsf{N}(T^k).$$

Moreover, since $TT^k = T^kT$, so from Theorem 3.8 we deduce that $T^{k+1} \in \Omega_1^r$. This completes the proof. □

Finally, as in [9, Lemma 5, P. 126], we obtain the following proposition.

Proposition 3.10. *Let* $n \in \mathbb{N}\backslash\{0\}$ *and* $A, T \in \mathscr{B}(\mathsf{X})$, *such that* $A^n = A^{2n-1} \in \Omega_1^\ell\ (= \Omega_1^r)$ *and* $TA = AT$.

1) *If* $T \in \Omega_n^\ell$ *and there exist* $C, D \in \mathscr{B}(\mathsf{X})$, *such that* $TD + CA = I$, *then* $TA \in \Omega_n^\ell$.

2) *If $T \in \Omega_n^r$ and there exist $C, D \in \mathscr{B}(\mathsf{X})$, such that $DT + AC = I$, then $TA \in \Omega_n^r$.*

Proof.
1) Let $S \in {}^{T}\mathsf{S}_n^{\ell}$ and $B \in {}^{A}\mathsf{S}_n^r$. Since $TD + CA = I$, then we see that

$$\begin{array}{rcl}(TA)^n SBA^{n-1}(TA) & = & A^n T^n S(CA + TD)BA^n T \\ & = & A^n T^n SCA^n T + A^n T^n DBA^n T.\end{array}$$

Consequently, we see that

$$(TA)^n SBA^{n-1}(TA) = A^n T^n S(I - TD)A^{n-1}T + A^n T^{n-1}(I - CA)BA^n T.$$

Using the fact that $TA = AT$, $A^n BA = A^n$ and $A^{2n-1} = A^n$, it follows that

$$(TA)^n SBA^{n-1}(TA) = A^n T^n - A^n T^{n-1} TDA^{n-1}T + A^n T^n - A^n T^{n-1} CAA^{n-1}T.$$

Finally, we obtain

$$(TA)^n SBA^{n-1}(TA) = (AT)^n + A^n T^n - A^n T^{n-1}(TD + CA)A^{n-1}T = (TA)^n.$$

Hence the result.
2) As in 1), we can deduce that $TA \in \Omega_n^r$. □

4. Adjoint and product in the case of Hilbert spaces

In this section, H denotes a complex Hilbert space of infinite dimension. For $T \in \mathscr{B}(\mathsf{H})$, we denote by T^* its adjoint. We recall that T is n-left pseudo-generalized invertible, for some $n \in \mathbb{N}$, if and only if its adjoint T^* is n-right pseudo-generalized invertible. In this section, if T is a pseudo-generalized invertible operator, we give some cases where the product TT^* still pseudo-generalized invertible.

In our starting result, we deal with normal right pseudo-generalized invertible operators.

Proposition 4.1. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{H})$ be a normal operator. If $T \in \Omega_n^r$ and there exists $S \in \mathscr{B}(\mathsf{H})$, such that $TST^n = T^n$ with TS self-adjoint, then*

$$(TT^*)^n \in \Omega_1^{\ell} \,(= \Omega_1^r).$$

Moreover, we have $SS^* \in {}^{TT^*}\mathsf{S}_n^{\ell} \cap {}^{TT^*}\mathsf{S}_n^r$ and $S^n(S^*)^n$ is a g_1-inverse of $(TT^*)^n$.

Proof.
Since $TST^n = T^n$ and TS is self-adjoint, then

$$\begin{array}{rcl}((TT^*)^n\, S^n(S^*)^n\, (TT^*)^n & = & (T^*)^n\, T^{n-1}(TS)\, S^{n-1}(S^*)^{n-1}(TS)^*\, (T^*)^{n-1}T^n \\ & = & T^{n-1}\, ((T^*)^n S^* T^*)\, S^{n-1}(S^*)^{n-1}(TST^n)\, (T^*)^{n-1} \\ & = & T^{n-1}\, (T^*)^n\, S^{n-1}(S^*)^{n-1}T^n\, (T^*)^{n-1}.\end{array}$$

Hence, by repeating the same process, we obtain

$$(TT^*)^n\, S^n(S^*)^n\, (TT^*)^n = (TT^*)^n.$$

This implies that $(TT^*)^n \in \Omega_1^{\ell}$ $(= \Omega_1^r)$ and TT^* is therefore n-left and n-right pseudo generalized invertible. Also, we have

$$\begin{array}{rcl}(TT^*)^n\, SS^*\, TT^* & = & (T^*)^n\, T^n\, S(TS)^*\, T \\ & = & (T^*)^n\, T^n\, STST \\ & = & (TT^*)^n.\end{array}$$

This implies that SS^* is a n-left pseudo-generalized inverse of TT^*, and also SS^* is a n-right pseudo-generalized inverse of TT^*. □

Similarly, for normal left pseudo-generalized invertible operators, we have

Proposition 4.2. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{H})$ be a normal operator. If $T \in \Omega_n^{\ell}$ and there exists $S \in \mathscr{B}(\mathsf{H})$, such that $T^nST = T^n$ and ST is self-adjoint, then*

$$(TT^*)^n \in \Omega_1^{\ell} \, (= \Omega_1^r).$$

Moreover, we have $S^*S \in {}^{TT^*}\mathsf{S}_n^{\ell} \cap {}^{TT^*}\mathsf{S}_n^r$ and $(S^*)^nS^n$ is a g_1-inverse of $(TT^*)^n$.

Proof.
As $T \in \Omega_n^{\ell}$, $T^n = T^nST$ and ST is self-adjoint, then $T^* \in \Omega_n^r$, $T^{*n} = T^* S^* T^{*n}$ and T^*S^* is self-adjoint. Since T is normal, the result can be deduced, from Proposition 4.1. □

It is natural to ask if Propositions 4.1 and 4.2 can be generalized to normal operators.

Question : The conclusion of Proposition 4.1 (resp. Proposition 4.2) holds if we suppose only that TS (resp. ST) is normal?

Now, let us give the following result.

Proposition 4.3. *Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \mathscr{B}(\mathsf{H})$, be a normal operator such that $T \in \Omega_n^{\ell} \cap \Omega_n^r$ and ${}^T\mathsf{S}_n^{\ell} \cap {}^T\mathsf{S}_n^r \neq \emptyset$. If there exists $S \in {}^T\mathsf{S}_n^{\ell} \cap {}^T\mathsf{S}_n^r$ such that TS is normal, then $TT^* \in \Omega_n^{\ell} \cap \Omega_n^r$ and SS^* is a n-left (resp. n-right) pseudo-generalized inverse of TT^*.*

Proof.
As $S \in {}^T\mathsf{S}_n^{\ell}$, we see that

$$(TT^*)^nSS^*TT^* = (T^*)^n \, T^{n-1} \, (TS) \, (TS)^* \, T.$$

Since TS is normal, it follows that

$$(TT^*)^nSS^*TT^* = T^{n-1}(T^*)^nS^*T^*TST = T^{n-1}(TST^n)^*TST.$$

Now, as $S \in {}^T\mathsf{S}_n^r$, we have

$$(TT^*)^nSS^*TT^* = T^{n-1}(T^*)^nTST = (TT^*)^n.$$

From the fact that TT^* and SS^* are self-adjoint, we deduce the result. □

Finally, in the same way, as in Proposition 4.3, we obtain the following result.

Proposition 4.4. *Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \mathscr{B}(\mathsf{H})$ be a normal operator such that $T \in \Omega_n^{\ell} \cap \Omega_n^r$ and ${}^T\mathsf{S}_n^{\ell} \cap {}^T\mathsf{S}_n^r \neq \emptyset$. If there exists $S \in {}^T\mathsf{S}_n^{\ell} \cap {}^T\mathsf{S}_n^r$ such that ST is normal, then $TT^* \in \Omega_n^{\ell} \cap \Omega_n^r$ and S^*S is a n-left (resp. n-right) pseudo-generalized inverse of TT^*.*

5. Some other results of product

In this section, we return to the case of Banach spaces. Other results of product related to pseudo-generalized invertible operators will be presented giving some applications to the notion of g-invertibility. First, recall that $T \in \mathscr{B}(\mathsf{X})$ is Drazin invertible if there exists $S \in \mathscr{B}(\mathsf{X})$ such that

$$T^nST = T^n, \; STS = S \text{ and } TS = ST. \tag{5.1}$$

In such a case, S is unique, called the Drazin inverse of T and denoted by T^D. For a better understanding of this notion see [2, 3, 5, 11] and we refer to [6], for a more general concept. Clearly, if T is Drazin invertible, then T is pseudo-generalized invertible and if $S \in \mathscr{B}(\mathsf{X})$ is a n-left (resp. n-right) pseudo-generalized inverse, such that $TS = ST$, then $(TS)^{n+1} = (TS)^n$. Consequently, $TS \in \Omega^{\ell} \cap \Omega^r$, In fact, more generally, we have the following remark.

Remark 5.1. *Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \Omega_n^{\ell}$ (resp. Ω_n^r) and S, $L \in \mathscr{B}(\mathsf{X})$, such that $SL \in {}^T\mathsf{S}_n^{\ell}$ (resp. ${}^T\mathsf{S}_n^r$). It is clear that if $TS = ST$, then $TS \in \Omega_n^{\ell}$ (resp. Ω_n^r) and if $TL = LT$, then $TL \in \Omega_n^{\ell}$ (resp. Ω_n^r).*

As an application, we see in the following that the notion of pseudo-generalized invertibility allows us to establish the g-invertibility in some special cases. Let us start with the following lemma, which is a generalization of [4, Theorem 3.8.7] proved for g-invertible operators.

Lemma 5.2. *Let $n \in \mathbb{N}\backslash\{0\}$, $T \in \mathscr{B}(\mathsf{X})$ and $U \in \mathscr{B}(\mathsf{X})$ be an invertible operator. If one of the following assertions holds :*

(1) *$T \in \Omega_n^\ell$ and $S \in {}^T\mathsf{S}_n^\ell$ such that $T^n S \in \Omega_1^\ell$ and $I + S(U - T)$ is invertible,*

(2) *$T \in \Omega_n^r$ and $S \in {}^T\mathsf{S}_n^r$ such that $T^n S \in \Omega_1^r$ and $I + (U - T)S$ is invertible,*

then $T^n \in \Omega_1^\ell$ $(= \Omega_1^r)$.

Proof.
If the first assertion holds and if we suppose that $V \in \mathscr{B}(\mathsf{X})$ is the inverse of $I + S(U - T)$, we see that

$$T^n = T^n(I + S(U - T))V = (T^n + T^n SU - T^n ST)V = T^n SUV.$$

As UV is invertible and $T^n S \in \Omega_1^\ell$, we obtain $T^n \in \Omega_1^\ell$.
Now, if the assertion (2) holds, then the result can be obtained in the same way. The proof is therefore complete. □.

For a subspace $\mathsf{M} \subseteq \mathsf{X}$, we denote by $T_{|\mathsf{M}}$ the restriction of T from M onto M.

Now, we discuss some cases in which the pseudo-generalized invertibility of two operators allows us to obtain the g-invertibility of their product. First, we start with the left pseudo-generalized invertibility.

Proposition 5.3. *Let $A, T \in \mathscr{B}(\mathsf{X})$ such that $\mathsf{N}(AT) \subseteq \mathsf{N}(T)$ and $\mathsf{N}(T)$ be a complemented subspace of X and let X_1be a topological complement of $\mathsf{N}(T)$. If*

(1) *$A(\mathsf{X}_1) \subseteq \mathsf{X}_1$ and $T(\mathsf{X}_1) \subseteq \mathsf{X}_1$,*

(2) *$T_1 = T_{|\mathsf{X}_1} \in \Omega_1^\ell$ and $A_1 = A_{|\mathsf{X}_1} \in \Omega_n^\ell$, where $n \in \mathbb{N}$,*

(3) *$T_1 A_1 = A_1 T_1$,*

then $AT \in \Omega_1^\ell$.

Proof.
Since $T_1 \in \Omega_1^\ell$, $A_1 \in \Omega_n^\ell$, $A_1 T_1 = T_1 A_1$ and $\mathsf{N}(T_1) = \{0\}$, then by Theorem 3.6, $A_1 T_1 \in \Omega_n^\ell$ in $\mathscr{B}(\mathsf{X}_1)$. Now, we have $(AT)_{|\mathsf{X}_1} = A_1 T_1$, $\mathsf{N}(AT) = \mathsf{N}(T)$ is complemented and

$$AT(\mathsf{X}_1) \subseteq A(\mathsf{X}_1) \subseteq \mathsf{X}_1,$$

where X_1 is a topological complement of $\mathsf{N}(AT)$. Then according to [7, Proposition 8.1], we get $AT \in \Omega_1^\ell$. □

A similar result in the case of right pseudo-generalized invertibility can be also obtained.

Proposition 5.4. *Let, $A, T \in \mathscr{B}(\mathsf{X})$ such that $A(\mathsf{R}(T)) = \mathsf{R}(T)$ and $\mathsf{R}(T)$ be complemented in X and let X_2 be a topological complement of $\mathsf{R}(T)$. If*

(1) *$\mathsf{X}_2 \subseteq \mathsf{N}(T)$,*

(2) *$T_2 = T_{|\mathsf{R}(T)} \in \Omega_1^r$ and $A_2 = A_{|\mathsf{R}(T)} \in \Omega_n^r$, where $n \in \mathbb{N}$,*

(3) *$T_2 A_2 = A_2 T_2$,*

then $AT \in \Omega_1^r$.

Proof.
Since $\mathsf{X}_2 \subseteq \mathsf{N}(T)$, it follows that

$$\mathsf{R}(T) = T(\mathsf{R}(T) \dotplus \mathsf{X}_2) = T(\mathsf{R}(T)) = \mathsf{R}(T_2).$$

So T_2 is onto and hence $\{0\}$ is the topological complement of $\mathsf{R}(T_2)$ in $\mathsf{R}(T)$. Therefore Theorem 3.8 allows us to deduce that $A_2T_2 \in \Omega_n^r$. Now, since $A(\mathsf{R}(T)) = \mathsf{R}(T)$, we get $(AT)_{|\mathsf{R}(T)} = A_2T_2 \in \Omega_n^r$. As X_2 is a topological complement of $\mathsf{R}(T)$ and $\mathsf{X}_2 \subseteq \mathsf{N}(T) \subseteq \mathsf{N}(AT)$, then by [7, Proposition 8.2], we deduce that $AT \in \Omega_1^r$. □

The last result of this section is the following proposition.

Proposition 5.5. *Let $T, S \in \mathscr{B}(\mathsf{X})$.*

1) *If $T, S \in \Omega^\ell$ such that $\alpha(T) < +\infty$ and $\alpha(S) < +\infty$, then $TS \in \Omega_1^\ell$.*

2) *If $T, S \in \Omega^r$ such that $\beta(T) < +\infty$ and $\beta(S) < +\infty$, then $TS \in \Omega_1^r$.*

Proof.

1) By [7, Proposition 4.1], we have $T, S \in \Omega_1^\ell$. Therefore, from [2, Theorem 4, P. 137] we deduce that $TS \in \Omega_1^\ell$.

2) Using the same argument, we get the stated result. □

6. Results on pseudo-generalized inverses

In this section for a left pseudo-generalized invertible operator T, we discuss in some special cases the relationship between ${}^T\mathsf{S}_n^\ell$ and ${}^T\mathsf{S}_k^\ell$. We start our investigation by establishing the following lemma, which will be used in subsequent proofs.

Lemma 6.1. *Let $n \in \mathbb{N}\backslash\{0, 1\}$, $T \in \Omega_1^\ell$ $(= \Omega_1^r)$.*

1) *If there exist $S \in {}^T\mathsf{S}_1^r$, $L \in {}^T\mathsf{S}_n^\ell$ and $F \in \mathscr{B}(\mathsf{X})$, such that $S(\mathsf{R}(F)) \subseteq \mathsf{N}(T^{n-1})$ with*

$$T = TLT + F,$$

then $L \in {}^T\mathsf{S}_{n-1}^\ell$.

2) *If there exist $S \in {}^T\mathsf{S}_1^r$, $L \in {}^T\mathsf{S}_n^r$ and $F \in \mathscr{B}(\mathsf{X})$, such that $S(\mathsf{R}(T^{n-1})) \subseteq \mathsf{N}(F)$ with*

$$T = TLT + F,$$

then $L \in {}^T\mathsf{S}_{n-1}^r$.

Proof.
1) Since $T = TLT + F$, it follows that

$$T^{n-1}STST = T^{n-1}STLTST + T^{n-1}SFST.$$

Now, as $TST = T$ and $S(\mathsf{R}(F)) \subseteq \mathsf{N}(T^{n-1})$, we deduce that

$$T^{n-1} = T^{n-1}STLT = T^{n-1}LT.$$

Hence $L \in {}^T\mathsf{S}_{n-1}^\ell$.
2) This assertion can be proved in the same way as 1). □

Now, we state the following proposition, giving a more general result.

Proposition 6.2. *Let $n \in \mathbb{N}\backslash\{0,1\}$, $T \in \Omega_1^\ell$ and $L \in {}^T\mathsf{S}_n^\ell$. If there exist $S \in {}^T\mathsf{S}_1^\ell$ and $F \in \mathscr{B}(\mathsf{X})$, such that $S(\mathsf{R}(F)) \subseteq \mathsf{N}(T^{n-k})$, for some $1 \le k \le n-1$ with $T = TLT + F$, then $L \in {}^T\mathsf{S}_{n-k}^\ell$.*

Proof.
• The case $n = 2$ is obvious.
• If $n > 2$. As $S(\mathsf{R}(F)) \subseteq \mathsf{N}(T^{n-k})$, we deduce that $S(\mathsf{R}(F)) \subseteq \mathsf{N}(T^{n-1})$. So, according to Lemma 6.1, we conclude that $L \in {}^T\mathsf{S}_{n-1}^\ell$. Now, if $n > 2$ and $k > 1$, taking into account that $L \in {}^T\mathsf{S}_{n-1}^\ell$ and $S(\mathsf{R}(F)) \subseteq \mathsf{N}(T^{n-k}) \subseteq \mathsf{N}(T^{n-2})$, by Lemma 6.1 we obtain that $L \in {}^T\mathsf{S}_{n-2}^\ell$. Finally, we repeat the same process k times, hence we see that $L \in {}^T\mathsf{S}_{n-k}^\ell$. □

In the same way, we obtain for right pseudo-generalized inverses :

Proposition 6.3. *Let $n \in \mathbb{N}\backslash\{0,1\}$, $T \in \Omega_1^r$ and $L \in {}^T\mathsf{S}_n^r$. If there exist $S \in {}^T\mathsf{S}_1^\ell$ and $F \in \mathscr{B}(\mathsf{X})$, such that $S(\mathsf{R}(T^{n-k})) \subseteq \mathsf{N}(F)$, for some $1 \le k \le n-1$ and $T = TLT + F$, then $L \in {}^T\mathsf{S}_{n-k}^r$.*

Next, some particular cases where the subsets of pseudo-generalized inverses coincide are given. First, we state the following lemma which we use to prove Propositions 6.5 and 6.6.

Lemma 6.4. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \mathscr{B}(\mathsf{X})$.*

1) *If $T \in \Omega_n^\ell$ and there exists $S \in {}^T\mathsf{S}_n^\ell$ such that $\mathsf{R}(S) \subseteq \mathsf{N}(I-T)$, then $T^n = T^k$, for all $k \ge n$.*

2) *If $T \in \Omega_n^r$ and there exists $S \in {}^T\mathsf{S}_n^r$ such that $\mathsf{R}(I-T) \subseteq \mathsf{N}(S)$, then $T^n = T^k$, for all $k \ge n$.*

Proof.
1) From $\mathsf{R}(S) \subseteq \mathsf{N}(I-T)$, it follows that $S = TS$. Therefore,

$$T^n ST = T^{n+1} ST.$$

Since $S \in {}^T\mathsf{S}_n^\ell$, we deduce that $T^n = T^{n+1}$. Consequently, $T^n = T^k$, for all $k \ge n$.
2) Can be proven using the same argument. □

Proposition 6.5. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \Omega_n^\ell$. If there exist $\lambda \in \mathbb{C}^*$ and $S \in {}^T\mathsf{S}_n^\ell$ such that $\mathsf{R}(S) \subseteq \mathsf{N}(\lambda - T)$, then*

$$T^n = \lambda^{n-k} T^k, \ \forall\, k \ge n$$

and

$${}^T\mathsf{S}_n^\ell = {}^T\mathsf{S}_k^\ell, \ \forall\, k \ge n.$$

Proof.
First, we know that

$$T \in \Omega_n^\ell \Longleftrightarrow \lambda T \in \Omega_n^\ell, \forall \lambda \in \mathbb{C}^*.$$

Hence, we have $\frac{T}{\lambda} \in \Omega_n^\ell$. Also, by hypothesis, we see that $\lambda S \in {}^{\frac{T}{\lambda}}\mathsf{S}_n^\ell$. Additionally, as $\mathsf{R}(S) = \mathsf{R}(\lambda S)$, we have

$$\mathsf{R}(\lambda S) \subseteq \mathsf{N}\Big(I - \frac{T}{\lambda}\Big).$$

Therefore, by Lemma 6.4, we deduce that

$$\Big(\frac{T}{\lambda}\Big)^n = \Big(\frac{T}{\lambda}\Big)^k, \ \forall k \ge n$$

which implies that

$$T^n = \lambda^{n-k} T^k, \ \forall k \ge n.$$

Now let $L \in {}^T\mathsf{S}_k^\ell$, where $k \geq n$. As $T^kLT = T^k$ and $T^n = \lambda^{n-k}T^k$, we see that

$$\lambda^{n-k}T^kLT = \lambda^{n-k}T^k.$$

Consequently,

$$T^nLT = T^n.$$

□

For right pseudo-generalized inverses, as in Proposition 6.5, we can prove the following result.

Proposition 6.6. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \Omega_n^r$. If there exist $\lambda \in \mathbb{C}^*$ and $S \in {}^T\mathsf{S}_n^r$ such that $\mathsf{R}(\lambda - T) \subseteq \mathsf{N}(S)$, then*

$$T^n = \lambda^{n-k}T^k, \ \forall\, k \geq n$$

and

$${}^T\mathsf{S}_n^r = {}^T\mathsf{S}_k^r, \ \forall\, k \geq n.$$

The next Lemmas will be used to prove Proposition 6.9.

Lemma 6.7. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \Omega_n^\ell$. If $I - T \in \Omega_1^\ell$, and ${}^T\mathsf{S}_n^\ell \cap {}^{I-T}\mathsf{S}_1^\ell \neq \emptyset$, then*

$${}^T\mathsf{S}_n^\ell = {}^T\mathsf{S}_k^\ell, \ \forall k \geq n.$$

Proof.
Let $L \in {}^T\mathsf{S}_n^\ell \cap {}^{I-T}\mathsf{S}_1^\ell$, then

$$I - T = (I-T)L(I-T) = L - TL - LT + TLT.$$

This implies that

$$T = I - L + TL + LT - TLT.$$

Therefore,

$$\begin{array}{rcl} T^{n+1} & = & T^n - T^nL + T^{n+1}L + T^nLT - T^{n+1}LT \\ & = & T^n - T^nL + T^{n+1}L + T^n - T^{n+1}. \end{array}$$

Hence,

$$\begin{array}{rcl} 2T^{n+2} & = & 2T^{n+1} - T^nLT + T^{n+1}LT \\ & = & 3T^{n+1} - T^n. \end{array}$$

Let $S \in {}^T\mathsf{S}_{n+1}^\ell$, then $S \in {}^T\mathsf{S}_{n+2}^\ell$. By multiplying this last equality, on the right hand side, by ST, we obtain

$$2T^{n+2} = 3T^{n+1} - T^nST.$$

Consequently, clearly $T^n = T^nST$ and $S \in {}^T\mathsf{S}_n^\ell$. Now, suppose that ${}^T\mathsf{S}_n^\ell = {}^T\mathsf{S}_k^\ell$, we will show that ${}^T\mathsf{S}_n^\ell = {}^T\mathsf{S}_{k+1}^\ell$, for all $k \geq n$.
First, we know that ${}^T\mathsf{S}_n^\ell \subseteq {}^T\mathsf{S}_{k+1}^\ell$. Let $S \in {}^T\mathsf{S}_{k+1}^\ell$, we have

$$2T^{k+2} = 3T^{k+1} - T^k.$$

Therefore,

$$2T^{k+2} = 3T^{k+1} - T^kST,$$

and so $T^k = T^kST$. This implies that $S \in {}^T\mathsf{S}_k^\ell = {}^T\mathsf{S}_n^\ell$. The proof is completed. □

Using the same argument as the previous lemma, we obtain :

Lemma 6.8. *Let $n \in \mathbb{N}\backslash\{0\}$ and $T \in \Omega_n^\ell$. If $I - T \in \Omega_1^r$ and ${}^T\mathsf{S}_n^r \cap {}^{I-T}\mathsf{S}_1^r \neq \emptyset$, then*

$${}^T\mathsf{S}_n^r = {}^T\mathsf{S}_k^r, \ \forall k \geq n.$$

As a final result, we give the following proposition.

Proposition 6.9. *Let* $n \in \mathbb{N}\backslash\{0\}$.

1) *If* $T \in \Omega_n^\ell$ *and there exists* $\lambda \in \mathbb{C}^*$ *such that* $\lambda - T \in \Omega_1^\ell$ *and if* ${}^T\mathsf{S}_n^\ell \cap {}^{\lambda-T}\mathsf{S}_1^\ell \neq \emptyset$, *then*

$$ {}^T\mathsf{S}_n^\ell = {}^T\mathsf{S}_k^\ell,\ \forall\, k \geq n. $$

2) *If* $T \in \Omega_n^r$ *and there exists* $\lambda \in \mathbb{C}^*$ *such that* $\lambda - T \in \Omega_1^r$ *and if* ${}^T\mathsf{S}_n^r \cap {}^{\lambda-T}\mathsf{S}_1^r \neq \emptyset$, *then*

$$ {}^T\mathsf{S}_n^r = {}^T\mathsf{S}_k^r,\ \forall\, k \geq n. $$

Proof.
1) First, since $T \in \Omega_n^\ell$ and $\lambda - T \in \Omega_1^\ell$, it follows that

$$ \frac{T}{\lambda} \in \Omega_n^\ell \text{ and } I - \frac{T}{\lambda} \in \Omega_1^\ell. $$

In addition, if we have ${}^T\mathsf{S}_n^\ell \cap {}^{\lambda-T}\mathsf{S}_1^\ell \neq \emptyset$, then there exists $S \in {}^T\mathsf{S}_n^\ell \cap {}^{\lambda-T}\mathsf{S}_1^\ell$. So $\lambda S \in {}^{\frac{T}{\lambda}}\mathsf{S}_n^\ell \cap {}^{I-\frac{T}{\lambda}}\mathsf{S}_1^\ell$. It follows that

$$ {}^{\frac{T}{\lambda}}\mathsf{S}_n^\ell \cap {}^{I-\frac{T}{\lambda}}\mathsf{S}_1^\ell \neq \emptyset $$

and hence, by Lemma 6.7,

$$ {}^{\frac{T}{\lambda}}\mathsf{S}_n^\ell = {}^{\frac{T}{\lambda}}\mathsf{S}_k^\ell,\ \forall\, k \geq n. $$

Now, let $L \in {}^T\mathsf{S}_k^\ell$, then $\lambda L \in {}^{\frac{T}{\lambda}}\mathsf{S}_k^\ell$ and therefore $\lambda L \in {}^{\frac{T}{\lambda}}\mathsf{S}_n^\ell$. Consequently, $L \in {}^T\mathsf{S}_n^\ell$.
2) This assertion can be proved in the same way as 1). □

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