Filomat 36:13 (2022), 4515–4523
https://doi.org/10.2298/FIL2213515G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# On the Regularized Trace of a Differential Operator of Sturm-Liouville Type

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**Abstract.** In this work, we study a spectral problem for the abstract Sturm-Liouville operator with a bounded operator coefficient $V(t)$ and with periodic boundary conditions on the interval $[0, \pi]$, and we present a regularized trace formula for this operator.

## 1. Introduction

An ordinary differential operator may not have a finite trace on infinite dimensional spaces. For this reason, the idea of a regularized trace formula is brought to light by Gelfand and Levitan in [8], where two scaler differential operators of Sturm-Liouville type are considered and a formula for the sum of the differences of the eigenvalues of these operators is obtained. Later, in besides these operators, many regularized trace formulas for different differential operators are found (see [1, 6, 7, 17]).
A similar regularized trace formula to the case of scaler differential operators is obtained for a differential operator with operator-valued coefficient in [14]. For a list of regularized trace formulas for differential operators with operator coefficient, we can refer the articles [2, 4, 5, 12, 13, 16]. The trace formulas for differential operators are used in inverse problems, [19] and in computing of the first eigenvalue of the related operator, [18]. Here, we study the regularized trace of the Sturm-Liouville operator with bounded operator coefficient and with periodic boundary conditions on $[0, \pi]$.

Let us give some basic definitions and properties to motivate our problem:
Let $\mathcal{H}$ be a separable Hilbert space. Consider a linear operator $A$ whose domain $D(A)$ is dense in $\mathcal{H}$ ($A$ is called densely defined operator in $\mathcal{H}$) and the inner product $(Au, v)$ for a given fix $v$ and every $u$ in $D(A)$. Then there exists an element $v^*$ for which

$$(Au, v) = (u, v^*)$$

holds. In the representation $v^* = A^*v$, $A^*$ is called the adjoint operator of $A$. By assumption, since $D(A)$ is dense in $\mathcal{H}$ the element $v^*$ is uniquely determined by the element $v$. We readily observe the following properties.

*(a)* $A^*$ is a linear operator.

2020 *Mathematics Subject Classification*. 47A10; 34L20
*Keywords*. Hilbert space, Trace-class operator, Spectrum, Regularized trace.
Received: 28 June 2021; Accepted: 09 August 2022
Communicated by Dragan S. Djordjević
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(b) If $A \subseteq B$, then $B^* \subseteq A^*$.
(c) The operator $A^*$ is closed even when $A$ is not closed.
(d) If $A$ has a closure $\overline{A}$, then $(\overline{A})^* = A^*$.
(e) If the operator $A^{**}$ exists, then $A \subseteq A^{**}$.

A linear operator $A$ defined on $D(A)$ is called symmetric, if for every $u, v \in D(A)$, the equality

$$(Au, v) = (u, Av)$$

holds. A symmetric operator $A$ has an extension as $A^* \supseteq A$.
A linear operator $A$ defined on $D(A)$ is called self-adjoint, if $A = A^*$. From this definition we see that every self-adjoint operator is symmetric. For the bounded operators these two notions are equivalent.
A number $\lambda$ is a regular point of an operator $A$ if the operator $R_\lambda = (A - \lambda I)^{-1}$ exists, is defined on $\mathcal{H}$ and is bounded. The spectrum of $A$ is set of all non-regular points and its discrete spectrum consists of the set of all eigenvalues. Especially, the discrete spectrum of a self-adjoint operator is a countable set of real numbers.
$\sigma_1(\mathcal{H})$ is the set of all compact operators $A$ defined on $\mathcal{H}$ satisfying the condition $\sum\limits_{k=1}^{\infty} s_k(A) < \infty$ where $s_k(A)$ $(k = 1, 2, \cdots)$ are the s-numbers of $A$. If $A \in \sigma_1(\mathcal{H})$, then it is called a trace-class or kernel operator.
If $A$ is a trace-class operator and $\{e_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is any orthonormal basis, then the series $\sum\limits_{j=1}^{\infty} \left(Ae_k, e_k\right)$ is convergent and the sum of the series does not depend on the choice of the basis $\{e_k\}_{k=1}^{\infty}$. The sum of this series is said to be matrix trace of the operator $A$ denoted by $trA$. The reader can find more details about the theory in the books [3, 9, 15].

Now, let us introduce our problem. Denote the inner product and the norm in $\mathcal{H}$ by $(.,.)$ and $\|.\|$, respectively. Let $\mathcal{H}_1 = L_2([0, \pi]; \mathcal{H})$ represent the set of all measurable functions $f$ defined on $[0, \pi]$ with their values in $\mathcal{H}$ such that, for every $g \in \mathcal{H}$, the scalar function $(f(t), g)$ is measurable in the finite interval $[0, \pi]$ and

$$\int_0^{\pi} \|f(t)\|^2 dx < \infty.$$

Define the operator $L$ on $\mathcal{H}_1$ by:

$$L(y) = -y''(t) + V(t)y(t); \qquad y(0) - y(\pi) = y'(0) - y'(\pi) = 0.$$

Assume that the operator function $V(t)$ has the following properties:

1. $V(t) \in C^2[0, \pi]$ (in the weak sense) and $V(t)$, $V'(t)$ and $V''(t)$ are self-adjoint, trace-class operators from $\mathcal{H}$ to $\mathcal{H}$.
2. $\|V\|_{\mathcal{H}_1} < 2$.
3. $\mathcal{H}_1$ has an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \left\|V\phi_n\right\|_{\mathcal{H}_1} < \infty$.
4. $\|V(t)\|_{\sigma_1(\mathcal{H})}$, $\|V'(t)\|_{\sigma_1(\mathcal{H})}$, $\|V''(t)\|_{\sigma_1(\mathcal{H})}$ are bounded and measurable functions on $[0, \pi]$.

Let $L_0$ denote the operator $L$ with $V(t) = 0$. Clearly $L_0$ is a self-adjoint linear operator on $\mathcal{H}_1$. The spectrum $\sigma(L_0)$ of $L_0$ is the set $\{(2m)^2\}_{m=0}^{\infty}$ and each $\lambda \in \sigma(L_0)$ is an eigenvalue of infinite multiplicity. The corresponding eigenfunctions to these eigenvalues are of the form:

$$\begin{aligned} \psi_{mn}^{(1)} &= \delta_m \cos 2mt \cdot \phi_n\,, \qquad m = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots \\ \psi_{mn}^{(2)} &= \delta_m \sin 2mt \cdot \phi_n\,, \qquad m = 1, 2, 3, \cdots; n = 1, 2, 3, \cdots \end{aligned} \tag{1.1}$$

where

$$\delta_m = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } m = 0 \\ \sqrt{\frac{2}{\pi}} & \text{if } m = 1, 2, \ldots. \end{cases}$$

## 2. Some relations on resolvents

Let $R^0_\lambda = (L_0 - \lambda I)^{-1}$ and $R_\lambda = (L - \lambda I)^{-1}$ be the resolvents of the operators $L_0$ and $L$, respectively. Note that the system (1.1) is an orthonormal basis for $\mathcal{H}_1$.

**Lemma 2.1.** *If $\lambda \notin \sigma(L_0)$ then $VR^0_\lambda \in \sigma_1(\mathcal{H}_1)$, that is, $VR^0_\lambda : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is a trace-class operator.*

*Proof.* It is enough to prove that

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\|VR^0_\lambda\psi^{(1)}_{mn}\right\|_{\mathcal{H}_1}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\|VR^0_\lambda\psi^{(2)}_{mn}\right\|_{\mathcal{H}_1}$$

is convergent (see Lemma 8.1,[10] ). Set $\mu_m = (2m)^2$. The system (1.1) gives

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\|VR^0_\lambda\psi^{(1)}_{mn}\right\|_{\mathcal{H}_1}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\|VR^0_\lambda\psi^{(2)}_{mn}\right\|_{\mathcal{H}_1}\\
&=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left|\mu_m-\lambda\right|^{-1}\left\|V\psi^{(1)}_{mn}\right\|_{\mathcal{H}_1}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\mu_m-\lambda\right|^{-1}\left\|V\psi^{(2)}_{mn}\right\|_{\mathcal{H}_1}\\
&=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left|\mu_m-\lambda\right|^{-1}\left[\delta_m^2\int_0^\pi\cos^2(2m)t\left\|V(t)\phi_n\right\|^2dt\right]^{1/2}\\
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\mu_m-\lambda\right|^{-1}\left[2/\pi\int_0^\pi\sin^2(2m)t\left\|V(t)\phi_n\right\|^2dt\right]^{1/2}\\
&=\sum_{n=1}^{\infty}|1-\lambda|^{-1}\left[1/\pi\int_0^\pi\left\|V(t)\phi_n\right\|^2dt\right]^{1/2}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\mu_m-\lambda\right|^{-1}\left[2/\pi\int_0^\pi\left\|V(t)\phi_n\right\|^2dt\right]^{1/2}\\
&\le|1-\lambda|^{-1}\sum_{n=1}^{\infty}\left\|V\phi_n\right\|_{\mathcal{H}_1}+\sum_{m=1}^{\infty}\left|\mu_m-\lambda\right|^{-1}\sum_{n=1}^{\infty}\left\|V\phi_n\right\|_{\mathcal{H}_1}<\infty.
\end{aligned}$$

Thus, the lemma follows. □

Using this lemma with conditions the (2) and (3) on $V(t)$, we can show that the spectrum $\sigma(L)$ of $L$ is a subset of the union of pairwise disjoint intervals $F_m = [\mu_m - \|V\|_{\mathcal{H}_1}, \mu_m + \|V\|_{\mathcal{H}_1}]$ ($m = 0, 1, 2, ...$) (see [11]). Each point of $\sigma(L)$ which is not equal to $\mu_m = (2m)^2$ is an isolated eigenvalue of finite multiplicity. However, $\mu_m$ itself is the possible eigenvalue of $L$ with either finite or infinite multiplicity. Moreover,

$$\lim_{n\to\infty}\lambda_{mn}=\mu_m$$

where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of $L$ in the interval $F_m$.

**Lemma 2.2.** *The operator-valued function $R_\lambda - R^0_\lambda$ is analytic in the resolvent set $\rho(L)$ of $L$ with respect to the $\sigma_1(\mathcal{H}_1)$ norm.*

*Proof.* Since $R^0_\lambda$ is a trace class operator and $R^0_{\lambda+\Delta\lambda} - R^0_\lambda = \Delta\lambda R^0_{\lambda+\Delta\lambda}R^0_\lambda$, we get:

$$D=\left\|\frac{VR^0_{\lambda+\Delta\lambda}-VR^0_\lambda}{\Delta\lambda}-V(R^0_\lambda)^2\right\|_{\sigma_1(\mathcal{H}_1)}\le\left\|VR^0_\lambda\right\|_{\sigma_1(\mathcal{H}_1)}\left\|R^0_{\lambda+\Delta\lambda}-R^0_\lambda\right\|_{\mathcal{H}_1}.$$

Thus $D \to 0$ as $\Delta\lambda \to 0$ and $\frac{d}{d\lambda}(VR^0_\lambda) = V(R^0_\lambda)^2$. This proves lemma. □

Let $\{\{\Psi^{(1)}_{mn}(t)\}^{\infty}_{m=0,n=1}, \{\Psi^{(2)}_{mn}(t)\}^{\infty}_{m=1,n=1}\}$ be orthonormal eigenfunctions corresponding to eigenvalues $\{\{\lambda^{(1)}_{mn}\}^{\infty}_{m=0,n=1}, \{\lambda^{(2)}_{mn}\}^{\infty}_{m=1,n=1}\}$ of $L$. Since the spectra of the operators $L_0$ and $L$ only consist of their eigenvalues and limit points, we have:

$$R^0_\lambda = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{B^{(1)}_{mn}}{\mu_m-\lambda}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B^{(2)}_{mn}}{\mu_m-\lambda},$$

$$R_\lambda = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\mathbf{B}^{(1)}_{mn}}{\lambda^{(1)}_{mn}-\lambda}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mathbf{B}^{(2)}_{mn}}{\lambda^{(2)}_{mn}-\lambda}$$

where

$$B^{(i)}_{mn} = (.,\psi^{(i)}_{mn})_{\mathcal{H}_1}\psi^{(i)}_{mn} \quad \text{and} \quad \mathbf{B}^{(i)}_{mn} = (.,\Psi^{(i)}_{mn})_{\mathcal{H}_1}\Psi^{(i)}_{mn} \quad (i=1,2).$$

In view of Lemma 2.2 and the last equalities above, it follows that for $i = 1,2$ the series $\sum\limits_{n=1}^{\infty}(\lambda^{(i)}_{pn}-\mu_p)$ is absolutely convergent. In this case, since $R_\lambda - R^0_\lambda \in \sigma_1[\mathcal{H}_1]$, for every $\lambda \in \rho(L)$ we get:

$$tr(R_\lambda - R^0_\lambda) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{1}{\lambda^{(1)}_{mn}-\lambda}-\frac{1}{\mu_m-\lambda}\Big)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{1}{\lambda^{(2)}_{mn}-\lambda}-\frac{1}{\mu_m-\lambda}\Big).$$

Multiplying both sides of the last equality by $\lambda/2\pi i$ and integrating over the circle $|\lambda| = b_p = \mu_p + 2p$ ($p = 1,2,\dots$) we find:

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\, tr(R_\lambda - R^0_\lambda)d\lambda = \sum_{m=0}^{p}\sum_{n=1}^{\infty}(\mu_m-\lambda^{(1)}_{mn})+\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\mu_m-\lambda^{(2)}_{mn}).$$

This relation can be rewritten in the form

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\mu_m-\lambda^{(1)}_{mn})+\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\mu_m-\lambda^{(2)}_{mn}) = \sum_{j=1}^{N}S^p_j + S_{pN} \tag{2.1}$$

where

$$S^p_j = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda\, tr\Big[R^0_\lambda(VR^0_\lambda)^j\Big]d\lambda \tag{2.2}$$

and

$$S_{pN} = \frac{(-1)^{N}}{2\pi i}\int_{|\lambda|=b_p}\lambda\, tr\Big[R_\lambda(VR^0_\lambda)^{N+1}\Big]\,d\lambda. \tag{2.3}$$

for $N \in \mathbf{N}$. Now, it is not difficult to see that

$$S^p_j = \frac{(-1)^{j}}{2\pi ij}\int_{|\lambda|=b_p} tr(VR^0_\lambda)^j d\lambda. \tag{2.4}$$

## 3. Finding of the regularized trace formula

We state and prove the regularized trace formula of the Sturm-Liouville operator $L$ in the following theorem. For simplicity, we will use the notation $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ for the inner product and the norm on $\mathcal{H}_1$ respectively.

**Theorem 3.1.** *If the conditions (1)-(4) on $V(t)$ are fulfilled, then we have*

$$\sum_{m=0}^{\infty}\Big[\sum_{n=1}^{\infty}\Big(\lambda^{(1)}_{mn}-4m^2\Big)-\frac{1}{\pi}\int_0^{\pi}trV(t)\,dt\Big]+\sum_{m=1}^{\infty}\Big[\sum_{n=1}^{\infty}\Big(\lambda^{(2)}_{mn}-4m^2\Big)-\frac{1}{\pi}\int_0^{\pi}trV(t)\,dt\Big] = 0. \tag{3.1}$$

*Proof.* The proof is based on both the system (1.1) and the relation (2.1). It will be done in four steps.

*Step 1.* Let us compute $S_1^p$: By (2.4), we have

$$\begin{aligned} S_1^p &= \frac{-1}{2\pi i}\int_{|\lambda|=b_p} tr(VR_\lambda^0)d\lambda \\ &= \frac{-1}{2\pi i}\int_{|\lambda|=b_p} \Big\{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(VR_\lambda^0\psi_{mn}^{(1)},\psi_{mn}^{(1)}\Big)_1 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(VR_\lambda^0\psi_{mn}^{(2)},\psi_{mn}^{(2)}\Big)_1\Big\}d\lambda. \end{aligned}$$

Since $|\big(VR_\lambda^0\psi_{mn}^{(i)},\psi_{mn}^{(i)}\big)_1| < \sqrt{\pi}\,|\mu_m-\lambda|^{-1}\|V(t)\phi_n\|$ and $V(t)$ satisfies the conditions (2)-(3), the series

$$\Theta_{m,1}(\lambda) = \sum_{n=1}^{\infty}\big(VR_\lambda^0\psi_{mn}^{(1)},\psi_{mn}^{(1)}\big)_1\,;\,\Theta_{m,2}(\lambda) = \sum_{n=1}^{\infty}\big(VR_\lambda^0\psi_{mn}^{(2)},\psi_{mn}^{(2)}\big)_1$$

and

$$\sum_{m=0}^{\infty}\Theta_{m,1}(\lambda)\,;\,\sum_{m=1}^{\infty}\Theta_{m,2}(\lambda)$$

are absolutely and uniformly convergent on the circle $|\lambda| = b_p$. Therefore, we have:

$$\begin{aligned} S_1^p &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big(V\psi_{mn}^{(1)},\psi_{mn}^{(1)}\Big)_1\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{d\lambda-\mu_m} + \sum_{m=1}^{p}\sum_{n=1}^{\infty}\Big(V\psi_{mn}^{(2)},\psi_{mn}^{(2)}\Big)_1\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{d\lambda-\mu_m} \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big(V\psi_{mn}^{(1)},\psi_{mn}^{(1)}\Big)_1 + \sum_{m=1}^{p}\sum_{n=1}^{\infty}\Big(V\psi_{mn}^{(2)},\psi_{mn}^{(2)}\Big)_1 \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}\Big(V(t)\phi_n,\phi_n\Big)dt + \frac{2}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}\Big(V(t)\phi_n,\phi_n\Big)dt \\ &= \frac{1}{\pi}\int_0^{\pi} trV(t)dt + \frac{2p}{\pi}\int_0^{\pi} trV(t)dt. \end{aligned}$$

Hence, we get:

$$S_1^p = \frac{2p+1}{\pi}\int_0^{\pi} trV(t)dt. \tag{3.2}$$

*Step 2.* Let us compute $S_2^p$ and $S_3^p$ : For $j = 2$, we have

$$\begin{aligned} S_2^p &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} tr(VR_\lambda^0)^2 d\lambda \\ &= \frac{1}{4\pi i}\int_{|\lambda|=b_p}\Big\{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big((VR_\lambda^0)^2\psi_{mn}^{(1)},\psi_{mn}^{(1)}\Big)_1 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big((VR_\lambda^0)^2\psi_{mn}^{(2)},\psi_{mn}^{(2)}\Big)_1\Big\}d\lambda \\ &= \frac{1}{4\pi i}\int_{|\lambda|=b_p}\Big\{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{\Big(V\psi_{mn}^{(1)},\psi_{kl}^{(1)}\Big)_1\Big(V\psi_{kl}^{(1)},\psi_{mn}^{(1)}\Big)_1}{(d\lambda-\mu_m)(d\lambda-\mu_k)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\Big(V\psi_{mn}^{(2)},\psi_{kl}^{(2)}\Big)_1\Big(V\psi_{kl}^{(2)},\psi_{mn}^{(2)}\Big)_1}{(d\lambda-\mu_m)(d\lambda-\mu_k)}\Big\}d\lambda. \end{aligned}$$

Moreover, for $m,k \le p$, we have:

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{(d\lambda-\mu_m)(d\lambda-\mu_k)} = 0. \tag{3.3}$$

This equality is also true for $m, k \geq p$. So, $S_2^p$ becomes

$$S_2^p = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{k=p+1}^{\infty}\sum_{l=1}^{\infty}\left(V\psi_{mn}^{(1)}, \psi_{kl}^{(1)}\right)_1\left(V\psi_{kl}^{(1)}, \psi_{mn}^{(1)}\right)_1\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{(d\lambda-\mu_m)(d\lambda-\mu_k)}$$
$$+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{k=p+1}^{\infty}\sum_{l=1}^{\infty}\left(V\psi_{mn}^{(2)}, \psi_{kl}^{(2)}\right)_1\left(V\psi_{kl}^{(2)}, \psi_{mn}^{(2)}\right)_1\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{(d\lambda-\mu_m)(d\lambda-\mu_k)}$$

or

$$S_2^p = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{k=p+1}^{\infty}\sum_{l=1}^{\infty}(\mu_m-\mu_k)^{-1}\left|\left(V\psi_{mn}^{(1)}, \psi_{kl}^{(1)}\right)_1\right|^2+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{k=p+1}^{\infty}\sum_{l=1}^{\infty}(\mu_m-\mu_k)^{-1}\left|\left(V\psi_{mn}^{(2)}, \psi_{kl}^{(2)}\right)_1\right|^2.$$

The last discussion gives:

$$|S_2^p| \leq \sum_{k=p+1}^{\infty}(\mu_k-\mu_p)^{-1}\sum_{l=1}^{\infty}\left[\|V\psi_{kl}^{(1)}\|_1^2+\|V\psi_{kl}^{(2)}\|_1^2\right].$$

or

$$|S_2^p| < const\sum_{k=p+1}^{\infty}(\mu_k-\mu_p)^{-1}$$

because for $i = 1, 2$

$$\sum_{l=1}^{\infty}\|V\psi_{kl}^{(i)}\|_1^2 \leq \sum_{l=1}^{\infty}\|V\phi_l\|_1^2 < const.$$

Here, it can be seen that

$$\sum_{k=p+1}^{\infty}(\mu_k-\mu_p)^{-1} = \sum_{k=p+1}^{\infty}(4k^2-4p^2)^{-1} < cp^{-1/2}\ (c > 0). \tag{3.4}$$

Therefore, we obtain:

$$\lim_{p\to\infty} S_2^p = 0.$$

In a similar manner we can show that

$$\lim_{p\to\infty} S_3^p = 0.$$

*Step 3.* Let us prove that

$$\lim_{p\to\infty} S_j^p = 0$$

for $j \geq 4$. First note that, we have:

$$\begin{aligned}\left\|VR_\lambda^0\right\|_{\sigma_1(\mathcal{H}_1)} &\leq \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\|VR_\lambda^0\psi_{mn}^{(1)}\right\|_1+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left\|VR_\lambda^0\psi_{mn}^{(2)}\right\|_1\\ &< c_3\sum_{m=0}^{\infty}|\mu_m-\lambda|^{-1}\ (c_3 > 0)\end{aligned} \tag{3.5}$$

with

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\|VR_{\lambda}^{0}\psi_{mn}^{(1)}\right\|_{1}\leq c_{1}\sum_{m=0}^{\infty}|\mu_{m}-\lambda|^{-1}\ (c_{1}>0),$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\|VR_{\lambda}^{0}\psi_{mn}^{(2)}\right\|_{1}\leq c_{2}\sum_{m=1}^{\infty}|\mu_{m}-\lambda|^{-1}\ (c_{2}>0).$$

Now, we claim that the right hand side in the last series of inequality (3.5) is finite. In fact, we have

$$\begin{aligned}\sum_{m=0}^{\infty}|\mu_{m}-\lambda|^{-1} &= \sum_{m=0}^{p}|\mu_{m}-\lambda|^{-1}+\sum_{m=p+1}^{\infty}|\mu_{m}-\lambda|^{-1}\\ &\leq \sum_{m=0}^{p}(|\lambda|-\mu_{m})^{-1}+\sum_{m=p+1}^{\infty}(\mu_{m}-|\lambda|)^{-1}\\ &\leq \sum_{m=0}^{p}\frac{1}{2}p^{-1}+\sum_{m=p+1}^{\infty}(\mu_{m}-\mu_{p}-2p)^{-1}\\ &\leq 1+\sum_{m=p+1}^{\infty}\left[\frac{1}{2}(\mu_{m}-\mu_{p})+\frac{1}{2}(\mu_{p+1}-\mu_{p})-2p\right]^{-1}\\ &\leq 1+\sum_{m=p+1}^{\infty}\frac{2}{\mu_{m}-\mu_{p}}.\end{aligned}$$

Thus by inequality (3.4) we get:

$$\sum_{m=0}^{\infty}|\mu_{m}-\lambda|^{-1}<c_{4}\ \ (c_{4}>0)$$

This gives

$$\left\|VR_{\lambda}^{0}\right\|_{\sigma_{1}(\mathcal{H}_{1})}<c_{5}\ (c_{5}>0)\,;\ |\lambda|=b_{p}=\mu_{p}+2p. \tag{3.6}$$

To complete this step we need to estimate $\left\|R_{\lambda}^{0}\right\|_{1}$ and $\|R_{\lambda}\|_{1}$ on the circle $|\lambda|=b_{p}=\mu_{p}+2p$. For $m\leq p$

$$|\mu_{m}-\lambda|\geq|\lambda|-\mu_{m}\geq 4p^{2}+2p-4m^{2}>p$$

and for $m\geq p+1$

$$|\mu_{m}-\lambda|\geq\mu_{m}-|\lambda|\geq 4(p+1)^{2}-4p^{2}-2p>p.$$

Hence we have:

$$|\mu_{m}-\lambda|^{-1}\leq p^{-1}\ ;\ |\lambda|=b_{p}=\mu_{p}+2p.$$

This implies

$$\left\|R_{\lambda}^{0}\right\|_{1}<p^{-1}. \tag{3.7}$$

Similarly, for sufficiently large $p$ we get:

$$\|R_{\lambda}\|_{1}<c_{6}p^{-1}\ (c_{6}>0)\,;\ |\lambda|=b_{p}=\mu_{p}+2p. \tag{3.8}$$

Hence

$$
\begin{aligned}
|S_j^p| &= \frac{1}{2\pi j}\Big|\int_{|\lambda|=b_p} tr(VR_\lambda^0)^j\, d\lambda\Big| \\
&\le \frac{1}{2\pi j}\int_{|\lambda|=b_p} \|VR_\lambda^0\|_1 \|(VR_\lambda^0)^{j-1}\|_1 |d\lambda| \\
&\le \frac{c_5}{2\pi j}\int_{|\lambda|=b_p} \|V\|_1^{j-1}\|R_\lambda^0\|_1^{j-1}|d\lambda| \quad \text{(by (3.6))} \\
&\le \frac{c_5}{2\pi j}\int_{|\lambda|=b_p} 2^{j-1}p^{1-j}|d\lambda| \qquad \text{(by (3.7))} \\
&\le const\cdot p^{3-j}.
\end{aligned}
$$

This gives:

$$\lim_{p\to\infty} S_j^p = 0 \ \text{for} j \ge 4.$$

*Step 4.* Here we will use some results of the previous step to show that $\lim_{p\to\infty} S_{pN} = 0$ for $N \ge 4$. From (2.3)

$$
\begin{aligned}
|S_{pN}| &= \frac{1}{2\pi}\Big|\int_{|\lambda|=b_p} \lambda\, tr\Big[R_\lambda(VR_\lambda^0)^{N+1}\Big]d\lambda\Big| \\
&\le \frac{1}{2\pi}\int_{|\lambda|=b_p} |\lambda|\Big|tr\Big[R_\lambda(VR_\lambda^0)^{N+1}\Big]\Big||d\lambda| \\
&\le b_p\int_{|\lambda|=b_p} \Big\|R_\lambda(VR_\lambda^0)^{N+1}\Big\|_{\sigma_1(\mathcal{H}_1)}|d\lambda| \\
&\le c_5\, c_6\, 2^N p^{-1-N}\, 2\pi(b_p)^2 \qquad \text{(by (3.6)-(3.8))} \\
&\le const\cdot p^{3-N}.
\end{aligned}
$$

This gives

$$\lim_{p\to\infty} S_{pN} = 0 \ \text{for}\, N \ge 4.$$

Using relation (2.1) and the results of the previous steps we conclude that the regularized trace formula (3.1) holds. □

**Funding:** This research received no external funding.
**Conflicts of Interest:** The authors declare no conflict of interest.

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