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Generalized Finite Difference Method for Solving Two-Interval Sturm-Liouville Problems with Jump Conditions

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Abstract. We consider a Sturm-Liouville problem defined on two disjoint intervals together with additional jump conditions across the common endpoint of these intervals. Based on Finite Difference Method (FDM) we have developed a new tecnique for solving such type nonstandard boundary value problems (BVP). To show applicability and effectiveness of the proposed generalization of FDM, we solved a simple but illustrative example. The obtained numerical solutions are graphically compared with the corresponding exact solutions.

1. Introduction

The advent of finite difference techniques in numerical applications began in the early 1950s and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and technology.

Many important theoretical and numerical results have been obtained during the last seven decades regarding the stability, accuracy and convergence of the FDM for different type initial and/or BVPs (see, [7, 15]).

The standard FDM is intended for solving one-interval initial and/or boundary value problems without jump conditions (see, for example, [1, 3, 6] and references cited therein).

Based on FDM, we have developed a new technique for solving two-interval Sturm-Liouville problems (SLPs), that included additional jump conditions across the common endpoint of these intervals (see, [4, 5, 9–12]).

Two-interval SLPs arise in solving many important problems of physics and engineering, such as heat and mass transfer problems, Earth's free oscillation problems, thermal conduction problems for a thin laminated plate when the plate composed by materials with different physical and chemical characteristics, vibrating string problems when the string loaded with additional point masses etc (see, for example, [2, 8, 13, 14]).

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2. Analysis of the Method

A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. This gives an algebraic system of equations to be solved in place of the differential equation, something that is easily solved on a computer.

Let us consider a linear boundary-value problem for Sturm-Liouville equation

$$y^{''} + p(x)y^{'} + q(x)y = f(x), \quad x \in [a,c) \cup (c,b] \tag{1}$$

subject to the boundary conditions at the endpoints $x = a$ and $x = b$, given by

$$y(a) = \alpha, \quad y(b) = \beta \tag{2}$$

and with additional jump conditions at the interior point of singularity $x = c$, given by

$$y(0-c) = my(0+c), y'(0-c) = ny'(0+c) \tag{3}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous functions on $[a,c) \cup (c,b]$ having finite limit values $p(c \pm 0), q(c \pm 0)$ and $f(c \pm 0)$respectively α, β m,n are real constants. To discretize the problem (1), (3) the definition range $[a,b]$ is divided into N equal ranges $[x_0,x_1], [x_1,x_2], ..., [x_{N-1},x_N]$, where

$$a = x_0 < x_1 < ... < x_N = b, \quad x_i = a + ih, \quad h = \frac{b-a}{N}. \tag{4}$$

By using the Taylor expansion

$$y(x_i + h) = y(x_i) \mp y^{'}(x_i) \mp \frac{h^2}{2} y^{''}(x_i) \mp \ldots.$$

We can express first and second derivative expressions in the boundary value problem as

$$y'(x) \approx \frac{1}{2}\left(D_+y(x) + D_-y(x)\right) \tag{5}$$

and

$$y''(x) \approx \frac{1}{h}\left(D_+y(x) - D_-y(x)\right), \tag{6}$$

where $D_+y(x)$ and $D_-y(x)$ denotes the forward finite difference and backward finite difference of the unknown solution $y(x)$.

Let us define the finite difference solution for $y(x)$ at all grid points $x_0, x_1, \cdots, x_N$ by $y_i = y(x_i)$ and substituting (4) and (5) in the boundary value problem (1)-(2) we have the following linear system of algebraic equations

$$\left(1 - \tfrac{1}{2}hp_i\right) y_{i-1} + \left(-2 + h^2 q_i\right) y_i + \left(1 + \tfrac{1}{2}hp_i\right) y_{i+1} = h^2 f(x_i)$$

$$1 \le i \le N-1, \qquad i = 1,2,3,...,N-1...$$

where

$$y_0 = \alpha, \quad y_N = \beta$$

Note that each equation of this system involves solution values at three nodal points x_{i-1} , x_i and x_{i+1}. The linear system of algebraic equations can be written in the matrix and vector form

$$My = B, \tag{7}$$

where M is a tridioganal matrix given by

$$M = \begin{pmatrix} -2+h^2q_1 & 1+\frac{1}{2}hp_1 & 0 & \cdots & 0 & 0 & 0 \\ 1-\frac{1}{2}hp_2 & -2+h^2q_2 & 1+\frac{1}{2}hp_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & -2+h^2q_{N-2} & 1+\frac{1}{2}hp_{N-2} & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1-\frac{1}{2}hp_{N-1} & -2+h^2q_{N-1} \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix} \quad and \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{pmatrix}.,$$

where

$$b_i = \begin{cases} h^2f_1 - \left(1-\frac{1}{2}hp_1\right), & i=1 \\ h^2f_i, & i=2,3,...,N-2 \\ h^2f_{N-1} - \left(1+\frac{1}{2}hp_{N-1}\right)\beta, & i=N-1 \end{cases}$$

The linear system of algebraic equations (7) is tridiagonal and can be solved efficiently by the Crout or Cholesky algoritm [3].

3. Convergence and Error Estimates of Finite Difference Method

When the FDM is applied to solve a boundary value problem, it is very important to know how accurate the numerical solution is compared to the exact solution.

Let $\tilde{Y} = (y_1, y_2, \ldots, y_n)$ denote the finite difference solution and $\tilde{y} = (y(x_1), y(x_2), \ldots, y(x_n))$ is the exact solution at the grid points $x_1, x_2, \ldots, x_n$. Then the vector

$$\tilde{e} = (y_1 - y(x_1), y_2 - y(x_2), \ldots, y_n - y(x_n)) = \tilde{Y} - \tilde{y} \tag{8}$$

is said to be the global error vector. You usually want to find an admissible upper bound for this error with respect to the infinite norm (so-called maximum norm), defined by $\| \tilde{e} \| = \max_{1\leqslant i\leqslant n} | y_i - y(x_i) |$ or p-norm ($p \geq 1$), defined by

$$\| \tilde{e} \|_p = \left(\sum_{i=1}^{n} | y_i - y(x_i) |^p \, (x_{i+1} - x_i) \right)^{1/2}$$

Denote $h_i := \max_{1\leqslant i\leqslant n}(x_{i+1} - x_i)$.

If $\| \tilde{e} \|_p$ converges to zero as $h \to \infty$, then a finite difference method is called convergent. Moreover, if there is $c \geq 0$ such that $\| \tilde{e} \|_p \leq Ch^q, q > 0$, the finite difference method is called q-th order accurate.

We shall show that the FDM solution converges to the exact solution of the BVP (1)- (2) when h converges to zero. Using formulas (5) and (6), one can show that the exact solution $\tilde{y} = (y(x_1), y(x_2), \ldots, y(x_n))$ satisfies the following linear system of equation

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i) + p_i \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} y^{(3)}(\eta_i) + q_i y(x_i) = f(x_i), \quad 1 \leqslant i \leqslant n. \quad (9)$$

On the other hand, the FDM solution $\tilde{Y} = (y_1, y_2, \ldots, y_n)$ satisfies the linear system of equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i = f_i, \quad 1 \leqslant i \leqslant n. \quad (10)$$

Substracting these equation one from the other, we get

$$\frac{e_{i+1} - 2e_i + e_{i-1}}{h^2} + p_i \frac{e_{i+1} - e_{i-1}}{2h} + q_i e_i = h^2 f_i, \quad 1 \leqslant i \leqslant n \quad (11)$$

where e_i is the global error $e_i := y(x_i) - y_i$. $h^2 f_i$ is the local truncation error at the grid point $x = x_i$ and

$$f_i = \frac{1}{12} y^{(4)}(\xi_i) - \frac{1}{6} y^{(3)}(\eta_i)$$

After multiplying both sides of (11) by h^2 and then collecting the corresponding terms, we have

$$\left(1 - \frac{h}{2} p_i\right) e_{i-1} + \left(-2 + h^2 q_i\right) e_i + \left(1 + \frac{h}{2} p_i\right) e_{i+1} = h^4 f_i \quad (12)$$

To estimate the magnitude of the error vector $\tilde{e}$, it is necessary to use an infinite norm $\| \tilde{e} \|_p, p \geq 1$ for some spesific value p.

We will apply the infinite norm $\| \tilde{e} \|_\infty$, because it is used to measure grid functions and is easily estimated.

The equation (12) can be written as

$$(2 + h^2 q_i) e_i = (1 - \frac{h}{2} p_i) e_{i+1} - (1 + \frac{h}{2} p_i) e_i + h^4 f_i.$$

Consequently,

$$\begin{aligned} | 2 + h^2 q_i \| e_i | &\leq | 1 - \frac{h}{2} p_i \| e_{i+1} | + mid 1 + \frac{h}{2} p_i \| e_i | + h^4 | f_i | \\ &\leq | 1 - \frac{h}{2} p_i \|| \tilde{e} \|_\infty + | 1 + \frac{h}{2} p_i \|| \tilde{e} \|_\infty + h^4 \| \tilde{f} \|_\infty, \end{aligned}$$

where $\| \tilde{f} \|_\infty = \max\limits_{1 \leqslant i \leqslant n} | f_i |$.

From this inequality it follows immediately that

$$| 2 + h^2 q_i \|| \tilde{e} \|_\infty \leq \left(| 1 - \frac{h}{2} p_i | + | 1 + \frac{h}{2} p_i | \right) \| \tilde{e} \|_\infty + h^4 \| \tilde{f} \|_\infty . \quad (13)$$

Since $q(x) < 0$, one can choose $h > 0$ small enough to satisfy

$$| 1 - \frac{h}{2} p_i | + | 1 + \frac{h}{2} p_i | = 2$$

and

$$| 2 + h^2 q_i |= 2 + h^2 | q_i |$$

for all $i = 1, 2, \ldots, n$.

Consequently, for sufficiently small $h > 0$ we have from (13) that

$$| q_i ||| \tilde{e} ||_\infty \leq h^2 || \tilde{f} ||_\infty . \tag{14}$$

Denoting $C = \frac{||\tilde{f}||_\infty}{\min\limits_{1\leqslant i\leqslant n}|q_i|}$, we obtain

$$|| \tilde{e} ||_\infty \leq Ch^2. \tag{15}$$

Hence, the FDM is convergent and 2-order accurate [7, 15].

4. The Generalized Finite Difference Method for Solving Two-Interval SLP with Jump Problems

Consider the following two-interval SLP, consisting of the differential equation

$$y^{''} + 2y^{'} + 4e^{4x} y = 0, \quad x \in [-1, 0) \cup (0, 1] \tag{16}$$

subject to the boundary conditions at the endpoints $x = -1$ and $x = 1$, given by

$$y(-1) = 2, \quad y(1) = 0 \tag{17}$$

together with jump conditions across the common endpoint $x = 0$, given by

$$y(0^-) = 2y(0^+), \quad y'(0^-) = 3y'(0^+) \tag{18}$$

At first we consider the problem (16)-(18) without jump conditions (18). It is easy to verify that the function

$$y = \frac{2sin\left(e^{2x} - e^2\right)}{csc\left(\frac{1-e^4}{e^2}\right)e^{2x+2}} \tag{19}$$

satisfies the equation (16) on whole $[-1, 0) \cup (0, 1]$ and both boundary conditions (17). For simplicity we will use the uniform cartesian grid $x_i = -1 + ih, i = 0, 1, \ldots, 50$ for $h = 0, 04$. In particular we have $x_0 = -1, x_{50} = 1$.

The central finite difference (CFD) approximation of the derivatives $y^{'}$ and $y^{''}$ are defined by

$$y'(x) \approx \frac{1}{2}\left(D_+ y(x) + D_- y(x)\right) \tag{20}$$

and

$$y''(x) \approx \frac{1}{h}\left(D_+ y(x) - D_- y(x)\right), \tag{21}$$

where $D_+ y(x)$ and $D_- y(x)$ denotes the forward finite difference and backward finite difference of $y(x)$.

By applying the CFD to the differential equation (16) at a typical grid point $x = x_i$ and denoting $y_i = y(x_i)$, we have the following finite difference equations

$$(1 - h)y_{i-1} + (-2 + 4h^2 e4x_i)y_i + (1 + h)y_{i+1} = 0, \quad i = 1, 2, \ldots, 49. \tag{22}$$

That is we have the linear algebraic system of equations with respect to the variables $y_1, y_2, \ldots, y_{49}$. The system of linear algebraic equations (22) can be written in a tridiagonal matrix-vector form

$$Ay = b \tag{23}$$

where

$$A = \begin{pmatrix} -2+4h^2e^{4x_1} & 1+h & 0 & \cdots & 0 & 0 & 0 \\ 1-h & -2+4h^2e^{4x_2} & 1+h & \cdots & 0 & 0 & 0 \\ 0 & 1-h & -2+4h^2e^{4x_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-h & -2+4h^2e^{4x_{48}} & 1+h \\ 0 & 0 & 0 & \cdots & 0 & 1-h & -2+4h^2e^{4x_{49}} \end{pmatrix},$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{48} \\ y_{49} \end{pmatrix}, \qquad b = \begin{pmatrix} -2-h \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

The solution of this system can be found by using MATLAB-Octave. The obtained numerical FDM solutions are graphically compared with the exact solution (19) (see, Figures 1,2,3 and 4).

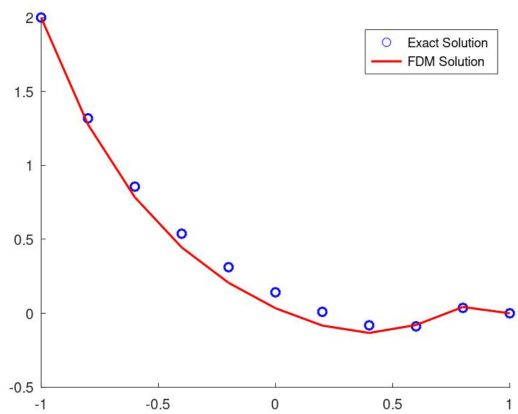

Figure 1: Graph of the FDM-solution and exact solution for the problem (16)-(17) for N=10

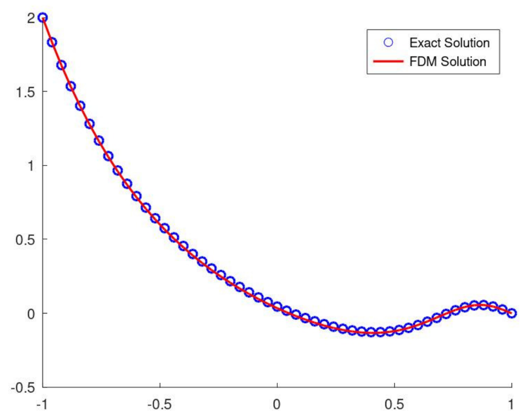

Figure 2: Graph of the FDM-solution and exact solution for the problem (16)-(17) for N=50

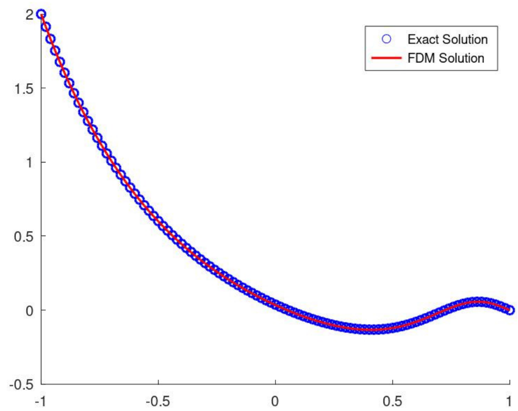

Figure 3: Graph of the FDM-solution and exact solution for the problem (16)-(17) for N=100

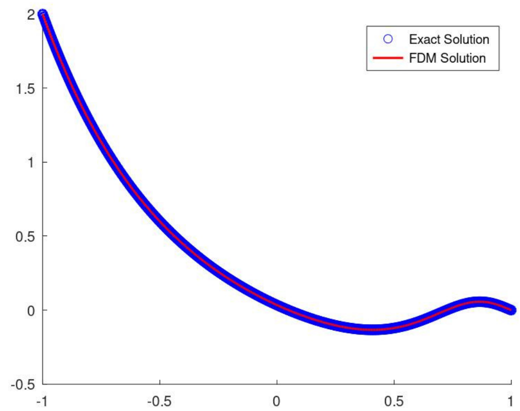

Figure 4: Graph of the FDM-solution and exact solution for the problem (16)-(17) for N=500

4.1. Remark

In figures 1,2,3 and 4 the exact solution (19) is compared with the numerical FDM solutions for $N = 10, 50, 100, 500$ respectively. It can be seen from these graphical illustrations that, the error between the FDM solutions and the exact solution decreases as the number of grid points N increases.

Now we will investigate the problem (16)-(18). If we select $N = 49$ and apply the jump conditions (18), then we have two additional algebraic equations

$$y_{24} - 2y_{25} = 0 \tag{24}$$

and

$$y_{22} - y_{24} - 3y_{25} + 3y_{27} = 0. \tag{25}$$

The solution of the algebraic system of linear equations (22), (24), (25) can be found by using MATLAB/Octave. The graphical illustration of the obtained results are given in the following figure.

5. Error Analysis

From equation (20) we have

$$h^2 \parallel f \parallel_\infty \geq \parallel e \parallel_\infty min \mid q(x_i) \mid \tag{26}$$

It is well-known that maximum absolute error of truncation error is related to $\parallel e \parallel_\infty min \mid q(x_i) \mid$. Table-1 shows that

$$\parallel e \parallel_\infty = 0.0027852$$

and

$$min \mid q(x_i) \mid = 4e^{4min(x_i)} = 0.00023945$$

Consequently

$$\parallel e \parallel_\infty min \mid q(x_i) \mid = 0.00023946$$

Naturally, the error decreases as the step length decreases.

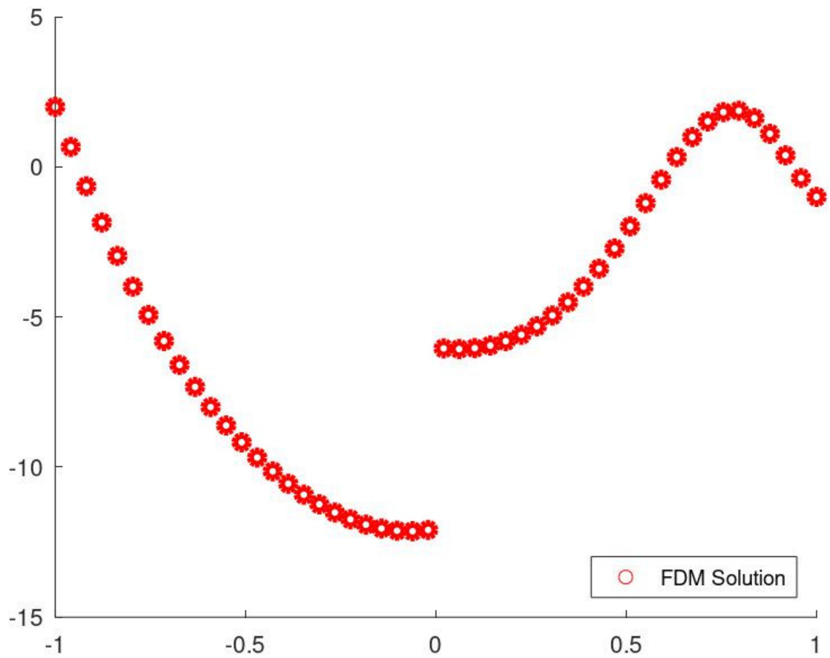

Figure 5: Graph of the FDM-solution for the problem (1)-(3)

Table 1: Maximum absolute error (MAE)

N	h	$\| e \|_\infty$
10	0.2	0.10745
50	0.04	0.011064
100	0.02	0.0027852
500	0.004	0.00011167
1000	0.002	0.000027918

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