Filomat 36:13 (2022), 4493–4503
https://doi.org/10.2298/FIL2213493S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Optimal Control for Non-Cooperative Systems Involving Fractional Laplace Operator

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Abstract. This study investigates the optimal control problem associated with $n \times n$ non-cooperative systems, including the fractional Laplace operator. The non-local issue is recast as a local problem. As a consequence, in these systems, the existence and uniqueness of the weak solution are established. A system's existence and optimal control conditions are also established.

1. Introduction

Let $O \subset \mathbb{R}^n$, $n \geq 1$, be an open, bounded and connected domain with smooth boundary ∂O. Then, for each $i = 1, 2, 3, ..., n$, the systems of the form

$$-\Delta\psi_i + \sum_{j=1}^{n} b_{ij}\psi_j = g_i \quad \text{in } O, \tag{1}$$

are said to be non-cooperative systems if for each $i, j = 1, 2, 3, ..., n$, the coefficients b_{ij} are given as follows:

$$b_{ij} = \begin{cases} 1, & i \geq j \\ -1, & i < j. \end{cases} \tag{2}$$

In this study, the $n \times n$ non-cooperative elliptic systems containing fractional Laplacian are explored in the context of optimal control. With zero Dirichlet conditions, we study the systems of the following form:

$$\begin{cases} (-\Delta)^{\alpha}\psi_i + \sum_{j=1}^{n} b_{ij}\psi_j = g_i & \text{in } O, \\ \psi_i = 0 & \text{on } \partial O, \end{cases} \tag{3}$$

where ψ_i is the state of the system, g_i is the external source and $(-\Delta)^{\alpha}$ is the fractional Laplacian for $\alpha \in (0, 1)$.

2020 *Mathematics Subject Classification*. Primary 46N10, 49J20, 35J55, 26A33
Keywords. Optimal control, Weak solution, Lax-Milgram Lemma, Fractional Laplace operator, Non-local operator, Sobolev spaces, Non-cooperative systems.
Received: 21 June 2021; Accepted: 14 September 2022
Communicated by Naseer Shahzad
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Due to their importance in several fields, the optimality requirements for partial differential equations (PDEs) with an integer order derivative have been extensively researched [13, 19, 21, 27, 28, 32]. They are often used in biology, ecology, economics, engineering, and finance. These findings have been used in both cooperative and non-cooperative systems [6, 9, 10, 24–26]. Compared to classical control problems, the study of fractional order control problems is very recent. However, it is becoming more well-known as a result of the significant roles that fractional differential equations (FDEs) play in physics, chemistry, and engineering. Recent studies in various fields have shown that FDEs are capable of properly describing the dynamics of a wide range of systems. For example, viscoelasticity and heat transfer in memory materials, anomalous diffusion in fractal media, non-local electrostatics, and image processing are just a few of the complex phenomena that may be studied using FDEs to characterize their behavior [8, 12, 15, 17, 30, 33]. There is a lot of study being done on time-fractional optimal control. Mophou utilized the Riemann—Liouville derivative to resolve a distributed optimal control problem for a time-fractional diffusion equation in [22, 23]. She also created the first-order optimality, existence, and uniqueness criteria for the model. According to [14], distributed control for a time-fractional differential system with the Schrödinger operator is studied, and the optimality conditions are then constructed. In [16], fractional control problem of Navier-Stokes equations are investigated, and the optimality conditions are derived by Lions approach. The subject is also quickly becoming space-fractional. Important findings on this subject were made by Antil and Otarola [1], who in their work found the optimality criteria for a model containing a space-fractional elliptic operator.

It is commonly known that the fractional Laplace operator is non-local, which means that when ξ approaches infinity, the values of ψ have an effect on the values of $(-\Delta)^{\alpha}\psi(\xi)$, $\xi \in \mathcal{O}$. Additionally, the support of the fractional Laplace operator $(-\Delta)^{\alpha}$ is non-compact even the support of ψ is compact. This fundamental flaw might lead to various challenges. In fact, situations involving $(-\Delta)^{\alpha}$, cannot be investigated using the traditional local PDE techniques. Caffarelli and Silvestre [4] showed that the fractional Laplace operator may be described as an operator that converts a Dirichlet boundary condition to a Neumann-type condition by employing an extension issue.

Consider the semi-infinite cylinder $\mathcal{R}$, which is defined as

$$\mathcal{R} = \{(\xi, \eta) : \xi \in \mathcal{O}, \eta \in (0, \infty)\} \subset \mathbb{R}^{n+1},\ n \geq 1, \tag{4}$$

where η is the extended variable and $\partial\mathcal{R} = \partial\mathcal{O} \times [0, \infty)$ the lateral boundary of $\mathcal{R}$. Assume that $\Psi = \{\Psi_i\}_{i=1}^{n}$ is a solution for the degenerate elliptic boundary value problem [7]

$$\begin{cases} \nabla.(\eta^{\beta}\nabla\Psi_i) = 0 & \text{in } \mathcal{R}, \\ \Psi_i = 0 & \text{on } \partial\mathcal{R}, \\ \frac{\partial\Psi_i}{\partial\nu} = \gamma_{\alpha}(g_i - \sum_{j=1}^{n} b_{ij}\psi_j) & \text{on } \mathcal{O}\times\{0\}, \end{cases} \tag{5}$$

where $\beta = 1-2\alpha \in (-1, 1)$, $\frac{\partial\Psi}{\partial\nu^{\beta}} = -\lim_{\eta\to 0^{+}} \eta^{\beta}\frac{\partial\Psi(\xi,\eta)}{\partial\eta}$, ν^{β} is the unit outer normal to $\mathcal{R}$ at $\mathcal{O}\times\{0\}$, $\lim_{\eta\to\infty} \Psi(\zeta, \eta) = 0$ and $\gamma_{\alpha} = 2^{\beta}\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}$ is a positive normalization constant.

We generalise several previously discovered conclusions for classical non-cooperative systems in this study. Indeed, we investigate $n \times n$ non-cooperative systems with fractional Laplacian. To solve the challenges caused by fractional Laplacian's non-locality, we employ the extension problem approach to convert the problem into a local extended problem [4, 29]. The Lax-Milgram Lemma is used to show the existence and uniqueness of the solution for the local system. Additionally, the Lions technique is utilized to determine the optimality criteria for both local and non-local systems. If $\alpha \to 1$, the results are comparable to the classical findings. The following is the order of this article: Section 2 offers different functional spaces to describe the $n \times n$ non-cooperative fractional systems and their extension. In section 3, we establish the weighted initial boundary value problem, the existence of the variational form of the initial boundary value problem is derived. In section 4, the control issue is posed for both the fractional systems and their extensions, then we drive the optimality requirements for both. Section 5 is dedicated to summary and debate.

2. Preliminaries and known results

The variational formulation of the previous model requires the Sobolev spaces. Two subsections make up this section. A brief overview of spectral fractional Sobolev spaces is given in Subsection 2.1. We review the concept of weighted Sobolev spaces and their embedding characteristics in subsection 2.2.

2.1. Fractional Sobolev spaces

Since the spectral fractional Laplace operator has admitted local boundary conditions on $\partial\mathcal{O}$, we formulate our problem using the spectral fractional Laplace operator. For more details about the properties of the fractional Laplace operator, we refer the reader to [2].

Definition 2.1. *[1, 3, 5]**(Eigenvalue Problem)** Assume that the domain $\mathcal{O}$ is Lipschitz bound and has homogeneous boundary conditions. An orthonormal basis of $L^2(\mathcal{O})$ is formed by the eigenfunctions $\{\varphi_l\}_{l\in\mathbb{N}}$ of the Laplace operator $-\Delta$, which are non-trivial solutions of the following system*

$$\begin{cases} -\Delta\varphi_l = \lambda_l\varphi_l & \text{in } \mathcal{O}, \\ \varphi_l = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{6}$$

Moreover, the corresponding eigenvalues $\{\lambda_l\}_{l\in\mathbb{N}}$ are all non-negative and form a non-decreasing sequence such that $\lambda_l \to \infty$.

In this case, the spectral decomposition for the Laplacian $-\Delta$ is given by

$$-\Delta\psi = \sum_{l=1}^{\infty} \lambda_l\psi_l\varphi_l \quad \text{with} \quad \psi_l = \int_{\mathcal{O}} \psi\varphi_l d\xi. \tag{7}$$

Definition 2.2. *[1, 5]**(The Spectral Fractional Laplacian)** Assume $C_0^\infty(\mathcal{O})$ is the space of infinitely differentiable functions on $\mathcal{O}$. Then, for $\psi \in C_0^\infty(\mathcal{O})$, the spectral fractional Laplacian with $\alpha \in (0,1)$ is defined as follows:*

$$(-\Delta)^\alpha\psi = \sum_{l=1}^{\infty} \lambda_l^\alpha\psi_l\varphi_l \quad \textit{with} \quad \psi_l = \int_{\mathcal{O}} \psi\varphi_l d\xi. \tag{8}$$

For any $\alpha \in (0,1)$, the spectral fractional Sobolev space can be introduced as follows: [1]

$$H^\alpha(\mathcal{O}) = \left\{ \psi = \sum_{l=1}^{\infty} \psi_l\varphi_l : \sum_{l=1}^{\infty} \lambda_l^\alpha\psi_l^2 < \infty \right\}. \tag{9}$$

Then,

$$(-\Delta)^\alpha : H^\alpha(\mathcal{O}) \to H^{-\alpha}(\mathcal{O}), \tag{10}$$

where $H^{-\alpha}(\mathcal{O})$ is the dual of $H^\alpha(\mathcal{O})$. Additionally, we have the embedding below

$$H^\alpha(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow H^{-\alpha}(\mathcal{O}). \tag{11}$$

Using the Cartesian product, we have

$$(H^\alpha(\mathcal{O}))^n \hookrightarrow (L^2(\mathcal{O}))^n \hookrightarrow (H^{-\alpha}(\mathcal{O}))^n. \tag{12}$$

2.2. Weighted Sobolev spaces

We need appropriate weighted Sobolev spaces with weight $|\eta|^\beta$ to examine the local issue (5). For more information, see [11, 18]. The space of all measurable functions defined on $\mathcal{R}$ is thus given by $L^2\left(|\eta|^\beta, \mathcal{R}\right)$ such that

$$\|\Psi\|^2_{L^2\left(|\eta|^\beta,\mathcal{R}\right)} := \int_{\mathcal{R}} \eta^\beta \Psi^2 < \infty. \tag{13}$$

Hence, the weighted Sobolev Space is defined as

$$\mathbf{H}^1\left(|\eta|^\beta, \mathcal{R}\right) = \left\{\Psi \in L^2\left(|\eta|^\beta, \mathcal{R}\right) : |\nabla\Psi| \in L^2\left(|\eta|^\beta, \mathcal{R}\right)\right\}, \tag{14}$$

which has the following norm in a Hilbert space:

$$\|\Psi\|_{\mathbf{H}^1\left(|\eta|^\beta,\mathcal{R}\right)} := \left(\|\Psi\|^2_{L^2\left(|\eta|^\beta,\mathcal{R}\right)} + \|\nabla\Psi\|^2_{L^2\left(|\eta|^\beta,\mathcal{R}\right)}\right)^{\frac{1}{2}}. \tag{15}$$

Moreover, we have the following weighted Sobolev space

$$\mathbf{H}^1_{0,L}\left(\eta^\beta, \mathcal{R}\right) = \left\{\Psi \in \mathbf{H}^1\left(\eta^\beta, \mathcal{R}\right) : \Psi = 0 \text{ on } \partial\mathcal{R}\right\}. \tag{16}$$

And, the following weighted Poincaré inequality is satisfied [1]

$$\int_{\mathcal{R}} \eta^\beta \Psi^2 \leq C \int_{\mathcal{R}} \eta^\beta |\nabla\Psi|^2, \quad \forall\, \Psi \in \mathbf{H}^1_{0,L}\left(\eta^\beta, \mathcal{R}\right), \tag{17}$$

where $C \geq 0$, depends on $\mathcal{O}$.

Definition 2.3. *[1] (The Trace Operator) For a function $\Psi \in \mathbf{H}^1\left(|\eta|^\beta, \mathcal{R}\right)$, the operator $\mathrm{T} : \mathbf{H}^1_{0,L}\left(\eta^\beta, \mathcal{R}\right) \to H^\alpha(\mathcal{O})$ is called the trace operator and satisfies*

$$\|\mathrm{T}\,\Psi\|_{H^\alpha(\mathcal{O})} \leq \rho\|\Psi\|_{\mathbf{H}^1_{0,L}\left(\eta^\beta,\mathcal{R}\right)},\ \rho > 0. \tag{18}$$

Furthermore, $\mathrm{T}\Psi = \Psi(\xi, 0) = \psi(\xi)$ is the trace of Ψ onto $\mathcal{O} \times \{0\}$.

Definition 2.4. *[4] The Dirichlet-to-Neumann operator on $\psi \in H^\alpha(\mathcal{O})$ is defined as follows:*

$$\psi \in H^\alpha(\mathcal{O}) \longmapsto \mathcal{N}(u) = \frac{\partial\Psi}{\partial\nu^\beta} \in H^{-\alpha}(\mathcal{O}), \tag{19}$$

where Ψ solves (5) *and ψ solves* (3).

Theorem 2.5. *[4] If $\alpha \in (0, 1)$ and $\psi \in H^\alpha(\mathcal{O})$, then*

$$\gamma_\alpha(-\Delta)^\alpha\psi = \mathcal{N}(\psi), \tag{20}$$

in the sense of distributions.

3. The weak solution

Multiply the first equation in (5) by test function $\phi_i(\xi, \eta) \in \mathbf{H}^1_{0,L}(\eta^\beta, \mathcal{R})$, and integrating over $\mathcal{R}$ yields

$$\int_{\mathcal{R}} \nabla.(\eta^\beta\nabla\Psi_i)\phi_i(\xi, \eta)d\xi d\eta = 0. \tag{21}$$

Applying Green's formula, we have

$$\int_{\mathcal{R}} \eta^{\beta} \nabla \Psi_i \nabla \phi_i(\xi, \eta) d\xi d\eta = -\int_{O\times\{0\}} \mathrm{T}\, \phi_i(\xi, \eta) \lim_{\eta\to 0^+} \eta^{\beta} \frac{\partial \Psi_i}{\partial \eta} d\xi. \tag{22}$$

Then, we get

$$\int_{\mathcal{R}} \eta^{\beta} \nabla \Psi_i \nabla \phi_i(\xi, \eta) d\xi d\eta = \int_{O\times\{0\}} \mathrm{T}\, \phi_i(\xi, \eta) \frac{\partial \Psi_i}{\partial \nu^{\beta}} d\xi. \tag{23}$$

By using the systems (5), equation (23) is equivalent to

$$\int_{\mathcal{R}} \eta^{\beta} \nabla \Psi_i \nabla \phi_i(\xi, \eta) d\xi d\eta = \int_{O\times\{0\}} \gamma_{\alpha}(g_i - \sum_{j=1}^{n} b_{ij}\psi_j)\, \mathrm{T}\, \phi_i(\xi, \eta) d\xi. \tag{24}$$

We may create a bilinear form on $\left(\mathbf{H}^1_{0,L}(\eta^{\beta}, O)\right)^n$ as the form

$$a(\Psi, \phi) = \sum_{i=1}^{n} \int_{\mathcal{R}} \eta^{\beta} \nabla \Psi_i \nabla \phi_i(\xi, \eta) d\xi d\eta + \gamma_{\alpha} \sum_{i,j=1}^{n} \int_{O\times\{0\}} b_{ij}\psi_j\, \mathrm{T}\, \phi_i(\xi, \eta) d\xi. \tag{25}$$

A linear form may also be defined as follows:

$$F(\phi) = \gamma_s \sum_{i=1}^{n} \int_{O\times\{0\}} g_i\, \mathrm{T}\, \phi_i(\xi, \eta) d\xi, \quad \forall\, \phi_i \in \mathbf{H}^1_{0,L}(\eta^{\beta}, \mathcal{R}). \tag{26}$$

Lemma 3.1. *The bilinear form* (25) *satisfies the coerciveness and boundedness conditions.*

Proof. In (25), replacing $\phi = \{\phi_i\}_{i=1}^n$ by $\Psi = \{\Psi_i\}_{i=1}^n$ yields

$$\begin{aligned} a(\Psi, \Psi) &= \sum_{i=1}^{n} \int_{\mathcal{R}} \eta^{\beta} \|\nabla \Psi_i\|^2 d\xi d\eta + \gamma_s \sum_{i=1}^{n} \int_{O\times\{0\}} \|\psi_i\|^2 d\xi \\ &\geq \sum_{i=1}^{n} \int_{\mathcal{R}} \eta^{\beta} \|\nabla \Psi_i\|^2 d\xi d\eta \\ &\geq \frac{1}{2} \sum_{i=1}^{n} \int_{\mathcal{R}} \eta^{\beta} \|\nabla \Psi_i\|^2 d\xi d\eta + \frac{1}{2C} \sum_{i=1}^{n} \int_{\mathcal{R}} \eta^{\beta} \|\Psi_i\|^2 d\xi d\eta \\ &\geq \frac{1}{2} \min\{1, \frac{1}{C}\} \left[\sum_{i=1}^{n} \|\Psi_i\|^2_{\mathbf{H}^1_{0,L}(\eta^{\beta}, \mathcal{R})} \right]. \end{aligned} \tag{27}$$

Therefore,

$$a(\Psi, \Psi) \geq \delta \|\Psi\|^2_{\left(\mathbf{H}^1_{0,L}(\eta^{\beta}, \mathcal{R})\right)^n}, \tag{28}$$

where $\delta = \dfrac{1}{2}\min\{1, \dfrac{1}{C}\} \geq 0$. Hence, the coerciveness condition is proved.

Also, the boundedness condition for the bilinear form can be proved as follows:

$$\begin{aligned} |a(\Psi, \phi)| &\leq \sum_{i=1}^{n} \int_{\mathcal{R}} |\eta^{\beta} \nabla \Psi_i \nabla \phi_i(\xi, \eta)| d\xi d\eta + \gamma_{\alpha} \sum_{i,j=1}^{n} \int_{O\times\{0\}} |b_{ij}\psi_j\, \mathrm{T}\, \phi_i(\xi, \eta)| d\xi \\ &\leq \sum_{i=1}^{n} \|\Psi_i\|_{\mathbf{H}^1(|\eta|^{\beta}, \mathcal{R})} \|\phi_i\|_{\mathbf{H}^1(|\eta|^{\beta}, \mathcal{R})} + \sum_{i,j=1}^{n} \|\psi_i\|_{H^{\alpha}(O)} \|\mathrm{T}\, \phi_j\|_{H^{\alpha}(O)}. \end{aligned} \tag{29}$$

By using the inequality (18) and the embedding of $\mathbf{H}^1_{0,L}\left(\eta^\beta,\mathcal{R}\right)$ in $\mathbf{H}^1(|\eta|^\beta,\mathcal{R})$, for a constant $M \geq 0$, we have

$$\begin{aligned}|a(\Psi,\phi)| &\leq M\left[\sum_{i,j=1}^{n}||\Psi_i||_{\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})}||\phi_j||_{\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{D}_{\mathcal{B}})}\right]\\ &\leq M||\Psi||_{\left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n}||\phi||_{\left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n}.\end{aligned} \tag{30}$$

Thus, all the assertions of the Lemma are satisfied.

Theorem 3.2. *For every $g = \{g_i\}_{i=1}^n \in (L^2(O))^n$, the systems* (5) *have a unique weak solution $\Psi = \{\Psi_i\}_{i=1}^n \in \left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n$. Moreover, there is a constant $\sigma > 0$, which does not depend on g, such that*

$$||\Psi||_{\left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n} \leq \sigma||g||_{(L^2(O))^n}. \tag{31}$$

Proof. From Lemma (3.1), we have that the bilinear form $a(\Psi,\phi)$ is coercive and bounded. Then, by applying the Lax-Milgram Lemma [19–21] in $\left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n$, there exists a unique element $\Psi \in \left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n$ satisfying the variational equation

$$a(\Psi,\phi) = F(\phi), \quad \forall\, \phi \in \left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n, \tag{32}$$

where the linear form $F(\phi)$ defined in (26) is bounded. Indeed, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|F(\phi)| &\leq \gamma_\alpha\sum_{i=1}^{n}||g_i||_{L^2(O)}||\operatorname{T}\phi_i||_{L^2(O)}\\ &\leq \gamma_\alpha\sum_{i=1}^{n}||g_i||_{L^2(O)}||\operatorname{T}\phi_i||_{H^\alpha(O)}.\end{aligned} \tag{33}$$

By using the inequality (18), we get

$$|F(\phi)| \leq \gamma_\alpha\rho\sum_{i=1}^{n}||g_i||_{L^2(O)}||\phi_i||_{\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})}. \tag{34}$$

Therefore, we have $||F|| \leq ||G||_{(L^2(O))^2}$, which yields the following estimate:

$$||\Psi||_{\left(\mathbf{H}^1_{0,L}(\eta^\beta,\mathcal{R})\right)^n} \leq \sigma||F|| \leq \sigma||g||_{(L^2(O))^n}. \tag{35}$$

Thus, (31) is proved.

4. The optimality conditions

The primary goal of this study is to formulate control problems. The adjoint state is used to solve the control issue. Furthermore, we generate the optimality conditions using the Lions approach [19, 21]. This section has two subsections. The optimality conditions for the $n \times n$ non-cooperative fractional systems are deduced in subsection 4.1. While the equivalent extended optimality conditions are established in subsection 4.2.

4.1. Fractional optimal control

Consider $(L^2(O))^n$ to be the control space. For a control $u = \{u_i\}_{i=1}^n \in (L^2(O))^n$, the state $\psi(u) = \{\psi_i(u)\}_{i=1}^n$ solves the systems

$$\begin{cases}(-\Delta)^\alpha \psi_i(u) + \sum_{j=1}^n b_{ij}\psi_j(u) = u_i & \text{in } O,\\ \psi_i = 0 & \text{on } \partial O.\end{cases} \tag{36}$$

The observation equations are as follows:

$$\mathbf{z}_i(u) = \psi_i(u), \quad i = 1, 2, 3, ..., n. \tag{37}$$

For the systems (36), given $\psi_d = \{\psi_{id}\}_{i=1}^n \in (L^2(O))^n$ and $v = \{v_i\}_{i=1}^n \in (L^2(O))^n$, we provide the following cost functional:

$$\mathtt{J}(v) = \sum_{i=1}^n \|\psi_i(v) - \psi_{id}\|^2_{L^2(O)} + (Nv, v)_{(L^2(O))^n}, \tag{38}$$

where $N = \{N_i\}_{i=1}^n \in \mathcal{L}((L^2(O))^n)^{1)}$ is Hermitian positive definite i.e.,

$$(Nv, v) \geq \mu \|v\|^2_{(L^2(O))^n}, \quad \mu > 0. \tag{39}$$

Let $\mathcal{U}_{ad} \subset L^2(O)$ be a closed and convex, then the following is the problem of control:

$$\begin{cases}\text{Finding } u \in (\mathcal{U}_{ad})^n,\\ \text{such that } \mathtt{J}(u) \leq \mathtt{J}(v), \quad \forall\, v \in (\mathcal{U}_{ad})^n.\end{cases} \tag{40}$$

Theorem 4.1. *There exists a unique element $u \in (\mathcal{U}_{ad})^n$, if the cost functional is provided by* (38) *and* (39) *is true. Furthermore, the following equations describe this control:*

$$\begin{cases}(-\Delta)^\alpha p_i + \sum_{j=1}^n b_{ji}p_j = \psi_i(u) - \psi_{id} & \text{in } O,\\ p_i = 0 & \text{on } \partial O,\end{cases} \tag{41}$$

together with

$$\sum_{i=1}^n (p_i, v_i - u_i) + (Nu, v - u)_{(L^2(O))^n} \geq 0, \ \ \forall\, v \in (\mathcal{U}_{ad})^n, \tag{42}$$

where $p = \{p_i\}_{i=1}^n \in (H^\alpha(O))^n$ is the adjoint state.

Proof. The cost functional (38) is strictly convex because of $N > 0$. Also, $\mathcal{U}_{ad}$ is nonempty, closed, bounded, and convex in $L^2(O)$. Thereby, the existence and uniqueness of the optimal control are established.

On the other hand, the element $u \in (\mathcal{U}_{ad})^n$ satisfy the following inequality:

$$\mathtt{J}^{'}(u).(v - u) \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^n, \tag{43}$$

or

$$\sum_{i=1}^n (\psi_i(u) - \psi_{id}, \psi_i(v) - \psi_i(u)) + \sum_{i=1}^n (Nu_i, v_i - u_i) \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^n. \tag{44}$$

[1)] $\mathcal{L}(X)$ is the space of all bounded and linear operators from X into itself.

Now, let $p = \sum_{k=1}^{\infty} p_k\varphi_k, \quad p_k = \int_{\mathcal{O}} p\varphi_k$ be the adjoint state of the systems (3), since

$$(\mathbf{A}\,\psi, p) = (\psi, \mathbf{A}^*\, p), \tag{45}$$

and hence,

$$\begin{aligned}
(\mathbf{A}\,\psi, p) &= \sum_{i=1}^{n}\left((-\Delta)^{\alpha}\psi_i + \sum_{j=1}^{n} b_{ij}\psi_j, p_i\right)\\
&= \sum_{i=1}^{n}\left(\sum_{l=1}^{\infty}\lambda_l^{\alpha}\psi_{l_i}\varphi_{l_i} + \sum_{l=1}^{\infty}\sum_{j=1}^{n} b_{ij}\psi_{l_j}\varphi_{l_j}, \sum_{k=1}^{\infty} p_{k_i}\varphi_{k_i}\right)\\
&= \sum_{i=1}^{n}\sum_{l,k=1}^{\infty}\lambda_l^{\alpha}\psi_{l_i}p_{k_i}(\varphi_{l_i},\varphi_{k_i}) + \sum_{i,j=1}^{n}\sum_{l,k=1}^{\infty} b_{ij}\psi_{l_j}p_{k_i}(\varphi_{k_i},\varphi_{l_j})\\
&= \sum_{i=1}^{n}\sum_{l=1}^{\infty}\lambda_l^{\alpha}\psi_{l_i}p_{l_i} + \sum_{i,j=1}^{n}\sum_{l,k=1}^{\infty} b_{ij}\psi_{l_j}p_{k_i}(\varphi_{k_i},\varphi_{l_j})\\
&= \sum_{i=1}^{n}\sum_{k=1}^{\infty}\lambda_k^{\alpha}\psi_{k_i}p_{k_i} + \sum_{i,j=1}^{n}\sum_{l,k=1}^{\infty} b_{ij}\psi_{l_j}p_{k_i}(\varphi_{k_i},\varphi_{l_j})\\
&= \sum_{i=1}^{n}\sum_{l,k=1}^{\infty}\lambda_k^{\alpha}\psi_{l_i}p_{k_i}(\varphi_{l_i},\varphi_{k_i}) + \sum_{i,j=1}^{n}\sum_{l,k=1}^{\infty} b_{ij}\psi_{l_j}p_{k_i}(\varphi_{k_i},\varphi_{l_j})\\
&= \sum_{i=1}^{n}\sum_{l,k=1}^{\infty}\lambda_k^{\alpha}\psi_{l_i}p_{k_i}(\varphi_{l_i},\varphi_{k_i}) + \sum_{i,j=1}^{n}\sum_{l,k=1}^{\infty} b_{ij}\psi_{l_j}p_{k_i}(\varphi_{k_i},\varphi_{l_j})\\
&= \sum_{i=1}^{n}\left(\sum_{l=1}^{\infty}\psi_{l_i}\varphi_{l_i}, \sum_{k=1}^{\infty}\lambda_k^{\alpha}p_{k_i}\varphi_{k_i} + \sum_{k=1}^{\infty}\sum_{j=1}^{n} b_{ji}p_{k_j}\varphi_{k_j}\right)\\
&= (\psi, \mathbf{A}^*\, p).
\end{aligned} \tag{46}$$

Therefore, (44) is equivalent to

$$\sum_{i=1}^{n}\left((-\Delta)^{s}p_i + \sum_{j=1}^{n} b_{ji}p_j, \psi_i(v) - \psi_i(u)\right) + \sum_{i=1}^{n}(N_iu_i, v_i - u_i) \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^n. \tag{47}$$

By using the equations. (36) and (46), we deduce

$$\sum_{i=1}^{n}(p_i, v_i - u_i) + \sum_{i=1}^{n}(N_iu_i, v_i - u_i)_{(L^2(\mathcal{O}))^n} \geq 0, \quad \forall\, v \in (\mathcal{U}_{ad})^n. \tag{48}$$

4.2. Extended optimal control

According to Theorem (2.5), if $\psi(q) \in (H^{\alpha}(\mathcal{O}))^n$ is a solution of (36) with $q = \{q_i\}_{i=1}^n \in (H^{-\alpha}(\mathcal{O}))^n$ and $\Psi(q) \in \left(\mathbf{H}^1_{0,L}(\eta^{\beta},\mathcal{R})\right)^n$ solves the following systems:

$$\begin{cases}\sum_{i=1}^{n}\nabla.(\eta^{\beta}\nabla\Psi_i(q)) = 0 & \text{in } \mathcal{R},\\ \Psi_i(q) = 0 & \text{on } \partial\mathcal{R},\\ \sum_{i=1}^{n}\frac{\partial\Psi_i(q)}{\partial\nu^{\beta}} = \gamma_{\alpha}(q_i - \sum_{j=1}^{n} b_{ij}\psi_j) & \text{on } \mathcal{O}\times\{0\}.\end{cases} \tag{49}$$

Then, we have

$$\mathrm{T}\,\Psi(q) = \psi(q). \tag{50}$$

And hence, the equivalence extended cost functional is given by

$$\min\ \mathbf{J}(q) = \sum_{i=1}^{n} \| \mathrm{T}\,\Psi_i(q) - \psi_{id}\|^2_{L^2(O)} + (Nq, q)_{(L^2(O))^n}, \quad \forall\, q \in (\mathcal{U}_{ad})^n. \tag{51}$$

Theorem 4.2. *If* (51) *gives the extended cost functional and the condition* (39) *is met, then there exists a unique optimal control* $r = \{r_i\}_{i=1}^n \in (\mathcal{U}_{ad})^n$. *Furthermore, the following equations describe this control:*

$$\begin{cases} \sum_{i=1}^n \nabla.(\eta^\beta \nabla \mathbf{P}_i) = 0 & in \quad \mathcal{R}, \\ \mathbf{P}_i = 0 & on \quad \partial_L \mathcal{R}, \\ \sum_{i=1}^n (\frac{\partial \mathbf{P}_i}{\partial \nu^\beta}, \mathrm{T}\,\Psi_i) = \sum_{i=1}^n (\mathrm{T}\,\mathbf{P}_i, \frac{\partial \Psi_i}{\partial \nu^\beta}) & on \quad O \times \{0\}, \end{cases} \tag{52}$$

together with

$$\gamma_\alpha \sum_{i=1}^n (\mathrm{T}\,\mathbf{P}_i, q_i - r_i) + (Nr, q - r) \geq 0, \quad \forall\, q \in (\mathcal{U}_{ad})^n, \tag{53}$$

where $\mathbf{P} = \{\mathbf{P}_i\}_{i=1}^n \in \left(\mathbf{H}^1_{0,L}(\eta^\beta, \mathcal{R})\right)^n$ *is the extended adjoint state.*

Proof. The element $r \in (\mathcal{U}_{ad})^n$ is optimal control if and only if

$$\mathbf{J}'(r).(q - r) \geq 0, \quad \forall\, q \in (\mathcal{U}_{ad})^n, \tag{54}$$

or its equivalent [19]

$$\sum_{i=1}^n (\mathrm{T}\,\Psi_i(q) - \psi_{id}, \mathrm{T}\,\Psi_i(q) - \mathrm{T}\,\Psi_i(r)) + \sum_{i=1}^n (N_i r_i, q_i - r_i), \quad \forall\, q \in (\mathcal{U}_{ad})^n. \tag{55}$$

Now, since $(\mathbf{A}\,\Psi, \mathbf{P}) = (\Psi, \mathbf{A}^*\,\mathbf{P})$, then

$$\begin{aligned}(\mathbf{A}\,\Psi, \mathbf{P}) &= \sum_{i=1}^n (\nabla.(\eta^\beta \nabla \Psi_i), \mathbf{P}_i) \\ &= \sum_{i=1}^n \int_{\mathcal{R}} \nabla.(\eta^\beta \nabla \Psi_i) \mathbf{P}_i d\xi d\eta. \end{aligned} \tag{56}$$

Via Green's formula, (56) is transformed to

$$\begin{aligned}(\mathbf{A}\,\Psi, \mathbf{P}) &= \sum_{i=1}^n \int_{\mathcal{R}} \nabla.(\eta^\beta \nabla \mathbf{P}_i) \Psi_i(\xi, \eta) d\xi d\eta - \sum_{i=1}^n \int_O \mathrm{T}\,\Psi_i \frac{\partial \mathbf{P}_i}{\partial \nu^\beta} + \sum_{i=1}^n \int_O \mathrm{T}\,\mathbf{P}_i \frac{\partial \Psi_i}{\partial \nu^\beta} \\ &= (\Psi, \mathbf{A}^*\,\mathbf{P}), \end{aligned} \tag{57}$$

and hence (52) is verified.

Introducing the following adjoint systems:

$$\begin{cases} \sum_{i=1}^n \nabla.(\eta^\beta \nabla \mathbf{P}_i) = 0 & \text{in} \quad \mathcal{R}, \\ \mathbf{P}_i = 0 & \text{on} \quad \partial \mathcal{R}. \\ \sum_{i=1}^n \frac{\partial \mathbf{P}_i}{\partial \nu^\beta} = \sum_{i=1}^n \mathrm{T}\,\Psi_i(q) - \psi_{id} & \text{on} \quad O \times \{0\}. \end{cases} \tag{58}$$

Then, (55) is reduced to

$$\sum_{i=1}^n \left(\frac{\partial \mathbf{P}_i}{\partial \nu^\beta}, \mathrm{T}\,\Psi_i(q) - \mathrm{T}_{\mathcal{B}}\,\Psi_i(r) \right) + \sum_{i=1}^n (N_i r_i, q_i - r_i) \geq 0, \quad \forall\, q \in (\mathcal{U}_{ad})^n. \tag{59}$$

Hence, we obtain to the following optimality condition

$$\gamma_\alpha \sum_{i=1}^{n} (\mathrm{T}\,\mathbf{P}_i, q_i - r_i) + (Nr, q - r) \geq 0, \quad \forall\, q \in (\mathcal{U}_{ad})^n. \tag{60}$$

Thereby, the proof is completed.

5. Summary and conclusion

In this paper, we look at the optimum control issue for $n \times n$ non-cooperative systems with fractional Laplacian. Because of the difficulties caused by the fractional Laplace operator's non-locality, we use an extension problem approach to extend our issue to local non-cooperative systems. The existence and uniqueness of a solution to the extended issue are shown using the Lax-Milgram Lemma. Furthermore, the Lions approach is used to show the optimality criteria for both fractional and extended control. If $\alpha \to 1$. The derived results are similar to the classical findings.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Groups Program under grant RGP.2/15/43.

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