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Novel Theorems for Metallic Structures on the Frame Bundle of the Second Order

Mohammad Nazrul Islam Khan[a]

[a]*Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudi Arabia.*

Abstract. It is well known that 'an almost complex structure' J that is $J^2 = -I$ on the manifold M is called 'an almost Hermitian manifold' (M, J, G) if $G(JX, JY) = G(X, Y)$ and proved that (F^2M, J^D, G^D) is 'an almost Hermitian manifold' on the frame bundle of the second order F^2M. The term 'an almost complex structure' refers to the general quadratic structure $J^2 = pJ + qI$, where $p = 0, q = -1$. However, this paper aims to study the general quadratic equation $J^2 = pJ + qI$, where $p,\ q$ are positive integers, it is named as a metallic structure. The diagonal lift of the metallic structure J on the frame bundle of the second order F^2M is studied and shows that it is also a metallic structure. The proposed theorem proves that the diagonal lift G^D of a Riemannian metric G is a metallic Riemannian metric on F^2M. Also, a new tensor field $\tilde{J}$ of type (1,1) is defined on F^2M and proves that it is a metallic structure. The 2-form and its derivative dF of a tensor field $\tilde{J}$ are determined. Furthermore, the Nijenhuis tensor $N_{\tilde{J}}$ of a metallic structure $\tilde{J}$ and the Nijenhuis tensor N_{J^D} of a tensor field J^D of type (1,1) on the frame bundle of the second order F^2M are calculated.

1. Introduction

In the framework of the geometry of frame bundles, it is classical to consider various geometric structures; for example, 'an almost complex', 'an almost Hermitian', 'an almost symplectic' structures, etc. using 'the complete', 'vertical', 'horizontal', and 'diagonal lifts' transforming structures on the base manifold to the frame bundle FM and the frame bundle of the second order F^2M. Bonome et. al. [5] defined 'an almost complex structure' J and proved that (FM, g^D, J) is 'an almost Hermitian manifold', where the diagonal lift g^D is a Riemannian metric on FM. The Nijenhuis tensor of 'an almost complex structure' J on FM has been calculated and established a necessary and sufficient condition for integrability of J. The author [24] introduced the notion of a tensor field $\tilde{J}$ of type (1,1) on the frame bundle FM and showed that it is a metallic structure. The Nijenhuis tensor, derivative, and co-derivative of the 2-form have been determined. Kowalski and Sekizawa [25] studied the properties of curvatures of a diagonal lift from an affine connection to the linear frame bundle. León and Salgado [11] initiated the study of a diagonal lift of tensor fields, an almost Hermitian manifold and the Kähler form to the frame bundle of the second order F^2M. Many authors have studied the geometry of the frame bundle and the frame bundle of the second order [7, 12, 26, 27].

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Email address: m.nazrul@qu.edu.sa (Mohammad Nazrul Islam Khan)

The various geometric structures; for example, 'an almost complex', 'an almost Hermitian', 'an almost symplectic' structures, etc. provide effective results while studying on the frame bundle and the frame bundle of the second order. The differential geometry of metallic structures on the manifold is one of the most studied subjects [2, 28–30, 35, 36]. The notion of the metallic means family was introduced by Spindel [33, 34] which consists of the Gold mean, the Silver mean, the Bronze mean, etc. Consider an equation $x^2 - px - q = 0, p$ and q are positive integers whose positive solution

$$\sigma_p^q = \frac{p + \sqrt{p^2 + 4q}}{2}$$

is called metallic means family. The polynomial structure of degree 2 that is the general quadratic equation on M given by

$$J^2 = pJ + qI, \tag{1}$$

where J is a tensor field of type (1,1), I is an identity tensor field and p and q are positive integers. Then J is called metallic structure which is a particular case of the polynomial structure of degree n introduced by Goldberg and Yano and Goldberg and Petridis [14, 15]. Hretreanu and Crasmareanu [17] initiated the study of metallic structure on Riemannian manifolds and established a necessary and sufficient condition for a submanifold to be a metallic Riemannian manifold. Various structures of metallic Riemannian manifolds for curvature have been studied by Blaga and Hretreanu [3, 4, 19]. Recently, the author [24] studied the frame bundle endowed with the metallic structure on an almost contact metric manifold and investigated that the diagonal lift g^D of a Riemannian metric g is a metallic Riemannian metric on FM. Many authors have studied the geometry of metallic structures [1, 8, 10, 16, 18–20, 22, 31].

The major focus of this work can be summarized as follows:

- The diagonal lift of a metallic structure is also a metallic structure on the frame bundle of the second order F^2M.
- The proposed theorem proves that the diagonal lift G^D of a Riemannian metric G is a metallic Riemannian metric on F^2M.
- Define a new tensor field $\tilde{J}$ of type (1,1) on F^2M and proves that it is a metallic structure.
- Some results on the 2-form, the derivative of 2-form, the Nijenhuis tensor $N_{\tilde{J}}$ of a metallic structure $\tilde{J}$ and the Nijenhuis tensor N_{J^D} of a tensor field J^D of type (1,1) on the frame bundle of the second order F^2M are calculated.

The structure of the remaining paper is organized as.

Section 2 focuses on definitions of the frame bundle FM and the frame bundle of the second order F^2M, the diagonal lift of tensor fields of type (1,1) and (0,2) on F^2M. In Section 3, the diagonal lift of the metallic structure J on the frame bundle of the second order F^2M is studied and shows that it is also a metallic structure. The proposed theorem proves that the diagonal lift G^D of G is a metallic Riemannian metric on F^2M. Also, a new tensor field $\tilde{J}$ of type (1,1) is defined and proves that it is a metallic structure. The 2-form and its derivative dF of a tensor field $\tilde{J}$ are determined. In Section 4, the Nijenhuis tensor $N_{\tilde{J}}$ of a metallic structure $\tilde{J}$ and the Nijenhuis tensor N_{J^D} of a tensor field J^D of type (1,1) on the frame bundle of the second order F^2M are calculated.

2. Preliminaries

Let $C^\infty(\mathfrak{R}^n)$ be the algebra of C^∞ functions on the Eulidean space $\mathfrak{R}^n$ whose coordinates are $(x^1, x^2, ..., x^n)$ and f and g be two elements of $C^\infty(\mathfrak{R}^n)$. Let M be an n-dimensional manifold and U and V be two neighborhoods at $x = 0 \in \mathfrak{R}^n$. The mappings $f : U \to M$ and $g : V \to M$ are said to define same r-jet if

$$\begin{aligned} f(0) &= g(0), \\ \left(\frac{\partial}{\partial x^\alpha}\right)(f) &= \left(\frac{\partial}{\partial x^\alpha}\right)(g), \\ \left(\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\right)(f) &= \left(\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\right)(g), \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ \left(\frac{\partial^r}{\partial x^\alpha \partial x^\beta \ldots . \partial x^\gamma}\right)(f) &= \left(\frac{\partial^r}{\partial x^\alpha \partial x^\beta \ldots . \partial x^\gamma}\right)(g), \end{aligned}$$

at $0 \in \mathfrak{R}^n$ for all $\alpha, \beta, \gamma = 1, 2, \ldots, p$.

"If f is a diffeomorphism of a neighborhood U of 0 onto a open subset of M then the r-jet $j^r f$ at $x = 0$ is called an r-frame at $x = f(0)$. The set of the r-frames of M is a principal bundle over M with projection π^r such that $\pi^r(j^r f) = f(0)$ and denoted by $F^r M$. If g is a diffeomorphism of a neighborhood V of 0 onto a open subset of M, then set of r-frames of M denoted by $G^r(n)$" [6, 11].

Let $j^r g$ and $j^r g'$ be two elements of $G^r(n)$ such that $(j^r g)(j^r g') = j^r(g \circ g')$. Then $G^r(n)$ is a Lie group with multiplication defined by the composition of jets. Let $j^r f$ and $j^r g$ be elements of $F^r M$ and $G^r(n)$. Then the multiplication of $j^r f$ and $j^r g$ defined as

$$(j^r f)(j^r g) = j^r(f \circ g).$$

Consider the frame bundle of the second order F^2M such that the canonical projection $\pi_1^2 : F^2M \to FM$ and $\pi_1^2(j^2 f) = j' f$ and the base M is covered by a system of coordinate neighborhood (U, x^i), where (x^i) is a local coordinate system defined in the neighborhood U. Thus $\{FU, (x^i, X^i_j)\}$ and $\{F^2U, (x^i, X^i_{jk})\}$ are the induced coordinate systems in FM and F^2M with groups $G'(n) = Gl(n)$ and $G^2(n)$, respectively. Where X^i_j and X^i_{jk} are local components of the vector X_j of the 1-frame and 2-frame respectively and $X^i_{jk} = X^i_{kj}$.

Let $g^2(n)$ be the Lie algebra of group $G^2(n)$ and (A, α) be an element of $g^2(n)$. Then a vector $\lambda(A, \alpha)$ on F^2M is called the fundamental vector field corresponding to (A, α), where $A \in gl(n)$ and $\alpha \in S^2(n), S^2(n)$ is the set of symmetric bilinear forms on $\mathfrak{R}^n$ such that $G^2(n) = gl(n) \times S^2(n)$ [6].

Definition 2.1 A coonection Γ in the bundle F^2M of 2-frames of M is called a connection of order 2 on M [6].

Definition 2.2 A curvature form Ω of Γ is a tensorial 2-form on F^2M of type $Ad(G^2(n))$ and given as

$$\Omega = \Omega_0 + \Omega_1,$$

where Ω_0 is a $gl(n)$ valued and Ω_1 is a $S^2(n)$ valued 2-form of F^2M [6].

2.1. Vector field of F^2M

Let F^2M be the frame bundle of the second order with the second order connection Γ on M. Let X be a vector field on M and X^H its horizontal lift. Then

$$X = X^i \frac{\partial}{\partial x^i}$$

and

$$X^H = X^i \left(\frac{\partial}{\partial x^i} - \Gamma^r_{il} x^l_j \frac{\partial}{\partial x^r_j} - (\Gamma^r_{il} x^l_{jk} + \Gamma^r_{ilm} x^l_j x^m_k) \right),$$

where Γ^r_{il} and Γ^r_{ilm} are the components of Γ on M [6].

Proposition 2.1. *Let F^2M be the frame bundle of the second order and X^H be the horizontal lift of X then*

$$[X^H, Y^H] = [X, Y] - \lambda(R(X, Y)^\circ) - 2\lambda\Omega_1(X^H, Y^H), \tag{2}$$

for all vector fields X, Y on M [6].

2.2. Diagonal lifts of tensor fields of type (1,1) and 1-form

Let F and τ be a tensor field of type (1,1) and 1-form, respectively on M. The diagonal lifts F^D and τ^D on F^2M are given by [6]

$$F^D = F^h_j D_h \otimes \eta^j + \delta^i_j F^h_k D^i_h \otimes \eta^k_j + \delta^i_j \delta^k_l F^h_m D^h_{jl} \otimes \eta^m_{ik}, \tag{3}$$

$$\tau^D = \tau_i \eta^i + \sum_{i=1}^{n} \tau_i \eta^i_j + \sum_{j,k=1}^{n} \tau_i \eta^i_{jk}, \tag{4}$$

where

$$\begin{aligned}
\eta^i &= dx^i, \\
\eta^i_j &= \Gamma^i_{rs} x^s_j dx^r + dx^i_j, \\
\eta^i_{jk} &= \{(\Gamma^i_{lr} x^r_{jk} + \Gamma^i_{lrs} x^r_j x^s_k) - y^s_m \Gamma^m_{lr}(x^r_j x^i_{sk} + x^r_k x^i_{js})\} dx^l - y^r_l (\delta^{sj} x^i_{rk} + \delta^{sk} x^i_{jr} dx^l_s + dx^i jk, \\
D_i &= \frac{\partial}{\partial x^i} - \Gamma^k_{il} x^l_r \frac{\partial}{\partial x^k_r} - (\Gamma^s_{im} x^m_{rk} + \Gamma^s_{iml} x^m_r x^l_k) \frac{\partial}{\partial x^s_{rk}}, \\
D^j_k &= \frac{\partial}{\partial x^i_j} + y^r_j x^m_{rs} \frac{\partial}{\partial x^m_{is}} + y^r_j x^m_{sr} \frac{\partial}{\partial x^m_{si}}, \\
D^i_{jk} &= \frac{\partial}{\partial x^i_{jk}},
\end{aligned}$$

are local components of η and D in F^2U.

The diagonal and horizontal lifts have the following formulas [6]:

$$\begin{aligned}
F^D X^H &= (FX)^H, \\
F^D(\lambda f) &= \lambda(F^\circ f), \\
F^D(\lambda g) &= \lambda(F^\circ g), \\
F^D(\lambda A) &= \lambda(F^\circ A), \\
F^D(\lambda \alpha) &= \lambda(F^\circ \alpha),
\end{aligned} \tag{5}$$

for any vector field X on M, all $A \in gl(n), \alpha \in S^2(n) \forall f : F^2M \to gl(n)$ and $g : F^2M \to S^2(n)$.

2.3. Diagonal lifts of tensor fields of type (0,2)

Let G be a tensor field of type (0,2) on M. The diagonal lift G^D of G to F^2M is a tensor field of type (0,2) has local components

$$G^D = G_{ij} \eta^j \otimes \eta^j + \delta_{kl} G_{ij} \eta^i_j \otimes \eta^j_l + \delta_{km} \delta_{lr} G_{ij} \eta^i_{kl} \otimes \eta^j_{mr}. \tag{6}$$

If rank of G is r, then G^D has rank $r(1 + n + \frac{n(n+1)}{2})$.

Let G° be globally defined functions on F^2M, then

$$G^\circ(A, B) = \delta^{rs} A^k_r B^l_s x^i_k x^j_l G_{ij}, \tag{7}$$

$$G^\circ(\alpha, \beta) = \delta^{rs} \alpha^k_r \beta^l_s x^i_k x^j_l G_{ij}, \tag{8}$$

where $A = A^i_j E^j_i, B = B^i_j E^j_i \in gl(n), \alpha = \alpha^i_{jk} E^{jk}_i$ and $\beta = \beta^i_{jk} E^{jk}_i \in S^2(n)$ [6].

Proposition 2.2. *The following identities are given by*

$$\begin{aligned} G^D(\lambda A, \lambda B) &= G^\circ(A, B), \\ G^D(\lambda A, \lambda\beta) &= G^\circ(\lambda\beta, \lambda A) = 0, \\ G^D(\lambda A, X^H) &= G^D(X^H, \lambda A) = 0, \quad (9) \\ G^D(\lambda\alpha, X^H) &= G^D(X^H, \lambda\alpha) = 0, \\ G^D(\lambda\alpha, \lambda\beta) &= G^\circ(\alpha, \beta), \quad (10) \\ G^D(X^H, Y^H) &= \{G(X, Y)\}^V, \end{aligned}$$

where $f^V = f \circ \pi^2$ *[6, 11].*

3. Proposed theorems for the metallic structure on the frame bundle of the second order

In this section, the diagonal lift of the metallic structure J on the frame bundle of the second order F^2M is studied and shows that it is also a metallic structure. The proposed theorem proves that the diagonal lift G^D of G is a metallic Riemannian metric on F^2M. Also, a new tensor field $\tilde{J}$ of type (1,1) is defined and proves that it is a metallic structure. The 2-form and its derivative dF of a tensor field $\tilde{J}$ are determined.

Let M be an n-dimensional manifold of class C^∞ and J be a tensor field of type (1,1) and I an identity tensor field on M. Then J satisfies

$$J^2 - pJ - qI = 0, \tag{11}$$

where p, q are positive integers and $\sigma_p^q = \frac{1}{2}(p + \sqrt{p^2 + 4q})$ is its positive solution. The tensor field J referred to as a metallic structure on M and (M, J) referred to as a metallic manifold [1, 21, 23].

Let G be a Riemannian metric on M such that

$$G(JX, Y) = G(X, JY), \tag{12}$$

or equivalent

$$G(JX, JY) = pG(X, JY) + qG(X, Y), \tag{13}$$

for all X and Y are vector fields on M. Then (M, J, G) is said to be a metallic Riemannian manifold.

The Nijenhuis tensor N_J of a metallic structure J is given by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \tag{14}$$

where X and Y are vector fields on M. The metallic structure J is called integrable if $N_J(X, Y) = 0$ [37, 38].

Now, state the following propositions [11]:

Proposition 3.1. *Let J and K be tensor fields of type (1,1) on M. Then*

$$\begin{aligned} (i)\ (JK)^D &= J^D K^D \quad (15) \\ (ii)\quad I^D &= I, \quad (16) \end{aligned}$$

where I is an identity tensor field.

Proposition 3.2. *If $P(t)$ is a polynomial of variable t, then*

$$(P(J)^D) = P(J^D), \tag{17}$$

where J is a tensor field of type (1,1) on M.

Theorem 3.3. *Let J be a tensor field of type (1,1) on the manifold M and F^2M its frame bundle of second order. Then the diagonal lift J^D of J is a metallic structure on F^2M if J is a metallic structure on M.*

Proof. Let J be a metallic structure. Applying diagonal lift on equation (11) and using Proposition (3.1) and Proposition (3.2), the obtained equation is

$$\begin{aligned} (J^2 - pJ - qI)^D &= 0, \\ (J^2)^D - qJ^D - qI^D &= 0, \\ (J^D)^2 - qJ^D - qI &= 0, \quad as \quad I^D. \end{aligned} \tag{18}$$

This shows that J^D is a metallic structure on F^2M.

Theorem 3.4. *Let F^2M its frame bundle of second order of the manifold M admits a tensor field J of type (1,1) given by equation (11), then the diagonal lift G^D of G is a metallic Riemannian metric on F^2M i.e.*

$$G^D(J^D\tilde{X}, J^D\tilde{Y}) = pG^D(\tilde{X}, J^D\tilde{Y}) + qG^D(\tilde{X}, \tilde{Y})$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on F^2M.

Proof. In order to prove that the diagonal lift G^D of G is a metallic Riemannian metric on F^2M. It is enough to check the identity

$$G^D(J^D\tilde{X}, J^D\tilde{Y}) = pG^D(\tilde{X}, J^D\tilde{Y}) + qG^D(\tilde{X}, \tilde{Y}) \tag{19}$$

in the following three cases:
(i) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^H$ in equation (19). Using equation (13) and Proposition (2.3), the obtained equation is

$$\begin{aligned} G^D(J^D X^H, J^D Y^H) &= G^D((JX)^H), (JY)^H), \\ &= (G(JX, JY))^V, \\ &= (pG(X, JY) + qG(X, Y))^V, \\ &= p(G(X, JY))^V + q(G(X, Y))^V, \\ &= pG^D(X^H, (JY)^H) + qG(X^H, Y^H). \end{aligned}$$

(ii) Setting $\tilde{X} = X^H$ and $\tilde{Y} = \lambda(A, \alpha)$ in equation (19). Using equation (13), Proposition (2.2) and Proposition (2.3), the obtained equation is

$$G^D(J^D X^H, J^D\lambda(A, \alpha)) = G^D((JX)^H, \lambda J^\circ(A, \alpha)) = 0.$$

(iii) Setting $\tilde{X} = \lambda(A, \alpha)$ and $\tilde{Y} = \lambda(B, \beta)$ in equation (19). Using equation (13), Proposition (2.2) and Proposition (2.3), the obtained equation is

$$\begin{aligned} G^D(J^D\lambda(A, \alpha), J^D\lambda(B, \beta)) &= G^D(\lambda J^\circ(A, \alpha), \lambda J^\circ(B, \beta)), \\ &= G^\circ(J^\circ(A, \alpha), J^\circ(B, \beta)) \\ &= pG^\circ((A, \alpha), J^\circ(B, \beta)) + qG^\circ((A, \alpha), (B, \beta)) \\ &= pG^D(\lambda(A, \alpha), J^D\lambda(B, \beta)) + qG^D(\lambda(A, \alpha), \lambda(B, \beta)), \end{aligned}$$

for vector fields X, Y on M, $A, B \in gl(n)$ and $\alpha, \beta \in S^2(n)$.

Let M be an n-dimensional manifold and F^2M its frame bundle of second order. Let X^H and λA be vector fileds on F^2M with respect to the second order connection Γ.

In [6], Cordero et al defined a tensor field $F_\alpha, \alpha = 1, 2, ..., n$ of type (1,1) on the frame bundle FM as

$$F_\alpha X^H = -X^{(\alpha)}, \quad F_\alpha X^{(\beta)} = \delta_\alpha^\beta X^H, \tag{20}$$

where X^H and $X^{(\alpha)}$ are 'the horizontal' and 'α−th vertical' lifts of a vector field X on M. It is proved that $F_\alpha^3 + F_\alpha = 0$ i.e. F-structure.

Also, Gezer and Kamran [13] defined a tensor field $\tilde{J}$ of type (1,1) on the tangent bundle TM of M by

$$\begin{aligned}\tilde{J}X^H &= \frac{1}{2}(\alpha X^H + (2\sigma_\alpha^\beta - \alpha)(X \otimes \tilde{E}^V),\\ \tilde{J}X^V &= \frac{1}{2}(\alpha(X \otimes \tilde{E}^V + (2\sigma_\alpha^\beta - \alpha)X^H),\\ \tilde{J}A^V &= \sigma_\alpha^\beta A^V,\end{aligned}$$

for any vector field X, tensor field A of type (1,1), $\tilde{E} = g \circ E$ and g a Riemannian metric on M. It is proved that $\tilde{J}$ is a metallic structure on TM.

From Cordero et al [6] and Gezer and Kamran [13], a tensor field $\tilde{J}$ of type (1,1) on F^2M is introduced as

$$\tilde{J}X^H = \frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda A\}, \tag{21}$$

$$\tilde{J}\lambda A = \frac{1}{2}\{p\lambda A + (2\sigma_p^q - p)X^H\}, \tag{22}$$

where $\sigma_p^q = \frac{p+\sqrt{p^2+4q}}{2}$.

Theorem 3.5. *Let F^2M be the frame bundle of the second order of the manifold M. Then the tensor field $\tilde{J}$ defined by equations (21) and (22) is a metallic structure on F^2M.*

Proof. In order to prove the tensor field $\tilde{J}$ is a metallic structure. It is enough to prove that $\tilde{J}^2 - p\tilde{J} - qI = 0$.

Let X^H and λA be vector fields on F^2M. Making use of equations (21) and (22), then

$$\begin{aligned}(\tilde{J}^2 - p\tilde{J} - qI)X^H &= \tilde{J}(\tilde{J}X^H) - p\tilde{J}X^H - qX^H\\ &= \tilde{J}\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda A\} - \frac{p}{2}\{pX^H + (2\sigma_p^q - p)\lambda A\} - qX^H\\ &= \frac{p}{2}\tilde{J}X^H + \frac{(2\sigma_p^q - p)}{2}\tilde{J}\lambda A\} - \frac{p^2}{2}X^H - \frac{p(2\sigma_p^q - p)}{2}\lambda A\} - qX^H\\ &= 0.\end{aligned}$$

Similarly, $(\tilde{J}^2 - p\tilde{J} - qI)\lambda A = 0$. This shows that $\tilde{J}^2 - p\tilde{J} - qI = 0$.

Hence, $\tilde{J}$ is a metallic structure on F^2M.

Let G^D be a metallic Riemannian metric and $\tilde{J}$ be a metallic structure on F^2M defined by equations (21) and (22). Let F be the 2-form on F^2M is given by

$$F(\tilde{X}, \tilde{Y}) = G^D(\tilde{X}, \tilde{J}\tilde{Y}). \tag{23}$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on F^2M.

Theorem 3.6. *Let F^2M be the frame bundle of the second order of the manifold M admits a metallic structure $\tilde{J}$ defined by equations (21) and (22), then 2-form F on F^2M given by*

$$\begin{aligned}&(i) & F(X^H, Y^H) &= \frac{p}{2}(G^D(X, Y))^V,\\ &(ii) & F(X^H, \lambda(A, \alpha)) &= \frac{(2\sigma_p^q - p)}{2}(G^D(X, Y))^V,\\ &(iii) & F(\lambda(A, \alpha), \lambda(B, \beta)) &= \frac{p}{2}(G^\circ(X, Y))^V.\end{aligned}$$

for all vector fields X on M, all $A \in gl(n), \alpha \in S^2(n)$.

Proof. (i) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^H$ in equation (23). Using equations (21) and (22) and Proposition (2.2), then

$$\begin{aligned} F(X^H, Y^H) &= G^D(X^H, \tilde{J}Y^H), \\ &= G^D(X^H, \frac{1}{2}\{pY^H + (2\sigma_p^q - p)\lambda A\}), \\ &= \frac{p}{2}G^D(X^H, Y^H) + \frac{(2\sigma_p^q - p)}{2}G^D(X^H, \lambda A), \\ &= \frac{p}{2}(G^D(X,Y))^V, \ \ as \ \ G^D(X^H, \lambda(A,\alpha) = 0. \end{aligned}$$

(ii) Setting $\tilde{X} = X^H$ and $\tilde{Y} = \lambda(A,\alpha)$ in equation (23). Using equations (21) and (22) and Proposition (2.2), then

$$\begin{aligned} F(X^H, \lambda(A,\alpha)) &= G^D(X^H, \tilde{J}\lambda(A,\alpha)) \\ &= G^D(X^H, \frac{1}{2}\{p\lambda(A,\alpha) + (2\sigma_p^q - p)Y^H\}) \\ &= \frac{p}{2}G^D(X^H, \lambda(A,\alpha)) + \frac{(2\sigma_p^q - p)}{2}G^D(X^H, Y^H), \\ &= \frac{(2\sigma_p^q - p)}{2}(G^D(X,Y))^V. \end{aligned}$$

(iii) Setting $\tilde{X} = \lambda(A,\alpha)$ and $\tilde{Y} = \lambda(B,\beta)$ in equation (23). Using equations (21) and (22) and Proposition (2.2), then

$$\begin{aligned} F(\lambda(A,\alpha), \lambda(B,\beta)) &= G^D(\lambda(A,\alpha), \tilde{J}\lambda(B,\beta)) \\ &= G^D(\lambda(A,\alpha), \frac{1}{2}\{p\lambda(B,\beta) + (2\sigma_p^q - p)Y^H\}) \\ &= \frac{p}{2}G^D(\lambda(A,\alpha), \lambda(B,\beta)) + \frac{(2\sigma_p^q - p)}{2}G^D(\lambda(A,\alpha), Y^H), \\ &= \frac{p}{2}G^\circ(\lambda(A,\alpha), \lambda(B,\beta)), \end{aligned}$$

where λA is a fundamental vector on F^2M.

Theorem 3.7. *Let F^2M be the frame bundle of the second order of the manifold M and the diagonal lift G^D of G. Then the derivative of the 2-form F is given by*

$$\begin{aligned} (i) \quad 3dF(X^H, Y^H, Z^H) &= \frac{p}{2}\{(XG(Y,Z))^V - (G([Y,Z],X))^V - (YG(X,Z))^V \\ &- (G([X,Z],Y))^V + (ZG(X,Y))^V - (G([X,Y],Z))^V\} \\ &+ 2\{F(\lambda\Omega(X^H,Y^H),Z^H) + F(\lambda\Omega(X^H,Z^H),Y^H) + F(\lambda\Omega(Y^H,Z^H),X^H)\}, \\ (ii) \ 3dF(X^H, Y^H, \lambda(A,\alpha)) &= \frac{2\sigma_p^q - p}{2}(XG(Y,Z))^V - \frac{2\sigma_p^q - p}{2}(YG(X,Z))^V + \frac{p}{2}(\lambda(A,\alpha)G(X,Y))^V \\ &- \frac{2\sigma_p^q - p}{2}(G([X,Y],Z))^V + 2\lambda F(\Omega(X^H,Y^H), \lambda(A,\alpha)), \end{aligned}$$

$$
\begin{aligned}
(iii)\ \ 3dF(X^H,\lambda(A,\alpha),\lambda(B,\beta)) &= \frac{p}{2}X^H G^\circ(\lambda(A,\alpha)],\lambda(B,\beta)) + \lambda(B,\beta)(G(X,Z))^V \\
&- \frac{2\sigma_p^q - p}{2}\lambda(A,\alpha)(G(X,Z))^V - F(\lambda[(A,\alpha),(B,\beta)],X^H), \\
(iv)\ \ 3dF(\lambda(A,\alpha),\lambda(B,\beta),\lambda(C,\gamma)) &= \frac{p}{2}\{\lambda(A,\alpha)G^\circ((B,\beta),(C,\gamma)) - \lambda(B,\beta)G^\circ((A,\alpha),(C,\gamma)) \\
&+ \lambda(C,\gamma)G^\circ((A,\alpha),(B,\beta))\} - \frac{p}{2}\{G^\circ([(A,\alpha),(B,\beta)],(C,\gamma)) \\
&- G^\circ([(A,\alpha),(C,\gamma)],(B,\beta)) + G^\circ([(B,\beta),(C,\gamma)],(A,\alpha))\},
\end{aligned}
$$

for all vector fields X on M, all $A \in gl(n,\Re)$, $\alpha \in S^2(n)$.

Proof. "The derivative dF of the 2-form F is given by [24]

$$
3dF(\tilde{X},\tilde{Y},\tilde{Z}) = \tilde{X}(F(\tilde{Y},\tilde{Z})) - \tilde{Y}(F(\tilde{X},\tilde{Z})) + \tilde{Z}(F(\tilde{X},\tilde{Y})) - F([\tilde{X},\tilde{Y}],\tilde{Z}) + F([\tilde{X},\tilde{Z}],\tilde{Y}) - F([\tilde{Y},\tilde{Z}],\tilde{X}), \tag{24}
$$

where $\tilde{X},\tilde{Y},\tilde{Z}$ being arbitrary vector fields on F^2M".

(i) Setting $\tilde{X} = X^H$, $\tilde{Y} = Y^H$, $\tilde{Z} = Z^H$ in equation (24) and using Theorem (3.4), then

$$
\begin{aligned}
3dF(X^H,Y^H,Z^H) &= X^H(F(Y^H,Z^H)) - Y^H(F(X^H,Z^H)) + Z^H(F(X^H,Y^H)) - F([X^H,Y^H],Z^H) \\
&+ F([X^H,Z^H],Y^H) - F([Y^H,Z^H],X^H), \\
&= \frac{p}{2}\{(XG(Y,Z))^V - (YG(X,Z))^V + (ZG(X,Y))^V\} \\
&- \frac{p}{2}\{(G([X,Y],Z))^V + (G([X,Z],Y))^V + (G([Y,Z],X))^V\} \\
&+ 2\{F(\lambda\Omega(X^H,Y^H),Z^H) + F(\lambda\Omega(X^H,Z^H),Y^H) + F(\lambda\Omega(Y^H,Z^H),X^H)\} \\
&= \frac{p}{2}\{(XG(Y,Z))^V - (G([Y,Z],X))^V - (YG(X,Z))^V \\
&- (G([X,Z],Y))^V + (ZG(X,Y))^V - (G([X,Y],Z))^V\} \\
&+ 2\{F(\lambda\Omega(X^H,Y^H),Z^H) + F(\lambda\Omega(X^H,Z^H),Y^H) + F(\lambda\Omega(Y^H,Z^H),X^H)\}.
\end{aligned}
$$

(ii) Setting $\tilde{X} = X^H$, $\tilde{Y} = Y^H$, $\tilde{Z} = \lambda(A,\alpha)$ in equation (24) and using Theorem (3.4), then

$$
\begin{aligned}
3dF(X^H,Y^H,\lambda(A,\alpha)) &= X^H(F(Y^H,\lambda(A,\alpha))) - Y^H(F(X^H,\lambda(A,\alpha))) + \lambda(A,\alpha)(F(X^H,Y^H)) \\
&- F([X^H,Y^H],\lambda(A,\alpha)) + F([X^H,\lambda(A,\alpha)],Y^H) - F([Y^H,\lambda(A,\alpha)],X^H), \\
&= \frac{2\sigma_p^q - p}{2}(XG(Y,Z))^V - \frac{2\sigma_p^q - p}{2}(YG(X,Z))^V + \frac{p}{2}(\lambda(A,\alpha)G(X,Y))^V \\
&- F([X,Y]^H,\lambda(A,\alpha)) + 2\lambda F(\Omega(X^H,Y^H),\lambda(A,\alpha)), \\
&= \frac{2\sigma_p^q - p}{2}(XG(Y,Z))^V - \frac{2\sigma_p^q - p}{2}(YG(X,Z))^V + \frac{p}{2}(\lambda(A,\alpha)G(X,Y))^V \\
&- \frac{2\sigma_p^q - p}{2}(G([X,Y],Z))^V + 2\lambda F(\Omega(X^H,Y^H),\lambda(A,\alpha)).
\end{aligned}
$$

(iii) Setting $\tilde{X} = X^H$, $\tilde{Y} = \lambda(A,\alpha)$, and $\tilde{Z} = \lambda(B,\beta)$ in equation (24) and using Theorem (3.4), then

$$
\begin{aligned}
3dF(X^H,\lambda(A,\alpha),\lambda(B,\beta)) &= X^H(F(\lambda(A,\alpha),\lambda(B,\beta))) - \lambda(A,\alpha)(F(X^H,\lambda(B,\beta))) + \lambda(B,\beta)(F(X^H,\lambda(A,\alpha)) \\
&- F([X^H,\lambda(A,\alpha)],\lambda(B,\beta)) + F([X^H,\lambda(B,\beta)],\lambda(A,\alpha)) - F([\lambda(A,\alpha),\lambda(B,\beta)],X^H), \\
&= \frac{p}{2}X^H G^\circ(\lambda(A,\alpha)],\lambda(B,\beta)) - \frac{2\sigma_p^q - p}{2}\lambda(A,\alpha)(G(X,Z))^V \\
&+ \lambda(B,\beta)(G(X,Z))^V - F(\lambda[(A,\alpha),(B,\beta)],X^H).
\end{aligned}
$$

(iv) Setting $\tilde{X} = \lambda(A,\alpha)$, $\tilde{Y} = \lambda(B,\beta)$, $\tilde{Z} = \lambda(C,\gamma)$in equation (24) and using Theorem (3.4), then

$$\begin{aligned} 3dF(\lambda(A,\alpha),\lambda(B,\beta),\lambda(C,\gamma)) &= \lambda(A,\alpha)(F(\lambda(B,\beta),\lambda(C,\gamma))) - \lambda(B,\beta)(F(\lambda(A,\alpha)\lambda(C,\gamma))) \\ &+ \lambda(C,\gamma)(F(\lambda(A,\alpha),\lambda(B,\beta))) - F([\lambda(A,\alpha),\lambda(B,\beta)],\lambda(C,\gamma)) \\ &+ F([\lambda(A,\alpha),\lambda(C,\gamma)]\lambda(B,\beta)) - F([\lambda(B,\beta),\lambda(C,\gamma)],\lambda(A,\alpha)), \\ &= \frac{p}{2}\{\lambda(A,\alpha)G^\circ((B,\beta),(C,\gamma)) - \lambda(B,\beta)G^\circ((A,\alpha),(C,\gamma)) \\ &+ \lambda(C,\gamma)G^\circ((A,\alpha),(B,\beta))\} - \frac{p}{2}\{G^\circ([(A,\alpha),(B,\beta)],(C,\gamma)) \\ &- G^\circ([(A,\alpha),(C,\gamma)],(B,\beta)) + G^\circ([(B,\beta),(C,\gamma)],(A,\alpha))\}, \end{aligned}$$

for all vector fields X on M, all $A \in gl(n), \alpha \in S^2(n)$.

4. Some calculations for the Nijenhuis tensor on the frame bundle of the second order

In this section, the Nijenhuis tensor of a metallic structure $\tilde{J}$ on the frame bundle of the second order F^2M is calculated.

"Let $\tilde{X}$ and $\tilde{Y}$ be vector fields on F^2M and $N_{\tilde{J}}$ be the Nijenhuis tensor of a tensor field $\tilde{J}$ of type (1,1) is given by [24]

$$N_{\tilde{J}}(\tilde{X},\tilde{Y}) = [\tilde{J}\tilde{X},\tilde{J}\tilde{Y}] - \tilde{J}[\tilde{J}\tilde{X},\tilde{Y}] - \tilde{J}[\tilde{X},\tilde{J}\tilde{Y}] + \tilde{J}^2[\tilde{X},\tilde{Y}]." \tag{25}$$

Theorem 4.1. *Let F^2M be the frame bundle of the second order of the manifold M admits a metallic structure $\tilde{J}$ defined by (21) and (22), then the Nijenhuis tensor $N_{\tilde{J}}$ of $\tilde{J}$ is given by*

$$\begin{aligned} (i) \qquad N_{\tilde{J}}(X^H,Y^H) &= (\frac{3p^2}{4}+q)[X,Y]^H - \frac{p^2+4p+4q}{2}\lambda(A,\alpha) + \frac{p(2\sigma_p^q - p)}{2}\lambda(A,\alpha), \\ (ii) \quad N_{\tilde{J}}(X^H,\lambda(A,\alpha)) &= \frac{p(2\sigma_p^q - p)}{2}(p-\tilde{J})([X,Y]^H - 2\lambda\Omega(X^H,Y^H)), \\ (iii)\ N_{\tilde{J}}(\lambda(A,\alpha),\lambda(B,\beta)) &= \left(\frac{p(2\sigma_p^q - p)}{2}\right)^2([X,Y]^H - 2\lambda\Omega(X^H,Y^H)). \end{aligned}$$

Proof. (i) Setting $\tilde{X} = X^H$ and $\tilde{Y} = Y^H$ in equation (25). The equation (26) is obtained by applying equations (21) and (22), and Proposition (2.2).

$$\begin{aligned} N_{\tilde{J}}(X^H,Y^H) &= [\tilde{J}X^H,\tilde{J}Y^H] - \tilde{J}[\tilde{J}X^H,Y^H] - \tilde{J}[X^H,\tilde{J}Y^H] + \tilde{J}^2[X^H,Y^H], \\ &= [\tilde{J}\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda(A,\alpha)\},\tilde{J}\frac{1}{2}\{pY^H + (2\sigma_p^q - p)\lambda(B,\beta)\}] \\ &- \tilde{J}[\tilde{J}\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda(A,\alpha)\},\frac{1}{2}\{pY^H + (2\sigma_p^q - p)\lambda(B,\beta)\}] \\ &- \tilde{J}[\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda(A,\alpha)\},\tilde{J}\frac{1}{2}\{pY^H + (2\sigma_p^q - p)\lambda(B,\beta)\}] \\ &+ \tilde{J}^2[\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda(A,\alpha)\},\frac{1}{2}\{pY^H + (2\sigma_p^q - p)\lambda(B,\beta)\}], \end{aligned}$$

$$N_{\tilde{J}}(X^H,Y^H) = (\frac{3p^2}{4}+q)[X,Y]^H - \frac{p^2+4p+4q}{2}\lambda(A,\alpha) + \frac{p(2\sigma_p^q - p)}{2}\lambda(A,\alpha). \tag{26}$$

(ii) Setting $\tilde{X} = X^H$ and $\tilde{Y} = \lambda(A,\alpha)$ in equation (25). The equation (27) is obtained by applying equations

(21), (22), (5), Proposition (2.1), and Proposition (2.2).

$$\begin{aligned}
N_{\tilde{J}}(X^H, \lambda(A,\alpha)) &= [\tilde{J}X^H, \tilde{J}\lambda(A,\alpha)] - \tilde{J}[\tilde{J}X^H, \lambda(A,\alpha)] - \tilde{J}[X^H, \tilde{J}\lambda(A,\alpha)] + \tilde{J}^2[X^H, \lambda(A,\alpha)],\\
&= [\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda(A,\alpha)\}, \frac{1}{2}\{p\lambda(B,\beta)\} + (2\sigma_p^q - p)Y^H]\\
&- \tilde{J}[\frac{1}{2}\{pX^H + (2\sigma_p^q - p)\lambda(A,\alpha)\}, \lambda(B,\beta)\}] - \tilde{J}[X^H, \frac{1}{2}\{p\lambda(A,\alpha) + (2\sigma_p^q - p)Y^H\}],\\
&+ \tilde{J}^2[X^H, \lambda(A,\alpha)],\\
&= \frac{p(2\sigma_p^q - p)}{2}(p - \tilde{J})([X,Y]^H - 2\lambda\Omega(X^H, Y^H)). \qquad (27)
\end{aligned}$$

(iii) Setting $\tilde{X} = \lambda(A,\alpha)$ and $\tilde{Y} = \lambda(B,\beta)$ in equation (25). The equation (28) is obtained by applying equations (21), (22), (5), Proposition (2.1), and Proposition (2.2).

$$\begin{aligned}
N_{\tilde{J}}(\lambda(A,\alpha), \lambda(B,\beta)) &= [\tilde{J}\lambda(A,\alpha), \tilde{J}\lambda(B,\beta)] - \tilde{J}[\tilde{J}\lambda(A,\alpha), \lambda(B,\beta)] - \tilde{J}[\lambda(A,\alpha), \tilde{J}\lambda(B,\beta)] + \tilde{J}^2[\lambda(A,\alpha), \lambda(B,\beta)],\\
&= [\frac{1}{2}\{p\lambda(A,\alpha) + (2\sigma_p^q - p)X^H\}, \frac{1}{2}\{p\lambda(A,\beta) + (2\sigma_p^q - p)Y^H\}]\\
&- \tilde{J}[\frac{1}{2}\{p\lambda(A,\alpha) + (2\sigma_p^q - p)X^H\}, \lambda(B,\beta)] - \tilde{J}[\lambda(A,\alpha), \frac{1}{2}\{p\lambda(A,\beta) + (2\sigma_p^q - p)Y^H]\\
&+ \tilde{J}^2[\lambda(A,\alpha), \lambda(B,\beta)],\\
&= \left(\frac{p(2\sigma_p^q - p)}{2}\right)^2([X,Y]^H - 2\lambda\Omega(X^H, Y^H)). \qquad (28)
\end{aligned}$$

Theorem 4.2. *Let F^2M be the frame bundle of the second order of the manifold M admitting a metallic structure J. The Nijenhuis tensor N_{J^D} of the diagonal lift J^D of J on F^2M is given by*

$$\begin{aligned}
(i)N_{J^D}(X^H, Y^H) &= (N_J(X,Y))^H - \lambda((R(JX,JY) - JR(JX,Y) - JR(X,JY)\\
&+ J^2R(X,Y))^\circ) - 2\lambda(\Omega_1((JX)^H, (JY)^H) - J^\circ\Omega_1((JX)^H, Y^H)\\
&- J^\circ\Omega_1(X^H, (JY)^H) + (J^2)^\circ\Omega_1(X^H, Y^H),\\
(ii)N_{J^D}(X^H, \lambda B) &= \lambda((\nabla_{JX}J - J\nabla_J J)^\circ B),\\
(iii)N_{J^D}(X^H, \lambda\beta) &= \lambda((\nabla_{JX}J - J\nabla_J J)^\circ\beta),\\
(iv)N_{J^D}(\lambda A, \lambda B) &= N_{J^D}(\lambda A, \lambda\beta) = N_{J^D}(\lambda\alpha, \lambda\beta) = 0,
\end{aligned}$$

for any vector fiels X, Y on M, any $A, B \in gl(n)$ and any $\alpha, \beta \in S^2(n)$.

Proof. Let $\tilde{X}$ and $\tilde{Y}$ be vector fields and N_{J^D} be the Nijenhuis tensor of of the diagonal lift J^D of J on F^2M is given by

$$N_{J^D}(\tilde{X}, \tilde{Y}) = [J^D\tilde{X}, J^D\tilde{Y}] - J^D[J^D\tilde{X}, \tilde{Y}] - J^D[\tilde{X}, J^D\tilde{Y}] + (J^D)^2[\tilde{X}, \tilde{Y}], \qquad (29)$$

for all vector fields X on M, all $A \in gl(n), \alpha \in S^2(n)$.

(i) Setting $\tilde{X} = X^H, \tilde{Y} = Y^H$ in (29) and using equation (5) and Proposition 2.1, we get

$$\begin{aligned}
N_{J^D}(X^H, Y^H) &= [J^DX^H, J^DY^H] - J^D[J^DX^H, Y^H] - J^D[X^H, J^DY^H] + (J^D)^2[X^H, Y^H],\\
&= [(JX)^H, (JY)^H] - J^D[(JX)^H, Y^H] - J^D[X^H, (JY)^H] + (J^D)^2[X^H, Y^H],\\
&= (N_J(X,Y))^H - \lambda((R(JX,JY) - JR(JX,Y) - JR(X,JY)\\
&+ J^2R(X,Y))^\circ) - 2\lambda(\Omega_1((JX)^H, (JY)^H) - J^\circ\Omega_1((JX)^H, Y^H)\\
&- J^\circ\Omega_1(X^H, (JY)^H) + (J^2)^\circ\Omega_1(X^H, Y^H).
\end{aligned}$$

Similarily, the proof of Theorem (4.2) i, ii, iii are obtained by applying equation (5) and Proposition 2.1.

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