Filomat 36:13 (2022), 4429–4441
https://doi.org/10.2298/FIL2213429Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Drazin Invertibility of the Sum of Two Elements in a Banach Algebra

Xiaolan Qin[a], Linzhang Lu[a,b]

[a] *School of Mathematical Science, Guizhou Normal University, P. R. China*
[b] *School of Mathematical Sciences, Xiamen University, P. R. China*

Abstract. In this paper, we study additive properties of the generalized Drazin inverse in a Banach algebra. We first show that $a + b \in \mathcal{A}^d$ under the condition that $a, b \in \mathcal{A}^d$, $aba^\pi = \lambda a^\pi bab^\pi a^\pi$, and then give some explicit expressions for the generalized Drazin inverse of the sum $a + b$ under some weaker conditions than those used in the previous papers. Some known results are extended.

1. Introduction

Generalized inverses of block matrices have important applications in automatics, probability, statistics, mathematical programming, numerical analysis, game theory, econometrics, control and so on [3, 4]. The Drazin inverse has applications in a number of areas such as control theory, Markov chains, singular differential and difference equations, iterative methods in numerical linear algebra. In 1979, Campbell and Meyer proposed the problem of finding a formula for the Drazin inverse of a 2×2 block matrix in terms of its various blocks, where the blocks on the diagonal are required to be square matrices [4]. At the present time, there is no known complete solution to this problem. In [13], Koliha extended the Drazin invertibility in the setting of Banach algebras with applications to bounded linear operators on a Banach space and deduced a formula for the generalized Drazin inverse of $a+b$ when $ab = ba = 0$. Later, Djordjević and Wei [12] gave the expression of $(a+b)^d$ under the assumption $ab = 0$ in the context of the Banach algebra of all bounded linear operators on an arbitrary complex Banach space. In [5], Castro-Gonzálezz and Koliha obtained a formula for $(a + b)^d$ under the conditions $a^\pi b = b, ab^\pi = a, b^\pi aba^\pi = 0$, which are weaker than $ab = 0$ in Banach algebras. In [9], Deng and Wei derived necessary and sufficient conditions for the existence of $(P + Q)^d$ under the assumption $PQ = QP$, where P, Q are bounded linear operators, and gave the expression of $(P + Q)^d$. In [8], Cvetković-Ilić et al. extended the result of [9] to Banach algebras. In [2], Benítez et al. investigated the explicit expressions for $(a + b)^d$ under the conditions $b^\pi a^\pi ba = 0$, $b^\pi aa^d baa^d = 0$ and $ab^\pi = a$, where a and b are generalized Drazin invertible in a unital Banach algebra. Using the assumption $aba^\pi = a^\pi b^\pi bab^\pi a^\pi$, a representation for $(a + b)^d$ was presented in [20]. More results on generalized Drazin inverse can be found in [11, 14, 16–19, 21].

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 16U80
Keywords. Banach algebra, generalized Drazin inverse, additive result
Received: 08 June 2021; Revised: 05 March 2022; Accepted: 09 July 2022
Communicated by Dijana Mosić
Corresponding author: Linzhang Lu
Research supported by NNSF of China (12161020).
Email addresses: qinxiaolan18@163.com (Xiaolan Qin), llz@gznu.edu.cn (Linzhang Lu)

In [15], Liu and Qin deduced the explicit expression for the generalized Drazin inverse of the sum $a + b$ under the condition $ab = a^{\pi}bab^{\pi}$, where a and b are generalized Drazin invertible in a complex Banach algebra. In [6], the corresponding results of [15] were further studied. In this paper, we extend these results and establish an explicit representation of the generalized Drazin inverse $(a+b)^d$ under the condition $aba^{\pi} = \lambda a^{\pi}bab^{\pi}a^{\pi}$. And we also give several representations for the generalized Drazin inverse of $a + b$ under some new conditions.

The paper is organized as follows. In section 2, we give some definitions and lemmas that are needed for our results. In section 3, we generalize some results of [15] and [6]. In section 4, we generalize some results of [7].

2. Preliminaries

Throughout this paper, $\mathcal{A}$ denotes a complex Banach algebra with the unit $\mathbb{1}$. λ stands for a nonzero complex number. For any $b \in \mathcal{A}$, the spectral radius of b is defined as $r(b) = \lim_{n\to\infty} \|b^n\|^{1/n}$ (see e.g. [1, Ch. 1]), or $r(b) = \max\{|\mu| : \mu \in \sigma(b)\}$, where $\sigma(b)$ is the spectrum of b, i.e., the set composed of complex numbers μ such that $b-\mu\mathbb{1}$ is not invertible. In a Banach algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is called quasinilpotent if $r(a) = 0$. In the following, $\mathcal{A}^{-1}$ and $\mathcal{A}^{qnil}$ denote the sets of all invertible and quasinilpotent elements in $\mathcal{A}$, respectively.

Let us recall that a generalized Drazin inverse of $a \in \mathcal{A}$ (introduced by Koliha in [13]) is an element $x \in \mathcal{A}$ which satisfies

$$xax = x, \qquad ax = xa, \qquad a - a^2x \in \mathcal{A}^{qnil}. \tag{2.1}$$

It has been proved in [13] that for any $a \in \mathcal{A}$, the set of $x \in \mathcal{A}$ satisfying (2.1) is empty or a singleton. If this set is a singleton, then we say that a is generalized Drazin invertible and x satisfying (2.1) is denoted by a^d. The set $\mathcal{A}^d$ consits of all $a \in \mathcal{A}$ such that a^d exists. For a complete treatment of the generalized Drazin inverse, see [10, Ch. 2].

Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be an idempotent. We denote $\overline{p} = \mathbb{1} - p$. Then we can write

$$a = pap + pa\overline{p} + \overline{p}ap + \overline{p}a\overline{p}.$$

Every idempotent $p \in \mathcal{A}$ induces a representation of an arbitrary element $a \in \mathcal{A}$ given by the following matrix:

$$a = \begin{bmatrix} pap & pa\overline{p} \\ \overline{p}ap & \overline{p}a\overline{p} \end{bmatrix}_p.$$

Let $a \in \mathcal{A}^d$ and $a^{\pi} = \mathbb{1} - aa^d$ be the spectral idempotent of a corresponding to 0. It is well known that $a \in \mathcal{A}$ can be represented in the following matrix form [10, Ch. 2]

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \tag{2.2}$$

where $p = aa^d$, a_1 is invertible in the algebra $p\mathcal{A}p$, a^d is its inverse in $p\mathcal{A}p$, and a_2 is quasinilpotent in the algebra $\overline{p}\mathcal{A}\overline{p}$. Thus, the generalized Drazin inverse of a can be expressed as

$$a^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Obviously, if $a \in \mathcal{A}^{qnil}$, then a is generalized Drazin invertible and $a^d = 0$.

The motivation for this article comes from [5, 6, 15]. In these papers, the authors considered some conditions on $a, b \in \mathcal{A}$ that allowed them to express $(a + b)^d$ in terms of a, a^d, b, b^d. Our aim in this paper is to consider the additive properties for the generalized Drazin inverse of the sum $a + b$ and give an explicit expression for $(a + b)^d$ under some new conditions.

The following several lemmas are needed for deriving our results. The first one was proved in [17] for matrices, has been extended to bounded linear operators in [11] and Banach algebra elements in [5].

Lemma 2.1. [5, Theorem 2.3] *Let $x, y \in \mathcal{A}$, and $p \in \mathcal{A}$ be an idempotent. Assume that x and y are represented as*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}_p, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}_p.$$

(i) *If $a \in (p\mathcal{A}p)^d$ and $b \in (\overline{p}\mathcal{A}\overline{p})^d$, then x and y are generalized Drazin invertible, and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix}_p, \qquad y^d = \begin{bmatrix} b^d & u \\ 0 & a^d \end{bmatrix}_p, \tag{2.3}$$

where

$$u = \sum_{n=0}^{\infty}(b^d)^{n+2}ca^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c(a^d)^{n+2} - b^d c a^d. \tag{2.4}$$

(ii) *If $x \in \mathcal{A}^d$ and $a \in (p\mathcal{A}p)^d$, then $b \in (\overline{p}\mathcal{A}\overline{p})^d$, and x^d and y^d are given by* (2.3) *and* (2.4).

Lemma 2.2. *(*[9, Theorem 1], [8, Theorem 2.1]*) Let $a, b \in \mathcal{A}^d$. If $ab = ba$, then $a + b \in \mathcal{A}^d$ if and only if $1 + a^d b \in \mathcal{A}^d$. In this case, we have*

$$(a+b)^d = a^d(1+a^d b)^d bb^d + b^\pi \sum_{n=0}^{\infty}(-b)^n (a^d)^{n+1} + \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi.$$

Lemma 2.3. [6, Lemma 2.3] *Let $a \in \mathcal{A}^{qnil}$, $b \in \mathcal{A}^d$. If $ab = \lambda bab^\pi$ and $\lambda \neq 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n.$$

3. Main result 1

Now we start the first of our main results, which is a generalization of [15, Theorem 4] and [6, Theorem 2.4].

Theorem 3.1. *Let $a, b \in \mathcal{A}^d$, $aba^\pi = \lambda a^\pi bab^\pi a^\pi$ and $\lambda \neq 0$. If $a^\pi ba^\pi$ (or $a^\pi b$ or ba^π or $aa^d baa^d$ or $aa^d b$ or baa^d) is generalized Drazin invertible, then the following conditions are equivalent:*
(i) $a + b \in \mathcal{A}^d$;
(ii) $w = aa^d(a+b) \in \mathcal{A}^d$;
(iii) $(a+b)aa^d \in \mathcal{A}^d$;
(iv) $aa^d(a+b)aa^d \in \mathcal{A}^d$.
In this case,

$$\begin{aligned}(a+b)^d = {} & w^d + \left(b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n\right)a^\pi - \left(b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n\right)a^\pi bw^d \\ & + \sum_{n=0}^{\infty}\left((b^d)^{n+2} + \sum_{k=0}^{\infty}(b^d)^{n+k+3}a(a+b)^k\right)a^\pi baa^d w^n w^\pi \\ & + \sum_{n=0}^{\infty} b^\pi (a+b)^n a^\pi b(w^d)^{n+2} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^d)^{k+1}a(a+b)^{n+k}a^\pi b(w^d)^{n+2}.\end{aligned} \tag{3.1}$$

Proof. First, suppose that $a \in \mathcal{A}^{qnil}$. Therefore, $a^\pi = \mathbb{1}$ and from $aba^\pi = \lambda a^\pi bab^\pi a^\pi$, we obtain $ab = \lambda bab^\pi$. Using Lemma 2.3, $a + b \in \mathcal{A}^d$. If a is quasinilpotent, then $a^d = 0$. Notice that $a^d = 0$ clearly implies $w = aa^d(a + b) = 0$, then $w^d = 0$. By the equalities $a^d = 0$, $a^\pi = \mathbb{1}$ and $w^d = 0$, (3.1) is equal to expression of Lemma 2.3. So Theorem 3.1 holds. Now, we assume that a is not quasinilpotent and we consider the matrix representations of a, a^d and b relative to the idempotent $p = aa^d$. We have

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \quad a^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix}_p, \quad b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_p,$$

where $a_1 \in (p\mathcal{A}p)^{-1}$, $a_2 = a - a^2a^d \in (\overline{p}\mathcal{A}\overline{p})^{qnil}$.

Since $aba^\pi = \lambda a^\pi bab^\pi a^\pi$, we have $aa^dba^\pi = \lambda a^d a^\pi bab^\pi a^\pi = 0$; hence, $b_2 = 0$. Thus, b can be represented as

$$b = \begin{bmatrix} b_1 & 0 \\ b_3 & b_4 \end{bmatrix}_p.$$

Using Lemma 2.1, if one of the elements $a^\pi ba^\pi$ or $a^\pi b$ or ba^π is generalized Drazin invertible, we conclude that $b_4 \in (\overline{p}\mathcal{A}\overline{p})^d$. Similarly, if one of the elements aa^dbaa^d or aa^db or baa^d is generalized Drazin invertible, then $b_1 \in (p\mathcal{A}p)^d$. Applying again Lemma 2.1, because $b \in \mathcal{A}^d$ and one of the above mentioned elements is generalized Drazin invertible, $b_1 \in (p\mathcal{A}p)^d$, $b_4 \in (\overline{p}\mathcal{A}\overline{p})^d$,

$$b^d = \begin{bmatrix} b_1^d & 0 \\ s & b_4^d \end{bmatrix}_p, \qquad \text{and} \qquad b^\pi = \begin{bmatrix} b_1^\pi & 0 \\ -(b_3b_1^d + b_4 s) & b_4^\pi \end{bmatrix}_p,$$

where

$$s = \sum_{n=0}^{\infty} (b_4^d)^{n+2} b_3 b_1^n b_1^\pi + \sum_{n=0}^{\infty} b_4^\pi b_4^n b_3 (b_1^d)^{n+2} - b_4^d b_3 b_1^d.$$

Since $aba^\pi = \lambda a^\pi bab^\pi a^\pi$, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & a_2b_4 \end{bmatrix}_p = aba^\pi = \lambda a^\pi bab^\pi a^\pi = \lambda \begin{bmatrix} 0 & 0 \\ 0 & b_4a_2b_4^\pi \end{bmatrix}_p,$$

we obtained $a_2b_4 = \lambda b_4a_2b_4^\pi$. By Lemma 2.3, we observe that $a_2 + b_4 \in (\overline{p}\mathcal{A}\overline{p})^d$ and

$$(a_2 + b_4)^d = b_4^d + \sum_{n=0}^{\infty} (b_4^d)^{n+2} a_2 (a_2 + b_4)^n.$$

Since $a_2 + b_4 \in (\overline{p}\mathcal{A}\overline{p})^d$, using Lemma 2.1, the following conditions are equivalent:

(*i*) $a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_3 & a_2 + b_4 \end{bmatrix}$ is generalized Drazin invertible;

(*ii*) $w(= aa^d(a + b) = aa^d(a + b)aa^d) = a_1 + b_1$ is generalized Drazin invertible;

(*iii*) $(a + b)aa^d$ is generalized Drazin invertible.

In this case,

$$(a + b)^d = \begin{bmatrix} a_1 + b_1 & 0 \\ b_3 & a_2 + b_4 \end{bmatrix}^d = \begin{bmatrix} w^d & 0 \\ u & (a_2 + b_4)^d \end{bmatrix}, \tag{3.2}$$

where

$$u = \sum_{n=0}^{\infty} ((a_2 + b_4)^d)^{n+2} b_3 w^n w^\pi + \sum_{n=0}^{\infty} (a_2 + b_4)^\pi (a_2 + b_4)^n b_3 (w^d)^{n+2} - (a_2 + b_4)^d b_3 w^d. \tag{3.3}$$

The equality $a_2b_4 = \lambda b_4a_2b_4^\pi$ implies $a_2b_4^d = 0$ and

$$\begin{aligned}(a_2+b_4)^\pi &= \overline{p} - (a_2+b_4)(a_2+b_4)^d\\ &= \overline{p} - b_4b_4^d - b_4\sum_{k=0}^{\infty}(b_4^d)^{k+2}a_2(a_2+b_4)^k = b_4^\pi - \sum_{k=0}^{\infty}(b_4^d)^{k+1}a_2(a_2+b_4)^k.\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty}(a_2+b_4)^\pi(a_2+b_4)^nb_3(w^d)^{n+2} = \sum_{k=0}^{\infty}b_4^\pi(a_2+b_4)^nb_3(w^d)^{n+2} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b_4^d)^{k+1}a_2(a_2+b_4)^{n+k}b_3(w^d)^{n+2}.$$

Observe that, by (3.2),

$$(a+b)^d = w^d + (a_2+b_4)^d + u.$$

Also we have (we have written with an asterisk any entry whose exactly expression is not necessary)

$$\begin{aligned}\sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^na^\pi &= \sum_{n=0}^{\infty}\begin{bmatrix}(b_1^d)^{n+2}a_1 & 0\\ * & (b_4^d)^{n+2}a_2\end{bmatrix}_p\begin{bmatrix}w^n & 0\\ * & (a_2+b_4)^n\end{bmatrix}_p\begin{bmatrix}0 & 0\\ 0 & \overline{p}\end{bmatrix}_p\\ &= \begin{bmatrix}0 & 0\\ 0 & \sum_{n=0}^{\infty}(b_4^d)^{n+2}a_2(a_2+b_4)^n\end{bmatrix}_p,\end{aligned}$$

$$b^da^\pi = \begin{bmatrix}b_1^d & 0\\ s & b_4^d\end{bmatrix}_p\begin{bmatrix}0 & 0\\ 0 & \overline{p}\end{bmatrix}_p = \begin{bmatrix}0 & 0\\ 0 & b_4^d\end{bmatrix}_p,$$

$$\begin{aligned}X_1 &= w^d + \left(b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n\right)a^\pi\\ &= \begin{bmatrix}w^d & 0\\ 0 & 0\end{bmatrix}_p + \begin{bmatrix}0 & 0\\ 0 & b_4^d + \sum_{n=0}^{\infty}(b_4^d)^{n+2}a_2(a_2+b_4)^n\end{bmatrix}_p = \begin{bmatrix}w^d & 0\\ 0 & (a_2+b_4)^d\end{bmatrix}_p,\end{aligned}$$

$$X_2 = \left(b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n\right)a^\pi bw^d = \begin{bmatrix}0 & 0\\ 0 & (a_2+b_4)^d\end{bmatrix}_p\begin{bmatrix}b_1w^d & 0\\ b_3w^d & 0\end{bmatrix}_p = \begin{bmatrix}0 & 0\\ (a_2+b_4)^db_3w^d & 0\end{bmatrix}_p,$$

$$\begin{aligned}X_3 &= \sum_{n=0}^{\infty}\left((b^d)^{n+2} + \sum_{k=0}^{\infty}(b^d)^{n+k+3}a(a+b)^k\right)a^\pi baa^dw^nw^\pi\\ &= \sum_{n=0}^{\infty}\left((b^d)^{n+2} + \sum_{k=0}^{\infty}(b^d)^{n+k+3}a(a+b)^k\right)a^\pi\left(a^\pi baa^d\right)w^nw^\pi\\ &= \sum_{n=0}^{\infty}\begin{bmatrix}0 & 0\\ 0 & (b_4^d)^{n+2} + \sum_{k=0}^{\infty}(b_4^d)^{n+k+3}a_2(a_2+b_4)^k\end{bmatrix}_p\begin{bmatrix}0 & 0\\ b_3 & 0\end{bmatrix}_p\begin{bmatrix}w^nw^\pi & 0\\ 0 & 0\end{bmatrix}_p\\ &= \sum_{n=0}^{\infty}\begin{bmatrix}0 & 0\\ 0 & \left((a_2+b_4)^d\right)^{n+2}\end{bmatrix}_p\begin{bmatrix}0 & 0\\ b_3w^nw^\pi & 0\end{bmatrix}_p\\ &= \begin{bmatrix}0 & 0\\ \sum_{n=0}^{\infty}\left((a_2+b_4)^d\right)^{n+2}b_3w^nw^\pi & 0\end{bmatrix}_p,\end{aligned}$$

$$X_4 = \sum_{n=0}^{\infty} b^{\pi}(a+b)^n a^{\pi} b (w^d)^{n+2} = \sum_{n=0}^{\infty} \begin{bmatrix} b_1^{\pi} & 0 \\ -(b_3 b_1^d + b_4 s) & b_4^{\pi} \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ (a_2+b_4)^n b_3 (w^d)^{n+2} & 0 \end{bmatrix}_p$$
$$= \begin{bmatrix} 0 & 0 \\ \sum_{n=0}^{\infty} b_4^{\pi}(a_2+b_4)^n b_3 (w^d)^{n+2} & 0 \end{bmatrix}_p,$$

$$X_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (b^d)^{k+1} a (a+b)^{n+k} a^{\pi} b (w^d)^{n+2}$$
$$= \begin{bmatrix} 0 & 0 \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (b_4^d)^{k+1} a_2 (a_2+b_4)^{n+k} b_3 (w^d)^{n+2} & 0 \end{bmatrix}_p.$$

It then follows that

$$X_1 + X_2 + X_3 + X_4 + X_5 = \begin{bmatrix} w^d & 0 \\ u & (a_2+b_4)^d \end{bmatrix} = (a+b)^d.$$

□

In Theorem 3.1, the condition $aba^{\pi} = \lambda a^{\pi} bab^{\pi} a^{\pi}$ is weaker than $ab = a^{\pi} bab^{\pi}$ of [15, Theorem 4] and $ab = \lambda a^{\pi} bab^{\pi}$ of [6, Theorem 2.4]. Indeed, it clear that $ab = \lambda a^{\pi} bab^{\pi}$ can imply $aba^{\pi} = \lambda a^{\pi} bab^{\pi} a^{\pi}$. However, in general, the converse is false. To show that our condition is strictly weaker than $ab = \lambda a^{\pi} bab^{\pi}$, we construct matrices a, b satisfying the condition $aba^{\pi} = \lambda a^{\pi} bab^{\pi} a^{\pi}$, but not $ab = \lambda a^{\pi} bab^{\pi}$ (or $ab = a^{\pi} bab^{\pi}$).

Example 3.2. *Let $\mathcal{A}$ be the Banach algebra of all complex 3×3 matrices, and take*

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Clearly

$$a^d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a^{\pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $b^3 = 0$, we have $b^d = 0$ and $b^{\pi} = I_3$. Hence

$$ab = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad aba^{\pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that

$$\lambda a^{\pi} bab^{\pi} = \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \lambda a^{\pi} bab^{\pi} a^{\pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have $aba^{\pi} = \lambda a^{\pi} bab^{\pi} a^{\pi}$, but $a^{\pi} bab^{\pi} \neq ab \neq \lambda a^{\pi} bab^{\pi}$.

In 2004, Castro and Koliha [5] assumed the following three conditions symmetric in $a, b \in \mathcal{A}^d$,

$$a^{\pi} b = b, ab^{\pi} = a, b^{\pi} aba^{\pi} = 0. \tag{3.4}$$

We observe that the conditions (3.4) and $aba^{\pi} = \lambda a^{\pi} bab^{\pi} a^{\pi}$ are independent. The following examples can illustrate this fact. The first example show that the conditions (3.4) hold, but the condition $aba^{\pi} = \lambda a^{\pi} bab^{\pi} a^{\pi}$ is not satisfied.

Example 3.3. *Let $\mathcal{A}$ be defined as in Example 3.2, and take*

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $a^2 = b^2 = 0$, we have $a^d = b^d = 0$ and $a^\pi = b^\pi = I_3$. It is now easy to see that the matrices a, b satisfy conditions $a^\pi b = b, ab^\pi = a$, and

$$ab = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad ba = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we observe that $b^\pi aba^\pi = 0$. While

$$\lambda a^\pi bab^\pi a^\pi = \lambda \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq aba^\pi.$$

In the next example, we construct matrices a, b in the algebra $\mathcal{A}$ of all complex 3×3 matrices such that $aba^\pi = \lambda a^\pi bab^\pi a^\pi$ is satisfied, but (3.4) is not satisfied.

Example 3.4. *Let $\mathcal{A}$ be defined as in Example 3.2, and take*

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $a^3 = b^3 = 0$, we have $a^d = b^d = 0$ and $a^\pi = b^\pi = I_3$. Then we get

$$b^\pi aba^\pi = aba^\pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad 3a^\pi bab^\pi a^\pi = 3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = aba^\pi.$$

While $b^\pi aba^\pi \neq 0$, so (3.4) is not satisfied.

If we assume that $a^\pi b = b$ in Theorem 3.1, the expression for $(a+b)^d$ will be exactly the same as in [5, Theorem 3.5]. In fact, if $a^\pi b = b$ in Theorem 3.1, then we can verify that $aba^\pi = \lambda a^\pi bab^\pi a^\pi$ is equivalent to $aba^\pi = \lambda bab^\pi a^\pi$, and so we have the following corollary.

Corollary 3.5. *Let $a, b \in \mathcal{A}^d$. If $a^\pi b = b$, $aba^\pi = \lambda bab^\pi a^\pi$ and $\lambda \neq 0$, then*

$$\begin{aligned}(a+b)^d = a^d &+ \left(b^d + \sum_{n=0}^{\infty} (b^d)^{n+2} a(a+b)^n\right) a^\pi - \left(b^d + \sum_{n=0}^{\infty} (b^d)^{n+2} a(a+b)^n\right) ba^d \\ &+ \sum_{n=0}^{\infty} b^\pi (a+b)^n b(a^d)^{n+2} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (b^d)^{k+1} a(a+b)^{n+k} b(a^d)^{n+2}.\end{aligned} \tag{3.5}$$

Proof. The assumptions $aba^\pi = \lambda bab^\pi a^\pi$ and $a^\pi b = b$ imply $aba^\pi = \lambda a^\pi bab^\pi a^\pi$. Hence Theorem 3.1 is applicable. Notice that the condition $a^\pi b = b$ is equivalent to $aa^d b = 0$. From this, we have $w = a^2 a^d$. By [15, Lemma 7], it follows that $w^d = a^d$ and $w^\pi = a^\pi$. Now, let us observe that the expression

$$\sum_{n=0}^{\infty} \left((b^d)^{n+2} + \sum_{k=0}^{\infty} (b^d)^{n+k+3} a(a+b)^k \right) a^\pi baa^d w^n w^\pi$$

given in (3.1) can be simplified. In view of the above relations and the equation $a^d a^\pi = 0$, we have

$$\sum_{n=0}^{\infty}\left((b^d)^{n+2}+\sum_{k=0}^{\infty}(b^d)^{n+k+3}a(a+b)^k\right)a^\pi baa^d w^n w^\pi = \sum_{n=0}^{\infty}\left((b^d)^{n+2}+\sum_{k=0}^{\infty}(b^d)^{n+k+3}a(a+b)^k\right)a^\pi baa^d (a^2a^d)^n a^\pi = 0.$$

Then the result follows by Theorem 3.1. □

Remark 3.1. *We observe that the expression for $(a+b)^d$ in (3.5) and in* (3.8), *[5, Theorem 3.5] are exactly the same.*

If $\mathcal{A}$ is a Banach algebra, then we can define another multiplication in $\mathcal{A}$ by $a \odot b = ba$. It is trivial to verify that $(\mathcal{A}, \odot)$ is a Banach algebra. If we apply Theorem 3.1 to this new algebra, we can immediately establish the following result.

Corollary 3.6. *Let $a, b \in \mathcal{A}^d$, $a^\pi b^\pi aba^\pi = \lambda a^\pi ba$ and $\lambda \neq 0$. If $a^\pi ba^\pi$ (or $a^\pi b$ or ba^π or $aa^d baa^d$ or $aa^d b$ or baa^d) is generalized Drazin invertible, then the following conditions are equivalent:*
(*i*) $a + b \in \mathcal{A}^d$;
(*ii*) $c = (a+b)aa^d \in \mathcal{A}^d$;
(*iii*) $aa^d(a+b) \in \mathcal{A}^d$;
(*iv*) $aa^d(a+b)aa^d \in \mathcal{A}^d$.
In this case,

$$\begin{aligned}(a+b)^d &= c^d + a^\pi\left(b^d+\sum_{n=0}^{\infty}(a+b)^n a(b^d)^{n+2}\right)-\sum_{n=0}^{\infty}c^d ba^\pi\left(b^d+(a+b)^n a(b^d)^{n+2}\right)\\ &\quad+\sum_{n=0}^{\infty}c^\pi c^n aa^d ba^\pi\left((b^d)^{n+2}+\sum_{k=0}^{\infty}(a+b)^k a(b^d)^{n+k+3}\right)\\ &\quad+\sum_{n=0}^{\infty}(c^d)^{n+2}ba^\pi(a+b)^n b^\pi-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(c^d)^{n+2}ba^\pi(a+b)^{n+k}a(b^d)^{k+1}.\end{aligned}$$

If aba commutes with a^d in Theorem 3.1, then an explicit expressions can be given for w^d and $(a+b)^d$ in terms of a, a^d, b, b^d.

Theorem 3.7. *Let $a, b \in \mathcal{A}^d$, $aba^\pi = \lambda a^\pi bab^\pi a^\pi$, $\lambda \neq 0$ and $abaa^d = aa^d ba$. If $a^\pi ba^\pi$ (or $a^\pi b$ or ba^π or $aa^d baa^d$ or $aa^d b$ or baa^d) is generalized Drazin invertible, then the following conditions are equivalent:*
(*i*) $a + b \in \mathcal{A}^d$;
(*ii*) $1 + a^d b \in \mathcal{A}^d$;
(*iii*) $aa^d(1 + a^d b) \in \mathcal{A}^d$;
(*iv*) $(1 + a^d b)aa^d \in \mathcal{A}^d$;
(*v*) $aa^d(1 + a^d b)aa^d \in \mathcal{A}^d$.
In this case,

$$(a+b)^d = w^d - b^d a^\pi bw^d + \left(b^d+\sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n\right)a^\pi \quad +\sum_{n=0}^{\infty}(b^d)^{n+2}a^\pi baa^d w^n w^\pi + \sum_{n=0}^{\infty}b^\pi b^n a^\pi b(w^d)^{n+2},$$

where $w = aa^d(a+b)$ and

$$w^d = a^d(1+a^d b)^d bb^d + aa^d b^\pi \sum_{n=0}^{\infty}(-b)^n (a^d)^{n+1}.$$

Proof. Using the same notation as in the proof of Theorem 3.1, observe that $a + b$ is generalized Drazin invertible if and only if $w = aa^d(a+b) = a_1 + b_1$ is generalized Drazin invertible. The condition $abaa^d = aa^d ba$ implies $a_2 b_3 = 0$ and $a_1 b_1 = b_1 a_1$. By Lemma 2.2, the following conditions are equivalent:

(i) $w = a_1 + b_1$ is generalized Drazin invertible;
(ii) $p + a_1^{-1}b_1 = aa^d(1 + a^d b)(= (1 + a^d b)aa^d = aa^d(1 + a^d b)aa^d)$ is generalized Drazin invertible;
(iii) $1 + a^d b$ is generalized Drazin invertible.
Hence, we can apply Lemma 2.2 to get the expression of w^d as

$$\begin{aligned} w^d = (a_1 + b_1)^d &= a_1^{-1}(1 + a_1^{-1}b_1)^d b_1 b_1^d + b_1^\pi \sum_{n=0}^{\infty}(-b_1)^n a_1^{-(n+1)} \\ &= a^d(1 + a^d b)^d bb^d + aa^d b^\pi \sum_{n=0}^{\infty}(-b)^n (a^d)^{n+1}. \end{aligned}$$

Since $a_2b_4 = \lambda b_4 a_2 b_4^\pi$, $a_2b_4^d = 0$ and $a_2b_3 = 0$, we have $a_2b_4^n b_3 = 0$ for any nonnegative integer n. Now we will simplify the expression of u given in (3.3). In effect, for every integer $n \geq 0$,

$$(a_2 + b_4)^n b_3 = b_4^n b_3 \qquad \text{and} \qquad a_2(a_2 + b_4)^n b_3 = 0,$$

then we have

$$u = \sum_{n=0}^{\infty}(b_4^d)^{n+2} b_3 w^n w^\pi + \sum_{n=0}^{\infty} b_4^\pi b_4^n b_3 (w^d)^{n+2} - b_4^d b_3 w^d.$$

Now we can prove that

$$b^d a^\pi b w^d = \begin{bmatrix} 0 & 0 \\ 0 & b_4^d \end{bmatrix}_p \begin{bmatrix} b_1 w^d & 0 \\ b_3 w^d & 0 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ b_4^d b_3 w^d & 0 \end{bmatrix}_p,$$

$$\begin{aligned} \sum_{n=0}^{\infty}(b^d)^{n+2} a^\pi b a a^d w^n w^\pi &= \sum_{n=0}^{\infty}(b^d)^{n+2} a^\pi \left(a^\pi b a a^d\right) w^n w^\pi \\ &= \sum_{n=0}^{\infty}\begin{bmatrix} 0 & 0 \\ 0 & (b_4^d)^{n+2} \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ b_3 & 0 \end{bmatrix}_p \begin{bmatrix} w^n w^\pi & 0 \\ 0 & 0 \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & 0 \\ \sum_{n=0}^{\infty}(b_4^d)^{n+2} b_3 w^n w^\pi & 0 \end{bmatrix}_p, \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} b^\pi b^n a^\pi b (w^d)^{n+2} &= \sum_{n=0}^{\infty}\begin{bmatrix} b_1^\pi & 0 \\ -(b_3 b_1^d + b_4 s) & b_4^\pi \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ b_4^n b_3 (w^d)^{n+2} & 0 \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & 0 \\ \sum_{n=0}^{\infty} b_4^\pi b_4^n b_3 (w^d)^{n+2} & 0 \end{bmatrix}_p. \end{aligned}$$

The rest of the proof follows in much the same way as the proof of Theorem 3.1. □

4. Main result 2

This section presents other of our main results, which extend some results of [7]. We start with an important special case.

Theorem 4.1. *Let $b \in \mathcal{A}^{qnil}$, $a \in \mathcal{A}^d$. If $aba^\pi = \lambda ba$ and $\lambda \neq 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a + b)^d = a^d + \sum_{n=0}^{\infty}(a^d)^{n+2} b (a + b)^n. \tag{4.1}$$

Proof. Using the same method of proof as in [6, Lemma 2.3], we can prove this result. $\square$

Now, we give another main result which generalizes [7, Theorem 2.3].

Theorem 4.2. *Let $a, b \in \mathcal{A}^d$. If $a = ab^\pi$, $b^\pi aba^\pi = \lambda b^\pi ba$ and $\lambda \neq 0$, then $a + b \in \mathcal{A}^d$ and*

$$\begin{aligned}(a+b)^d = {} & (b^d + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n)a^\pi + b^\pi\left(a^d + \sum_{n=0}^{\infty}(a^d)^{n+2}b(a+b)^n\right)\\ & - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^d)^{n+2}a^{n+1}(a^d)^{k+1}b(a+b)^k - \sum_{n=0}^{\infty} b^d a(a^d)^{n+2}b(a+b)^n.\end{aligned} \tag{4.2}$$

Proof. If b is quasinilpotent, then the result follows Theorem 4.1. Hence, we assume that b is neither invertible nor quasinilpotent and we consider the following matrix representations of b and a relative to the idempotent $p = bb^d$. We have

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p,$$

where $b_1 \in (p\mathcal{A}p)^{-1}$, $b_2 = b - b^2b^d \in (\overline{p}\mathcal{A}\overline{p})^{qnil}$. So we get

$$b^d = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p, \quad b^\pi = \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p.$$

We will use the condition $a = ab^\pi$. Since

$$ab^\pi = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p = \begin{bmatrix} 0 & a_2 \\ 0 & a_4 \end{bmatrix}_p = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p,$$

we get $a_1 = a_3 = 0$.

Thus, a and $a + b$ can be represented as

$$a = \begin{bmatrix} 0 & a_2 \\ 0 & a_4 \end{bmatrix}_p, \quad a + b = \begin{bmatrix} b_1 & a_2 \\ 0 & b_2 + a_4 \end{bmatrix}_p. \tag{4.3}$$

Then we have

$$a^d = \begin{bmatrix} 0 & a_2(a_4^d)^2 \\ 0 & a_4^d \end{bmatrix}_p, \qquad a^\pi = \begin{bmatrix} p & -a_2a_4^d \\ 0 & a_4^\pi \end{bmatrix}_p.$$

From $b^\pi aba^\pi = \lambda b^\pi ba$ and

$$b^\pi aba^\pi = \begin{bmatrix} 0 & 0 \\ 0 & a_4b_2a_4^\pi \end{bmatrix}_p, \qquad \lambda b^\pi ba = \lambda \begin{bmatrix} 0 & 0 \\ 0 & b_2a_4 \end{bmatrix}_p,$$

we obtained $a_4b_2a_4^\pi = \lambda b_2a_4$. By using Theorem 4.1, we get $a_4 + b_2 \in (\overline{p}\mathcal{A}\overline{p})^d$ and

$$(a_4 + b_2)^d = a_4^d + \sum_{n=0}^{\infty}(a_4^d)^{n+2}b_2(a_4 + b_2)^n.$$

By employing Lemma 2.1(ii) for the representation of $a + b$ given in (4.3) we get

$$(a+b)^d = \begin{bmatrix} b_1^{-1} & u \\ 0 & (a_4 + b_2)^d \end{bmatrix}, \tag{4.4}$$

where

$$u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_2 (a_4 + b_2)^n (a_4 + b_2)^\pi + \sum_{n=0}^{\infty} b_1^\pi b_1^n a_2 ((a_4 + b_2)^d)^{n+2} - b_1^{-1} a_2 (a_4 + b_2)^d.$$

Observe that since $b_1 \in (p\mathcal{A}p)^{-1}$, then $b_1^\pi = 0$.

Hence, the expression of u reduces to

$$u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_2 (a_4 + b_2)^n (a_4 + b_2)^\pi - b_1^{-1} a_2 (a_4 + b_2)^d.$$

From $a_4 b_2 a_4^\pi = \lambda b_2 a_4$, we get $b_2 a_4^d = 0$. Hence

$$\begin{aligned}(a_4 + b_2)^\pi &= \overline{p} - (a_4 + b_2)(a_4 + b_2)^d = \overline{p} - (a_4 + b_2)\left(a_4^d + \sum_{n=0}^{\infty} (a_4^d)^{n+2} b_2 (a_4 + b_2)^n\right) \\ &= \overline{p} - a_4 a_4^d - a_4 \sum_{n=0}^{\infty} (a_4^d)^{n+2} b_2 (a_4 + b_2)^n = a_4^\pi - \sum_{n=0}^{\infty} (a_4^d)^{n+1} b_2 (a_4 + b_2)^n,\end{aligned}$$

and

$$b_1^{-(n+2)} a_2 (a_4 + b_2)^n (a_4^d)^{k+1} b_2 (a_4 + b_2)^k = b_1^{-(n+2)} a_2 a_4^n (a_4^d)^{k+1} b_2 (a_4 + b_2)^k.$$

So we get

$$\begin{aligned}u = &\sum_{n=0}^{\infty} b_1^{-(n+2)} a_2 (a_4 + b_2)^n a_4{}^\pi - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_1^{-(n+2)} a_2 a_4^n (a_4^d)^{k+1} b_2 (a_4 + b_2)^k \\ &- b_1^{-1} a_2 a_4^d - b_1^{-1} a_2 \sum_{n=0}^{\infty} (a_4^d)^{n+2} b_2 (a_4 + b_2)^n.\end{aligned} \tag{4.5}$$

Observe that (4.4) and $b_1^{-1} = b^d$ yield

$$(a + b)^d = b^d + (a_4 + b_2)^d + u. \tag{4.6}$$

Also we have (we have written with an asterisk any entry whose exactly expression is not necessary)

$$\begin{aligned}b^\pi (a^d)^{n+2} b (a + b)^n &= \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \begin{bmatrix} 0 & * \\ 0 & (a_4^d)^{n+2} \end{bmatrix}_p \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p \begin{bmatrix} b_1^n & * \\ 0 & (a_4 + b_2)^n \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (a_4^d)^{n+2} b_2 (a_4 + b_2)^n \end{bmatrix}_p,\end{aligned}$$

$$b^\pi a^d = \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \begin{bmatrix} 0 & a_2 (a_4^d)^2 \\ 0 & a_4^d \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ 0 & a_4^d \end{bmatrix}_p,$$

$$\begin{aligned}b^d a (a^d)^{n+2} b (a + b)^n &= \begin{bmatrix} 0 & b_1^{-1} a_2 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} 0 & * \\ 0 & (a_4^d)^{n+2} b_2 (a_4 + b_2)^n \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & b_1^{-1} a_2 (a_4^d)^{n+2} b_2 (a_4 + b_2)^n \\ 0 & 0 \end{bmatrix}_p,\end{aligned}$$

$$b^d a^\pi = \begin{bmatrix} b_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} p & -a_2 a_4^d \\ 0 & a_4^\pi \end{bmatrix}_p = \begin{bmatrix} b_1^{-1} & -b_1^{-1} a_2 a_4^d \\ 0 & 0 \end{bmatrix}_p,$$

$$\begin{aligned}(b^d)^{n+2} a(a+b)^n a^\pi &= \begin{bmatrix} b_1^{-(n+2)} & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} 0 & a_2 \\ 0 & a_4 \end{bmatrix}_p \begin{bmatrix} b_1^n & * \\ 0 & (a_4+b_2)^n \end{bmatrix}_p \begin{bmatrix} p & -a_2 a_4^d \\ 0 & a_4^\pi \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & b_1^{-(n+2)} a_2 (a_4+b_2)^n a_4^\pi \\ 0 & 0 \end{bmatrix}_p,\end{aligned}$$

$$\begin{aligned}&(b^d)^{n+2} a^{n+1} (a^d)^{k+1} b(a+b)^k \\ &= \begin{bmatrix} 0 & b_1^{-(n+2)} a_2 a_4^n \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} 0 & * \\ 0 & (a_4^d)^{k+1} \end{bmatrix}_p \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p \begin{bmatrix} b_1^k & * \\ 0 & (a_4+b_2)^k \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & b_1^{-(n+2)} a_2 a_4^n (a_4^d)^{k+1} b_2 (a_4+b_2)^k \\ 0 & 0 \end{bmatrix}_p.\end{aligned}$$

From (4.5) and (4.6), it follows (4.2). □

Corollary 4.3. *Let $a \in \mathcal{A}^{qnil}$, $b \in \mathcal{A}^d$. If $ab^\pi = a$, $b^\pi ab = \lambda b^\pi ba$ and $\lambda \neq 0$, then*

$$(a+b)^d = b^d + \sum_{n=0}^{\infty} (b^d)^{n+2} a(a+b)^n.$$

Proof. If a is quasinilpotent, then $a^d = 0$ and $a^\pi = \mathbb{1}$. Applying Theorem 4.2, we get the result. □

In [7, Theorem 2.3], authors gave an explicit representation for $(a+b)^d$ under conditions $ab^\pi = a$ and $b^\pi ab = b^\pi ba$. Theorem 4.2 extends it to more general setting. The following example can illustrate this fact.

Example 4.4. *Let $\mathcal{A}$ be defined as in Example 3.2, and take*

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix}.$$

Since $a^3 = b^3 = 0$, we have $a^d = b^d = 0$ and $a^\pi = b^\pi = I_3$. Then we have

$$b^\pi ab = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b^\pi ba = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, we can see that $a = ab^\pi, b^\pi ab = \frac{1}{5} b^\pi ba$, the conditions of Theorem 4.2 hold. However, the conditions of [7, Theorem 2.3] are not satisfied.

In the rest of the paper, we look for simplifying equation (4.2) for $(a+b)^d$ under some stronger hypotheses than Theorem 4.2. The results of the preceding theorem, in particular the matrix representations, suggest that we should retain the condition $a = ab^\pi$, while replacing $b^\pi aba^\pi = \lambda b^\pi ba$ by a stronger hypothesis.

First, we extend [7, Corollary 2.1].

Theorem 4.5. *Let $a, b \in \mathcal{A}^d$. If $a = ab^\pi$, $aba^\pi = \lambda ba$ and $\lambda \neq 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = a^d + b^d + \sum_{n=0}^{\infty} (a^d)^{n+2} b(a+b)^n + \sum_{n=0}^{\infty} (b^d)^{n+2} a(a+b)^n. \tag{4.7}$$

Proof. Applying Theorem 4.1, similarly as in the proof of Theorem 4.2, we have that $a + b \in \mathcal{A}^d$ and $(a + b)^d$ is represented as in (4.7). □

The final theorem gives the simpler expression of $(a + b)^d$ under the conditions that $a = ab^\pi$ and $b^\pi aba^\pi = \lambda ba$.

Theorem 4.6. *Let $a, b \in \mathcal{A}^d$. If $a = ab^\pi$, $b^\pi aba^\pi = \lambda ba$ and $\lambda \neq 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a + b)^d = a^d + b^d + \sum_{n=0}^{\infty} (a^d)^{n+2} b(a + b)^n.$$

Proof. The assumption $b^\pi aba^\pi = \lambda ba$ implies $b^\pi aba^\pi = \lambda b^\pi ba$. According to Theorem 4.2, we complete the proof. □

Acknowledgements

The authors would like to thank the referee for his/her helpful suggestions to the improvement of this paper. The very detailed comments improve many proofs of the paper, e.g., Theorem 3.1, Corollary 3.5 and Theorem 4.2.

References

[1] F.F. Bonsall, J. Duncan, Complete Normed Algebras, Springer Verlag, 1973.
[2] J. Benítez, X. Liu, Y. Qin, Representations for the generalized Drazin inverse in a Banach algebra, Bulletin of Mathematical Analysis and Applications 5(1) (2013) 53-64.
[3] S.L. Campbell, Singular Systems of Differential Equations, Pitman, San Francisco-London-Melbourne, 1980.
[4] S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Pitman, London, 1979.
[5] N. Castro-González, J.J. Koliha, New additive results for the g-Drazin inverse, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085-1097.
[6] H. Chen, M. Sheibani, Expressions for the g-Drazin inverse in a Banach algebra, arXiv:1912.02642v1 [math.RA] (2019).
[7] D.S. Cvetković-Ilić, D.S. Djordjević, Y. Wei, Additive results for the generalized Drazin inverse in a Banach algebra, Linear Algebra Appl. 418 (2006) 53-61.
[8] D.S. Cvetković-Ilić, X. Liu, Y. Wei, Some additive results for the generalized Drazin inverse in a Banach algebra, Electron. J. Linear Algebra 22 (2011) 1049-1058.
[9] C. Deng, Y. Wei, New additive results for the generalized Drazin inverse, J. Math. Anal. Appl. 370 (2010) 313-321.
[10] D. Djordjević, V. Rakočević, Lectures on Generalized inverses, University of Niš, 2008.
[11] D.S. Djordjević, P.S. Stanimirović, On the generalized Drazin inverse and generalized resolvent, Czechoslovak Math. J. 51 (126) (2001) 617-634.
[12] D.S. Djordjević, Y. Wei, Additive results for the generalized Drazin inverse, J. Aust. Math. Soc. 73 (2002) 115-125.
[13] J.J. Koliha, A generalized Drazin inverse, Glasgow Math J. 38 (1996) 367-381.
[14] Y. Liao, J. Chen, J. Cui, Cline's formula for the generalized Drazin inverse, Bull. Malays. Math. Sci. Soc. 37 (2014) 37-42.
[15] X. Liu, X. Qin, Representations for the generalized Drazin inverse of the sum in a Banach algebra and its application for some operator matrices, The Scientific World Journal (2015), Article ID 156934, 8 pages.
[16] X. Liu, X. Qin, J. Benítez, New additive results for the generalized Drazin inverse in a Banach algebra, Filomat 30 (2016) 2289-2294.
[17] C.D. Meyer, Jr., N.J. Rose, The index and the Drazin inverse of block triangular matrices, SIAM J. Appl. Math. 33(1) (1977) 1-7.
[18] D. Mosić, Additive results for the generalized Drazin inverse in a Banach algebra, Bull. Malays. Math. Sci. Soc. 40 (2017) 1465-1478.
[19] D. Mosić, H. Zou, J. Chen, The generalized Drazin inverse of the sum in a Banach algebra, Ann. Funct. Anal. 8(1) (2017) 90-105.
[20] D. Mosić, D. Zhang, On the generalized Drazin inverse of the sum in a Banach Algebra, Quaestiones Math. (2018) 1-10. DOI: 10.2989/16073606. 2018.1517832
[21] H. Zou, D. Mosić, J. Chen, Generalized Drazin invertiblity of the product and sum of two elements in a Banach algebra and its applications, Turkish. J. Math. 41 (2017) 548-563.