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On a Recent Result Concerning the Drazin Inverse of Matrices

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Abstract. In this paper, we investigate the recent paper of Shakoor et al. [A. Shakoor, I. Ali, S. Wali, A. Rehman, Some formulas on the Drazin inverse for the sum of two matrices and block matrices, Bull. Iran. Math. Soc. 48 (2022) 351-366]. Here we prove that the main, additive result from the mentioned paper is actually a corollary of one known result. Furthermore, we give new representations for the Drazin inverse of anti–triangular block matrix, which generalize some representations from current literature on the topic.

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices and let $A \in \mathbb{C}^{n\times n}$. By $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\operatorname{rank}(A)$ we denote the range, the null space and the rank of matrix A, respectively. Furthermore, by $\operatorname{ind}(A)$ we denote the smallest nonnegative integer k, such that $\operatorname{rank}(A^{k+1}) = \operatorname{rank}(A^k)$, called the index of A. If $\operatorname{ind}(A) = k$, then there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies the following relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

The matrix A^d is called the Drazin inverse of A [1, 2]. The concept of Drazin inverse was introduced by Drazin in 1958, in associative rings and semigroups (see [3]).

In this paper we use notation $A^\pi = I - AA^d$ to denote the projection on $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$. Also, we agree that $A^0 = I$, for every complex matrix A. Furthermore, if the lower limit of a sum is greater than its upper limit, we define the sum to be 0.

Suppose $P, Q \in \mathbb{C}^{n\times n}$. In 1958, in celebrated paper of Drazin [3], additive properties of the Drazin inverse are studied in associative rings and semigroups. In the matrix concept, the result which was offered by Drazin is $(P+Q)^d = P^d + Q^d$, when $PQ = QP = 0$. Hartwig, Wang and Wei reopened this problem in 2001 in the matrix concept and generalized the result given by Drazin, to a case when $PQ = 0$ [4]. Since then, this topic attracts a great attention and there are a plenty of papers on this subject (see [5–17]). However, there is no formula for $(P+Q)^d$ without any side condition for matrices P and Q, so this problem remains open.

In 2015, Višnjić derived a formula for $(P+Q)^d$ under conditions $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$ (see [19, Theorem 2.1]). In a recent paper, Shakoor et al. offered a formula for $(P+Q)^d$, which

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is valid when $P^2QP = 0$ and $PQ^2 = 0$ (see [18, Theorem 3.1]). Obviously, the result of Shakoor et al. is a special case of a result of Višnjić. Furthermore, we will prove that mentioned result of Shakoor is actually a corollary of the result [19, Theorem 2.1].

The problem of finding the Drazin inverse of the sum of two matrices is closely related to the problem of finding the Drazin inverse of 2×2 complex block matrix $\widetilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A and D are square matrices, not necessarily of the same size. This problem was opened in 1979, by Campbell and Meyer [20], and since then it is a topic of great significance, due to its applications in several areas, such as differential and difference equations and perturbation theory of the Drazin inverse (see [4, 20–25]). Many authors have studied this problem and offered some formulas for $\widetilde{M}^d$, when blocks of matrix $\widetilde{M}$ satisfy some certain conditions (for example, see [2, 5–9, 11–15, 17, 19, 23]). However, in the present there is no formula for $\widetilde{M}^d$, with no side conditions for blocks of matrix $\widetilde{M}$, so this problem is still an open one. Meanwhile, a general expression for the Drazin inverse of a 2×2 block triangular matrix (either $B = 0$ or $C = 0$) is known [26, 27]. Furthermore, in 1983 Campbel [22] posed a problem of finding the Drazin inverse of anti–triangular block matrix (where $D = 0$), as an application in solving second order differential equations. It is shown that anti–triangular block matrices and its Drazin inverses are involved in applications like saddle-point problems, optimization problems and graph theory [28–30].

Consider the upper anti–triangular block matrix:

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1}$$

where A is a square complex matrix and 0 is a square null matrix (sizes of matrices A and 0 does not have to be equal). Many authors studied the problem of finding the Drazin inverse of matrix M, defined by (1), and offered a formulas for M^d, under some specific conditions for blocks of M. Here we list some of them:

(i) $BC = 0$ [31, Corollary 4.3];

(ii) $BCA = 0$ [6, Theorem 4.5];

(iii) $A^\pi AB = 0$ and $BCAA^d = 0$ [29, Theorem 3.8];

(iv) $A^\pi BCA = 0$ and $BCAA^d = 0$ [10, Theorem 2.3];

(v) $AA^\pi BC = 0$, $CA^\pi BC = 0$ and $ABCA^d = 0$ [10, Theorem 2.6];

(vi) $ABC = 0$ [5, Corollary 3.9] and [29, Theorem 3.3];

(vii) $BCA^\pi = 0$ and $AA^dBC = 0$ [29, Theorem 3.6];

(viii) $ABCA^\pi = 0$ and $AA^dBC = 0$ [10, Theorem 2.1];

(ix) $BCA^\pi A = 0$, $BCA^\pi B = 0$ and $A^dBCA = 0$ [10, Theorem 2.4]

In section 3 of this paper, we derive some new representations for M^d. Namely, in Theorem 3.1 we offer a representation for M^d under conditions $AA^\pi BCA = 0$, $CA^\pi BCA = 0$ and $A^dBCA^d = 0$, which generalizes representations given under conditions (i)–(v) from the previous list. Furthermore, in Theorem 3.2 a representation for M^d is given under conditions $ABCA^\pi A = 0$, $ABCA^\pi B = 0$ and $A^dBCA^d = 0$, which are weaker than conditions (i), (vi)–(ix) from the list above.

Before we give our results, we state the following auxiliary lemmas.

Lemma 1.1. [1, Exercise 4.33] *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. *Then* $(AB)^d = A((BA)^2)^dB$.

Lemma 1.2. [4, Theorem 2.1] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *If* $PQ = 0$ *then*

$$(P+Q)^d = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1} + \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi.$$

2. Additive properties of the Drazin inverse for matrices

Through this section we will assume that $P, Q \in \mathbb{C}^{n\times n}$. In [19, Theorem 2.1], a formula for $(P+Q)^d$ is derived under conditions $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$. Recently, Shakoor et al. studied additive properties of the Drazin inverse for matrices (in the skew field concept) and offered the formula for $(P+Q)^d$, when matrices P and Q satisfy $P^2QP = 0$ and $PQ^2 = 0$ (see [18, Theorem 3.1]). Obviously, conditions from [19, Theorem 2.1] are weaker than conditions from [18, Theorem 3.1]. Furthermore, the formula for $(P+Q)^d$, which is given in [18, Theorem 3.1], is a corollary of the result [19, Theorem 2.1]. Before we prove this, we state the result from [19, Theorem 2.1].

Theorem 2.1. [19, Theorem 2.1] *If $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$ then*

$$(P+Q)^d = (Y_1 + Y_2)(P+Q), \tag{2}$$

where

$$Y_1 = \sum_{i=0}^{\mathrm{ind}((P+Q)Q)-1} ((P+Q)Q)^{\pi}((P+Q)Q)^i \left(((P+Q)P)^d\right)^{i+1}, \tag{3}$$

$$Y_2 = \sum_{i=0}^{\mathrm{ind}((P+Q)P)-1} \left(((P+Q)Q)^d\right)^{i+1} ((P+Q)P)^i((P+Q)P)^{\pi}, \tag{4}$$

furthermore, for $n \in \mathbb{N}$

$$\left(((P+Q)P)^d\right)^n = \sum_{i=0}^{\mathrm{ind}(QP)-1} (QP)^{\pi}(QP)^i \left(P^d\right)^{2(i+n)} + \sum_{i=0}^{\mathrm{ind}(P^2)-1} \left((QP)^d\right)^{i+n} P^{2i}P^{\pi} - \sum_{i=1}^{n-1} ((QP)^d)^i \left(P^d\right)^{2(n-i)}, \tag{5}$$

$$\left(((P+Q)Q)^d\right)^n = \sum_{i=0}^{\mathrm{ind}(Q^2)-1} Q^{\pi}Q^{2i}\left((PQ)^d\right)^{i+n} + \sum_{i=0}^{\mathrm{ind}(PQ)-1} \left(Q^d\right)^{2(i+n)} (PQ)^i(PQ)^{\pi} - \sum_{i=1}^{n-1} \left(Q^d\right)^{2i} \left((PQ)^d\right)^{n-i}, \tag{6}$$

and

$$((P+Q)P)^{\pi} = (QP)^{\pi}P^{\pi} - \sum_{i=0}^{\mathrm{ind}(QP)-2} (QP)^{\pi}(QP)^{i+1}\left(P^d\right)^{2(i+1)} - \sum_{i=0}^{\mathrm{ind}(P^2)-2} \left((QP)^d\right)^{i+1} P^{2(i+1)}P^{\pi}, \tag{7}$$

$$((P+Q)Q)^{\pi} = Q^{\pi}(PQ)^{\pi} - \sum_{i=0}^{\mathrm{ind}(Q^2)-2} Q^{\pi}Q^{2(i+1)}\left((PQ)^d\right)^{i+1} - \sum_{i=0}^{\mathrm{ind}(PQ)-2} \left(Q^d\right)^{2(i+1)} (PQ)^{i+1}(PQ)^{\pi}. \tag{8}$$

As a special case of Theorem 2.1, we have the following corollary.

Corollary 2.2. *If $P^2QP = 0$ and $PQ^2 = 0$, then:*

$$(P+Q)^d = Q(Y_1+Y_2) + Q\left(Y_1((P+Q)P)^d + ((P+Q)Q)^d\, Y_2\right)PQ + Y_3 + Y_3\left(((P+Q)P)^d\right)PQ + Q^d, \tag{9}$$

where Y_1 and Y_2 are defined in (3) and (4) respectively, in addition $(((P+Q)P)^d)^n$ and $(((P+Q)Q)^d)^n$, for $n \in \mathbb{N}$, are defined in (5) and (6) respectively, furthermore $((P+Q)P)^{\pi}$ and $((P+Q)Q)^{\pi}$ are as given in (7) and (8) respectively, and

$$Y_3 = P((P+Q)P)^d - Q((P+Q)Q)^d. \tag{10}$$

Proof. Since $(P+Q)^d = (P+Q)((P+Q)^d)^2 = ((P+Q)^2)^d(P+Q)$, according to the proof of [19, Theorem 2.1], we have that:

$$\begin{aligned}(P+Q)^d &= \left(P\left(((P+Q)P)^d\right)^2 - Q((P+Q)Q)^d((P+Q)P)^d + Q\sum_{i=0}^{l-1}((P+Q)Q)^\pi((P+Q)Q)^i\left(((P+Q)P)^d\right)^{i+2}\right.\\ &\quad \left. +Q\sum_{i=0}^{s-1}\left(((P+Q)Q)^d\right)^{i+2}((P+Q)P)^i((P+Q)P)^\pi\right)(P+Q)^2,\end{aligned}$$

where $s = \operatorname{ind}((P+Q)P)$ and $l = \operatorname{ind}((P+Q)Q)$. Since $(P+Q)^2 = (P+Q)P + (P+Q)Q$ and $PQ^2 = 0$, after some computations we get that:

$$\begin{aligned}(P+Q)^d &= P((P+Q)P)^d - Q((P+Q)Q)^d + Q\sum_{i=0}^{t-1}((P+Q)Q)^\pi((P+Q)Q)^i\left(((P+Q)P)^d\right)^{i+1}\\ &\quad +Q\sum_{i=0}^{s-1}\left(((P+Q)Q)^d\right)^{i+1}((P+Q)P)^i((P+Q)P)^\pi + P\left(((P+Q)P)^d\right)^2 PQ\\ &\quad -Q((P+Q)Q)^d((P+Q)P)^d PQ + Q\sum_{i=0}^{t-1}((P+Q)Q)^\pi((P+Q)Q)^i\left(((P+Q)P)^d\right)^{i+2}PQ\\ &\quad +Q\sum_{i=0}^{s-1}\left(((P+Q)Q)^d\right)^{i+2}((P+Q)P)^i((P+Q)P)^\pi PQ + Q\left(((P+Q)Q)^d\right)^2 Q^2.\end{aligned}$$

Applying equality (6) for $n = 2$, we get that $Q\left(((P+Q)Q)^d\right)^2 Q^2 = Q^d$. Now, using the notations for Y_1, Y_2 and Y_3, which are given in (3), (4) and (10), respectively, we obtain the formula for $(P+Q)^d$, as it is defined in (9). □

In the following exercise we will analyze the main result form the paper of Shakoor et al.[18]. Actually, we will prove that the formula for $(P+Q)^d$, which is given in [18, Theorem 3.1], is equal to the formula for $(P+Q)^d$ derived in Corollary 2.2.

Exercise 2.3. [18, Theorem 3.1] *If $P^2QP = 0$ and $PQ^2 = 0$, then:*

$$\begin{aligned}(P+Q)^d =&(P^2+PQ)^dP + Q(Q^2+PQ)^d + QTP + \Big(\left((P^2+PQ)^d\right)^2\\ &+ QT(P^2+PQ)^d + Q(Q^2+PQ)^dT\Big)P^2Q,\end{aligned}\tag{11}$$

where for $n \in \mathbb{N}$:

$$\begin{aligned}\left((P^2+PQ)^d\right)^n =& \sum_{i=0}^{t-1}P(QP)^\pi\left(QP^3+(QP)^2\right)^i\left(P^d\right)^{4i+2+2n}(P+Q) + \sum_{i=0}^{t-1}P\left((QP)^d\right)^{2i+1+n}P^{4i}P^\pi(P+Q)\\ &+\sum_{i=0}^{t-1}P\left((QP)^d\right)^{2i+2+n}P^{4i+2}P^\pi(P+Q) - \sum_{j=0}^{n-1}P\left((QP)^d\right)^{n-j}\left(P^d\right)^{2(j+1)}(P+Q),\end{aligned}\tag{12}$$

$$\left((PQ+Q^2)^d\right)^n = \sum_{i=0}^{t-1}Q^\pi Q^{2i}\left((PQ)^d\right)^{i+n} + \sum_{i=0}^{t-1}\left(Q^d\right)^{2(i+n)}(PQ)^i(PQ)^\pi - \sum_{j=1}^{n-1}\left(Q^d\right)^{2j}\left((PQ)^d\right)^{n-j},\tag{13}$$

$$\begin{aligned}(P^2+PQ)^\pi =&P^\pi - P^dQ - \sum_{i=0}^{t-1}P\left((QP)^d\right)^{2i+1}P^{4i}P^\pi(P+Q) - \sum_{i=0}^{t-1}PQP(QP)^\pi\left(QP^3+(QP)^2\right)^i\left(P^d\right)^{4(i+1)}(P+Q)\\ &-\sum_{i=0}^{t-1}P\left((QP)^d\right)^{2i+2}P^{4i+2}P^\pi(P+Q) + PQP(QP)^d\left(P^d\right)^2(P+Q),\end{aligned}\tag{14}$$

$$(PQ+Q^2)^\pi = (PQ)^\pi - \sum_{i=0}^{t-1} Q^\pi Q^{2(i+1)} \left((PQ)^d\right)^{i+1} - \sum_{i=0}^{t-1} Q\left(Q^d\right)^{2i+1}(PQ)^i(PQ)^\pi, \tag{15}$$

$$\begin{aligned} T = &\sum_{i=0}^{t-1}\left((PQ+Q^2)^d\right)^{i+2}(P+Q)(P^2+PQ)^i(P^2+PQ)^\pi + \sum_{i=0}^{t-1}(PQ+Q^2)^\pi(PQ+Q^2)^i(P+Q)\left((P^2+PQ)^d\right)^{i+2} \\ &-(PQ+Q^2)^d(P+Q)(P^2+PQ)^d, \end{aligned} \tag{16}$$

and $t=\max\{\mathrm{ind}(P^2),\mathrm{ind}(Q^2),\mathrm{ind}(PQ),\mathrm{ind}(QP)\}$. *We will prove that the formula for* $(P+Q)^d$*, given in* (11)*, is equal to the formula given in* (9).

Proof. Through this exercise, we will use Lemma 1.1. First, we will analyze the expression for $((P^2+PQ)^d)^n$, where $n\in\mathbb{N}$, which is defined in (12). Since $P^2QP=0$, we get that

$$(QP^3+(QP)^2)^k = (QP)^{2(k-1)}QP^3+(QP)^{2k}, \quad \text{for } k\in\mathbb{N}.$$

After some computations, we obtain

$$\sum_{i=0}^{t-1}(QP)^\pi\left(QP^3+(QP)^2\right)^i\left(P^d\right)^{4i+2+2n} = \sum_{i=0}^{\mathrm{ind}(QP)-1}(QP)^\pi(QP)^i\left(P^d\right)^{2(i+1+n)}$$

Also, we get that

$$\sum_{i=0}^{t-1}\left((QP)^d\right)^{2i+1+n}P^{4i}P^\pi + \sum_{i=0}^{t-1}\left((QP)^d\right)^{2i+2+n}P^{4i+2}P^\pi = \sum_{i=0}^{\mathrm{ind}(P^2)-1}\left((QP)^d\right)^{i+1+n}P^{2i}P^\pi.$$

Furthermore,

$$\sum_{j=0}^{n-1}\left((QP)^d\right)^{n-j}\left(P^d\right)^{2(j+1)} = \sum_{j=1}^{n}\left((QP)^d\right)^j\left(P^d\right)^{2(n+1-j)}.$$

Therefore, we have the following expression for $((P^2+PQ)^d)^n$:

$$\begin{aligned}\left((P^2+PQ)^d\right)^n = P\Bigg(&\sum_{i=0}^{\mathrm{ind}(QP)-1}(QP)^\pi(QP)^i\left(P^d\right)^{2(i+1+n)} + \sum_{i=0}^{\mathrm{ind}(P^2)-1}\left((QP)^d\right)^{i+1+n}P^{2i}P^\pi \\ &-\sum_{j=1}^{n}((QP)^d)^j\left(P^d\right)^{2(n+1-j)}\Bigg)(P+Q).\end{aligned}$$

Hence, we get $((P^2+PQ)^d)^n = ((P(P+Q))^d)^n = P\cdot Z\cdot(P+Q)$, where Z is equal to the right hand side of the equality given in (5), for $n+1$. Therefore, we get that $((P(P+Q))^d)^n = P(((P+Q)P)^d)^{n+1}(P+Q)$ is valid, where $((P(P+Q))^d)^n$ is defined as in (12) and $(((P+Q)P)^d)^{n+1}$ is defined as in (5).

Now, we will analyze the expression for $(P^2+PQ)^\pi$ given in (14). We get that

$$\sum_{i=0}^{t-1}(QP)^\pi\left(QP^3+(QP)^2\right)^i\left(P^d\right)^{4i+4} = \sum_{i=0}^{\mathrm{ind}(QP)-1}(QP)^\pi(QP)^i\left(P^d\right)^{2(i+2)}$$

and

$$\sum_{i=0}^{t-1}\left((QP)^d\right)^{2i+1}P^{4i}P^\pi + \sum_{i=0}^{t-1}\left((QP)^d\right)^{2i+2}P^{4i+2}P^\pi = \sum_{i=0}^{\mathrm{ind}(P^2)-1}\left((QP)^d\right)^{i+1}P^{2i}P^\pi.$$

Therefore,

$$\begin{aligned}(P^2+PQ)^\pi &= P^\pi - P^dQ - PQP\Big(\sum_{i=0}^{\mathrm{ind}(QP)-1}(QP)^\pi(QP)^i\left(P^d\right)^{2(i+2)}\\ &+\sum_{i=0}^{\mathrm{ind}(P^2)-1}\left((QP)^d\right)^{i+2}P^{2i}P^\pi+(QP)^d\left(P^d\right)^2\Big)(P+Q),\end{aligned}\tag{17}$$

On the other hand, we have that

$$(P^2+PQ)^\pi = I - P(P+Q)(P(P+Q))^d = I - P(P+Q)P\left(((P+Q)P)^d\right)^2(P+Q).$$

If we compute $I - P(P+Q)P(((P+Q)P)^d)^2(P+Q)$, using the formula for $(((P+Q)P)^d)^n$ given in (5) and condition $P^2QP=0$, we get exactly the formula (17).

Obviously, the expression for $((PQ+Q^2)^d)^n = (((P+Q)Q)^d)^n$ given in (13) is equal to the expression given in (6). Also, we can easily check that the formula for $((P+Q)Q)^\pi$ given in (15) is equal to the formula which is offered in (8).

Now, we will express matrix T, defined in (16), in terms of matrices $(P+Q)P$ and $(P+Q)Q$ and its Drazin inevrses. Using Lemma 1.1, after some computation, we get

$$\begin{aligned}T=&\Big(\sum_{i=0}^{l-1}\left(((P+Q)Q)^d\right)^{i+2}((P+Q)P)^i((P+Q)P)^\pi+\sum_{i=0}^{s-1}((P+Q)Q)^\pi((P+Q)Q)^i\left(((P+Q)P)^d\right)^{i+2}\\ &-((P+Q)Q)^d((P+Q)P)^d\Big)(P+Q),\end{aligned}$$

where $s=\mathrm{ind}((P+Q)P)$ and $l=\mathrm{ind}((P+Q)Q)$. Our next intention is to express matrix $(P+Q)^d$, given in (11), in terms of $(P+Q)P$, $(P+Q)Q$, $((P+Q)P)^d$ and $((P+Q)Q)^d$. The elements of the sum, which is on the right hand side of the equality (11), are as follows:

$$\begin{aligned}(P^2+PQ)^dP &= P((P+Q)P)^d,\\ Q(Q^2+PQ)^d &= Q((P+Q)Q)^d,\\ QTP &= Q\sum_{i=0}^{l-1}\left(((P+Q)Q)^d\right)^{i+1}((P+Q)P)^i((P+Q)P)^\pi\\ &\quad+Q\sum_{i=0}^{s-1}((P+Q)Q)^\pi((P+Q)Q)^i\left(((P+Q)P)^d\right)^{i+1}-Q((P+Q)Q)^d,\\ \left((P^2+PQ)^d\right)^2P^2Q &= P\left(((P+Q)P)^d\right)^2PQ,\end{aligned}$$

$$\begin{aligned}QT(P^2+PQ)^dP^2Q &= Q\sum_{i=0}^{s-1}((P+Q)Q)^\pi((P+Q)Q)^i\left(((P+Q)P)^d\right)^{i+2}PQ\\ &\quad-Q((P+Q)Q)^d((P+Q)P)^dPQ,\\ Q(Q^2+PQ)^dTP^2Q &= Q\sum_{i=0}^{l-1}\left(((P+Q)Q)^d\right)^{i+2}((P+Q)P)^i((P+Q)P)^\pi-Q\left(((P+Q)Q)^d\right)^2PQ.\end{aligned}$$

Applying formula (6) for $n = 2$, we get $Q(((P+Q)Q)^d)^2PQ = Q((P+Q)Q)^d - Q^d$. Finally, using notations for Y_1, Y_2 and Y_3, which are given in (3), (4) and (10), respectively, we get that the formula (11) is equal to the formula (9). □

The following example illustrates that the result given in [19, Theorem 2.1] is more general than a result given in [18, Theorem 3.1]. Actually, we give two matrices P and Q, which do not satisfy the conditions from [18, Theorem 3.1], but do satisfy the conditions from [19, Theorem 2.1].

Example 2.4. Consider the matrices:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $PQ^2 \neq 0$, matrices P and Q do not satisfy the conditions of [18, Theorem 3.1]. Furthermore, we have the following list of conditions, which are not satisfied, and therefore some known formulas for $(P+Q)^d$ can not be applied:

(i) $PQ \neq 0$, so we can not apply the formula for $(P+Q)^d$ from [4, Theorem 2.1];
(ii) $Q^2 \neq 0$, hence the formula for $(P+Q)^d$ from [5, Theorem 2.2] fail to apply;
(iii) $P^2Q \neq 0$, therefore we can not use the formulas from [6, Theorem 2.3], [7, Theorem 2.2], [9, Theorem 3.1], [11, Theorem 3.2] and [18, Theorem 3.2];
(iv) $PQ^2 \neq 0$, consequently the formulas for $(P+Q)^d$ from [7, Theorem 2.1], [11, Theorem 3.1], [10, Theorem 4.1] and [13, Theorem 3.1] can not be applied;
(v) $PQP \neq 0$, hence the formulas from [12, Theorem 4], [12, Theorem 5] and [14, Theorem 2.1] fail to apply;
(vi) $QPQ \neq 0$, therefore the formula from [13, Theorem 2.1] can not be used to derive $(P+Q)^d$;
(vii) $(PQ)^2 \neq 0$, so we can not use the formula from [8, Corollary 2.3];
(viii) $P^3Q \neq 0$, consequently the formulas from [9, Theorem 3.2] and [15, Theorem 3.1] can not be applied;
(ix) $QP^3 \neq 0$, hence we can not apply the formula from [15, Theorem 3.2];
(x) $(P+Q)P(P+Q)P \neq 0$, so the formulas from [16, Theorem 3.1] and [17, Corollary 3.2] fail to apply;
(xi) $(P+Q)^3P(P+Q)^3P \neq 0$, therefore the formula from [17, Corollary 3.3] can not be used to obtain $(P+Q)^d$.

However, we have that $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$. Hence, matrices P and Q satisfy the conditions of [19, Theorem 2.1], so we can apply the formula (2). We have that $\operatorname{ind}((P+Q)P) = 2$, $\operatorname{ind}((P+Q)Q) = 2$ and $((P+Q)P)^k = ((P+Q)P)^2$, $((P+Q)Q)^k = ((P+Q)Q)^2$, for any integer $k \geq 2$. Therefore, $((P+Q)P)^d = ((P+Q)P)^2$ and $((P+Q)Q)^d = ((P+Q)Q)^2$. Hence,

$$((P+Q)P)^d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad ((P+Q)Q)^d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying formula (2), we get

$$\begin{aligned} (P+Q)^d &= \Big(((P+Q)Q)^\pi((P+Q)P)^d + ((P+Q)Q)^\pi(P+Q)Q\left(((P+Q)P)^d\right)^2 \\ &\quad + ((P+Q)Q)^d((P+Q)P)^\pi \\ &\quad + \left(((P+Q)Q)^d\right)^2(P+Q)P((P+Q)P)^\pi\Big)(P+Q). \end{aligned}$$

Consequently, we obtain $(P+Q)^d$:

$$(P+Q)^d = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -3 & 1 & 1 & -4 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 2.5. Notice that Theorem 3.2 from [18] is the symmetrical formulation of Theorem 3.1 form the same paper, which we have studied in Exercise 2.3. In addition, Theorem 2.2 from [19] is the symmetrical formulation of Theorem 2.1. As we have proved in Exercise 2.3, [18, Theorem 3.1] is a corollary of [19, Theorem 2.1]. Therefore, we can conclude that [18, Theorem 3.2] is a corollary of [19, Theorem 2.2].

3. Representations for the Drazin inverse of anti–triangular block matrix

In this section we derive new representations for the Drazin inverse of anti–triangular block matrix M, defined by (1), which generalize some known representations.

Theorem 3.1. *Let M be matrix defined by (1). If $AA^\pi BCA = 0$, $CA^\pi BCA = 0$ and $A^dBCA^d = 0$, then*

$$\begin{aligned} M^d =M\Bigg((P^d)^2&\begin{bmatrix} (A^\pi BC)^\pi & -AA^\pi B(CA^\pi B)^d \\ 0 & (CA^\pi B)^\pi \end{bmatrix} \\ &+\sum_{i=0}^{t-1}(P^d)^{2i+4}\begin{bmatrix} (A^\pi BC)^{i+1}(A^\pi BC)^\pi & AA^\pi B(CA^\pi B)^i(CA^\pi B)^\pi \\ 0 & (CA^\pi B)^{i+1}(CA^\pi B)^\pi \end{bmatrix} \\ &+\sum_{i=0}^{r-1}P^\pi P^{2i}\begin{bmatrix} ((A^\pi BC)^d)^{i+1} & AA^\pi B((CA^\pi B)^d)^{i+2} \\ 0 & ((CA^\pi B)^d)^{i+1} \end{bmatrix}\Bigg), \end{aligned}$$

where

$$\begin{aligned} P &= \begin{bmatrix} A & AA^dB \\ C & 0 \end{bmatrix}, \\ (P^d)^n &= \begin{bmatrix} (A^d)^{n-1}T & (A^d)^{n+1}B \\ C(A^d)^nT & C(A^d)^{n+2}B \end{bmatrix}, \\ T &= A^d+\sum_{j=0}^{l-1}(A^d)^{j+3}BCA^j, \end{aligned}$$

for every $n \in \mathbb{N}$, and $t = \max\{\operatorname{ind}(CA^\pi B), \operatorname{ind}(A^\pi BC)-1\}$, $r = \operatorname{ind}(P^2)$, $l = \operatorname{ind}(A)$.

Proof. Consider the splitting of matrix M:

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A & AA^dB \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}.$$

If we denote by $P = \begin{bmatrix} A & AA^dB \\ C & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}$, we have that $PQP^2 = 0$ and $Q^2 = 0$. Therefore, matrices P and Q satisfy conditions of [19, Corollary 2.2], so we have:

$$\begin{aligned} M^d = M\Bigg(&\sum_{i=0}^{r-1}P^\pi P^{2i}\left(\left((PQ)^d\right)^{i+1}+\left((QP)^d\right)^{i+1}\right) \\ &+\sum_{i=0}^{s-1}(P^d)^{2(i+1)}\left((PQ)^i(PQ)^\pi+(QP)^i(QP)^\pi\right)-(P^d)^2\Bigg), \end{aligned} \tag{18}$$

where $r = \operatorname{ind}(P^2)$ and $s = \max\{\operatorname{ind}(PQ), \operatorname{ind}(QP)\}$. Hence, we should derive expressions for P^d, $(PQ)^d$ and $(QP)^d$.

First, we will focus on obtaining P^d. Since $A^\pi AB_1 = 0$ and $B_1CAA^d = 0$, where $B_1 = AA^dB$, we have that matrix P satisfy the conditions of [29, Theorem 3.8] and after applying this theorem we get:

$$P^d = \begin{bmatrix} T & (A^d)^2B \\ CA^dT & C(A^d)^3B \end{bmatrix},$$

where

$$T = A^d + \sum_{j=0}^{l-1}(A^d)^{j+3}BCA^j,$$

and $l = \mathrm{ind}(A)$. By induction, we get:

$$T^n = (A^d)^{n-1}T,$$

and

$$(P^d)^n = \begin{bmatrix} T^n & (A^d)^{n+1}B \\ CA^dT^n & C(A^d)^{n+2}B \end{bmatrix} = \begin{bmatrix} (A^d)^{n-1}T & (A^d)^{n+1}B \\ C(A^d)^nT & C(A^d)^{n+2}B \end{bmatrix}, \tag{19}$$

for every $n \in \mathbb{N}$. Furthermore, after computation and using Lemma 1.2 we get:

$$(PQ)^n = \begin{bmatrix} 0 & AA^\pi B(CA^\pi B)^{n-1} \\ 0 & (CA^\pi B)^n \end{bmatrix}, \text{ for } n \in \mathbb{N}, \tag{20}$$

$$((PQ)^d)^n = \begin{bmatrix} 0 & AA^\pi B((CA^\pi B)^d)^{n+1} \\ 0 & ((CA^\pi B)^d)^n \end{bmatrix}, \text{ for } n \in \mathbb{N}, \tag{21}$$

$$(PQ)^\pi = \begin{bmatrix} I & -AA^\pi B(CA^\pi B)^d \\ 0 & (CA^\pi B)^\pi \end{bmatrix}, \tag{22}$$

$$(QP)^n = \begin{bmatrix} (A^\pi BC)^n & 0 \\ 0 & 0 \end{bmatrix}, \text{ for } n \in \mathbb{N}, \tag{23}$$

$$((QP)^d)^n = \begin{bmatrix} ((A^\pi BC)^d)^n & 0 \\ 0 & 0 \end{bmatrix}, \text{ for } n \in \mathbb{N}, \tag{24}$$

$$(QP)^\pi = \begin{bmatrix} (A^\pi BC)^\pi & 0 \\ 0 & I \end{bmatrix}. \tag{25}$$

Substituting (19) – (25) into (18), we get that the statement of the theorem is true. □

Another representation for M^d is offered in the following theorem.

Theorem 3.2. *Let M be matrix of a form (1). If $ABCA^\pi A = 0$, $ABCA^\pi B = 0$ and $A^dBCA^d = 0$, then*

$$\begin{aligned} M^d = \Bigg(& \begin{bmatrix} (BCA^\pi)^\pi & 0 \\ -(CA^\pi B)^d CA^\pi A & (CA^\pi B)^\pi \end{bmatrix} (P^d)^2 \\ & + \sum_{i=0}^{t-1} \begin{bmatrix} (BCA^\pi)^\pi (BCA^\pi)^{i+1} & 0 \\ (CA^\pi B)^\pi (CA^\pi B)^i CA^\pi A & (CA^\pi B)^\pi (CA^\pi B)^{i+1} \end{bmatrix} (P^d)^{2i+4} \\ & + \sum_{i=0}^{r-1} \begin{bmatrix} ((BCA^\pi)^d)^{i+1} & 0 \\ ((CA^\pi B)^d)^{i+2} CA^\pi A & ((CA^\pi B)^d)^{i+1} \end{bmatrix} P^{2i} P^\pi \Bigg) M, \end{aligned}$$

where

$$P = \begin{bmatrix} A & B \\ CA^dA & 0 \end{bmatrix},$$
$$(P^d)^n = \begin{bmatrix} (A^d+V)(A^d)^{n-1} & (A^d+V)(A^d)^nB \\ C(A^d)^{n+1} & C(A^d)^{n+2}B \end{bmatrix},$$
$$V = \sum_{j=0}^{l-1} A^jBC(A^d)^{j+3},$$

for every $n \in \mathbb{N}$, *and* $t = \max\{\mathrm{ind}(CA^\pi B), \mathrm{ind}(BCA^\pi) - 1\}$, $r = \mathrm{ind}(P^2)$, $l = \mathrm{ind}(A)$.

Proof. If we split matrix M as

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ CA^dA & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix},$$

and denote by $P = \begin{bmatrix} A & B \\ CA^dA & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix}$, we have that $P^2QP = 0$ and $Q^2 = 0$. Therefore, matrices P and Q satisfy the conditions of [19, Corollary 2.1]. Furthermore, matrix P satisfy conditions of [29, Theorem 3.6]. Using the similar method as in the proof of Theorem 3.1, we complete the proof. □

As we have noticed in Introduction, representations for M^d from Theorem 3.1 and 3.2 generalize certain representations from [5, 6, 10, 29, 31].

Remark 3.3. In [18], authors studied the problem of finding the Drazin inverse of a 2×2 block matrix $M_1 = \begin{bmatrix} A & B \\ C & CA^dB \end{bmatrix}$, i.e. of a block matrix with generalized Schur complement equal to zero. Furthermore, authors noticed that representations for M_1^d can be obtained using the additive formula from the same paper, when the following conditions are satisfied:

(i) $ABCA^\pi A = 0$, $ABCA^\pi B = 0$ and $CA^\pi BCA^\pi = 0$;

(ii) $AA^\pi BCA = 0$, $CA^\pi BCA = 0$ and $A^\pi BCA^\pi B = 0$.

We remark that in [19, Theorem 3.1], a formula for M_1^d is already derived when $ABCA^\pi A = 0$ and $ABCA^\pi B = 0$ hold. Therefore, the condition $CA^\pi BCA^\pi = 0$, given in (i) from the previous list, is superfluous. Furthermore, we have that the condition $A^\pi BCA^\pi B = 0$, given in (ii) from the list above, is also superfluous. Namely, in [19, Theorem 3.2] a representation for M_1^d is obtained under conditions $AA^\pi BCA = 0$ and $CA^\pi BCA = 0$, without the third condition $A^\pi BCA^\pi B = 0$.

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