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On a Class of Toeplitz and Little Hankel Operators on $L_a^2(\mathbb{U}_+)$

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Abstract. In this paper we establish certain algebraic properties of Toeplitz operators and a class of little Hankel operators defined on the Bergman space of the upper half plane. We show that if K is a compact operator on $L_a^2(\mathbb{U}_+), M(s) = \frac{i-s}{i+s}, \tau_a(s) = \frac{(c-1)+sd}{(1+c)s-d}$ where $a = c + id \in \mathbb{D}, s \in \mathbb{U}_+$ and $Jf(s) = f(-\bar{s})$ then $\lim_{|a|\to 1^-} \|K - T_{J(M\circ\tau_a)}KT^*_{M\circ\tau_a}\| = 0$ and for $\varphi, \psi \in h^\infty(\mathbb{D})$, if $\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}$ is compact, then $\lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w])\| = 0$, where $d_{\overline{w}}(s) = \dfrac{1}{\sqrt{\pi}}\dfrac{w+i}{\overline{w}-i}\dfrac{(-2i)Im\, w}{(s+w)^2}, w \in \mathbb{U}_+, \hbar_\varphi$ is the little Hankel operator on $L_a^2(\mathbb{U}_+)$ with symbol φ and α_s is a function defined on $\mathbb{U}_+$ with $|\alpha_s| = 1$, for all $s \in \mathbb{U}_+$. Applications of these results are also obtained.

1. Introduction

Let $L_a^2(\mathbb{D})$ be the Bergman space, the Hilbert space of functions, analytic on $\mathbb{D}$ and square integrable with respect to the measure dA. It is well known that $L_a^2(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ with the set of functions $\{\sqrt{n+1}z^n\}$ as an orthonormal basis. For $\varphi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator $\mathcal{T}_\varphi$ on $L_a^2(\mathbb{D})$ by $\mathcal{T}_\varphi f = P(\varphi f)$. For $\psi \in L^\infty(\mathbb{D})$, the little Hankel operator $\mathcal{S}_\psi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ with symbol ψ is defined by $\mathcal{S}_\psi f = P\mathcal{J}(\psi f), f \in L_a^2(\mathbb{D})$, where P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$ and $\mathcal{J}$ is the mapping from $L^2(\mathbb{D}, dA)$ into itself such that $\mathcal{J}f(z) = f(\bar{z})$. For details see [13]. Let $\varphi \in L^\infty(\mathbb{D})$. We define the Hankel operator $\hbar_\varphi$ with symbol φ on $L_a^2(\mathbb{D})$ by $\hbar_\varphi f = P(U\varphi f)$, where U is the operator defined on $L^2(\mathbb{D}, dA)$ by $(Uf)(w) = \overline{w}(\mathcal{J}f)(w) = \overline{w}f(\overline{w})$, where $\mathcal{J}f(w) = f(\overline{w}), f \in L^2(\mathbb{D}, dA)$. Let $h^\infty(\mathbb{D})$ is the space of all bounded harmonic functions on $\mathbb{D}$. Define a unitary operator U_a on L^2 by $U_a f(w) = (f \circ \varphi_a(w))k_a(w), a \in \mathbb{D}$.

Let $L^2(\mathbb{U}_+, d\widetilde{A})$ denote the space of complex valued, absolutely square integrable, Lebesgue measurable functions on the upper half plane $\mathbb{U}_+ = \{z = x + iy \in \mathbb{C} : y > 0\}$ where $d\widetilde{A} = dxdy$ is the area measure on $\mathbb{U}_+$. The space $L^2(\mathbb{U}_+, d\widetilde{A})$ is Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{U}_+} f(s)\overline{g(s)}d\widetilde{A}(s).$$

The Bergman space of the upper half plane denoted as $L_a^2(\mathbb{U}_+)$ is the closed subspace of $L^2(\mathbb{U}_+, d\widetilde{A})$ consisting of those functions in $L^2(\mathbb{U}_+, d\widetilde{A})$ that are analytic. The function $K_w(s) = -\frac{1}{\pi(\overline{w}-s)^2}, w, s \in \mathbb{U}_+$ is the reproducing

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kernel for $L_a^2(\mathbb{U}_+)$ at w. For more details see [5]. Let P_+ be the orthogonal projection from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $L_a^2(\mathbb{U}_+)$ is given by $(P_+f)(w) = \langle f, K_w \rangle$. Let $L^\infty(\mathbb{U}_+)$ be the space of all complex valued, essentially bounded, Lebesgue measurable functions on $\mathbb{U}_+$. Define for $\varphi \in L^\infty(\mathbb{U}_+)$, $\|\varphi\|_\infty = ess \sup_{s \in \mathbb{U}_+} |\varphi(s)| < \infty$. The space $L^\infty(\mathbb{U}_+)$ is a Banach space with respect to the essential supremum norm. For $\varphi \in L^\infty(\mathbb{U}_+)$, we define the Toeplitz operator T_φ on $L_a^2(\mathbb{U}_+)$ by $T_\varphi f = P_+(\varphi f)$. The Toeplitz operator T_φ is bounded and $\|T_\varphi\| \leq \|\varphi\|_\infty$. The big Hankel operator H_φ from $L_a^2(\mathbb{U}_+)$ into $(L_a^2(\mathbb{U}_+))^\perp$ is defined by $H_\varphi f = (I - P_+)(\varphi f), f \in L_a^2(\mathbb{U}_+)$. The little Hankel operator h_φ from $L_a^2(\mathbb{U}_+)$ into $\overline{(L_a^2(\mathbb{U}_+))} = \{\overline{f} : f \in L_a^2(\mathbb{U}_+)\}$ is defined by $h_\varphi f = \overline{P}_+(\varphi f)$, where $\overline{P}_+$ is the orthogonal projection operator from $L^2(\mathbb{U}_+, d\widetilde{A})$ onto $\overline{L_a^2(\mathbb{U}_+)}$. For $\psi \in L^\infty(\mathbb{U}_+)$, define the operator $S_\psi : L_a^2(\mathbb{U}_+) \to L_a^2(\mathbb{U}_+)$ as $S_\psi f = P_+ J(\psi f)$, where $J : L^2(\mathbb{U}_+, d\widetilde{A}) \to L^2(\mathbb{U}_+, d\widetilde{A})$ is defined by $Jf(s) = f(-\overline{s})$. The operator S_ψ is unitarily equivalent to h_φ for some $\varphi \in L^\infty(\mathbb{U}_+)$. Hence both the operators h_φ and S_ψ are referred to as little Hankel operator on $L_a^2(\mathbb{U}_+)$. These operators on the Hardy space of the right half plane have been intensively studied in [11]. Let $\mathcal{L}(L_a^2(\mathbb{U}_+))$ be the set of all bounded linear operators from $L_a^2(\mathbb{U}_+)$ into itself. Let $\mathcal{LF}(L_a^2(\mathbb{U}_+))$ and $\mathcal{LC}(L_a^2(\mathbb{U}_+))$ be the set of all finite rank operators and set of all compact operators in $\mathcal{L}(L_a^2(\mathbb{U}_+))$ respectively.

For $\varphi \in L^\infty(\mathbb{U}_+)$, the Hankel operator $\hbar_\varphi$ with symbol φ on $L_a^2(\mathbb{U}_+)$ by $\hbar_\varphi f = P_+(V\varphi f)$, where V is the operator defined on $L^2(\mathbb{U}_+, d\widetilde{A})$ by $(VG)(s) = -\alpha(s)\overline{Ms}G(-\overline{s})$, where $G \in L_a^2(\mathbb{U}_+), M(s) = \frac{i-s}{i+s}$ and α is a function defined on $\mathbb{U}_+$ with $|\alpha(s)| = 1$, for all $s \in \mathbb{U}_+$.

Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$ with bounded harmonic symbols behave more like Toeplitz operators on the Hardy space [13]. Similarly little Hankel operators on the Bergman space $L_a^2(\mathbb{D})$ have similar properties as Hankel operators on the Hardy space. For details see [11], [13]. There are many algebraic relation between Toeplitz and Hankel operators on the Hardy space. For example, if $f, g \in L^\infty(\mathbb{T})$ where $\mathbb{T}$ is the unit circle then $T_{fg} = T_f T_g + H_{\tilde{f}} H_g$ and $H_{\widetilde{fg}} = T_f H_g + H_{\tilde{f}} T_g$ and for each $z \in \mathbb{D}$, $T_{\widetilde{\varphi_z}} H_g T_f T_{\overline{\varphi_z}} = H_g T_f - [H_g T_f k_z] \otimes k_z + [H_g k_z] \otimes [T_{\varphi_z} H_f^* k_{\overline{z}}]$ and $T_{\widetilde{\varphi_z}} T_f H_g T_{\overline{\varphi_z}} = T_f H_g - [T_f H_g k_z] \otimes k_z - [H_{\tilde{f}} k_z] \otimes [T_{\varphi_z} H_g^* k_{\overline{z}}]$ where T_φ and H_ψ are the Toeplitz and Hankel operators defined on the Hardy space $H^2(\mathbb{T})$ by $T_\varphi f = P_{H^2}(\varphi f)$ and $H_\psi f = P_{H^2}(E(\psi f))$ respectively where $Ef(w) = \overline{w} f(\overline{w})$. For more details see [9]. But no strong connection between Toeplitz and little Hankel operators on the Bergman space has been established yet. In this paper we establish certain algebraic properties of Toeplitz and little Hankel operators defined on $L_a^2(\mathbb{U}_+)$.

Barria and Halmos [1], Feintuch [7], [8], Das [2] established various assymptotic properties of Toeplitz and Hankel operators on the Hardy space and in the Bergman space. In this work, we present certain asymptotic properties of Toeplitz and little Hankel operators on $L_a^2(\mathbb{U}_+)$.

Toeplitz and Hankel operators on the Hardy space also occur in several other guises. For example, Toeplitz operators on the Hardy space of the disk can be represented as a Weiener-Hopf operators [3], [12] and Hankel operators on the Hardy space of the disk are unitarily equivalent to certain Hankel integral operators [11] on the Hardy space of the right half plane. Thus the two theories are equivalent and a given result can be stated in terms of Toeplitz operators on the Hardy space or for Weiner-Hopf operators on the Hardy space of the upper half plane. Similarly the results for Hankel operators on the Hardy space of the disk can be stated in terms of Hankel integral operators defined on the Hardy space of the right half plane. But despite this identification, it is necessary to study Hankel operators, Hankel integral operators, Toeplitz operators and Wiener-Hopf operators since certain problems are more natural in one setting than the other.

In [9], Guo studied certain algebraic, asymptotic properties of Toeplitz and Hankel operators defined on the Hardy space. In this paper we extend the results of Guo for Toeplitz and little Hankel operators defined on the Bergman space of the upper half plane.

The organization of the paper is as follows. In section 2, we introduce the elementary functions

$d_{\overline{w}}(s), D(w,s), D_{\overline{w}}(s)$ which plays important roles in describing certain algebraic properties of Toeplitz operator T_G and the class of little Hankel operators $\hbar_G$ where $G \in L^\infty(\mathbb{U}_+)$. In section 3 with the help of the unitary operators V_a, we describe the symbol correspondence between the Toeplitz operators and little Hankel operators defined on $L^2_a(\mathbb{D})$ and $L^2_a(\mathbb{U}_+)$. Further, we show that $\hbar_{\overline{\theta_a}} = -c(d_w \otimes d_{\overline{w}})$, where $w = M^{-1}\overline{a}, \overline{a} \in \mathbb{D}$, $\theta_a = (M \circ \tau_a)(s)$ and c is a constant. In section 4 we prove the main results of the paper. We establish certain algebraic properties of Toeplitz operators and little Hankel operators defined on $L^2_a(\mathbb{U}_+)$. We show that $T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\overline{\theta_a}} = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - c^2([\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}d_{\overline{w}}] \otimes d_{\overline{w}}) + c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\varphi\circ M}d_w])$ and $T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}} = T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} - c^2([T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes d_{\overline{w}}) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w])$ hold. Using these results we then show that if K is a compact operator then $\|K - T_{J(\theta_a)}KT^*_{\theta_a}\| \to 0$ as $|a| \to 1^-$. As an application of these results, we show that if $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ and assume T is a finite sum of finite products of Toeplitz operators with symbols $\varphi_{ij} \in L^\infty(\mathbb{U}_+)$. That is, $T = \sum_{i=1}^{n}\prod_{j=1}^{m_i} T_{\varphi_{ij}}$. If $\varphi_{ij} \circ M^{-1} \in C(\overline{\mathbb{D}})$ for all $i = 1, \cdots, n, j = 1, \cdots, m_i$ then $T = T_\varphi + K$ where $\varphi \circ M^{-1} \in C(\overline{\mathbb{D}})$ and $K \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ is a compact operator. If in addition $\varphi \in V(\mathbb{U}_+)$ then $\lim_{|a|\to 1^-} \|T - T^*_{M\circ\tau_a}TT_{M\circ\tau_a}\| = 0$.

2. Preliminaries

In this section we introduce the elementary functions $d_{\overline{w}}(s), D(w,s), D_{\overline{w}}(s)$ which plays important role in describing certain algebraic properties of Toeplitz operator T_G and the class of little Hankel operators $\hbar_G$ where $G \in L^\infty(\mathbb{U}_+)$.

Define $M : \mathbb{U}_+ \to \mathbb{D}$ by $M(s) = \frac{i-s}{i+s} = z$. Then M is one-to-one, onto and $M^{-1} : \mathbb{D} \to \mathbb{U}_+$ is given by $M^{-1}(z) = i\frac{1-z}{1+z}$. Thus M is its self inverse. Further $M'(s) = \frac{-2i}{(i+s)^2}$ and $(M^{-1})'(z) = \frac{-2i}{(1+z)^2}$. Let $W : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{U}_+)$ be defined by $(Wg)(s) = g(Ms)\frac{(2i)}{\sqrt{\pi}(i+s)^2}$. The map W is one-to-one and onto. Hence W^{-1} exists and $W^{-1} : L^2_a(\mathbb{U}_+) \to L^2_a(\mathbb{D})$ is given by $(W^{-1}G)(z) = (2i)\sqrt{\pi}G(M^{-1}(z))\frac{1}{(1+z)^2}$.

For $a \in \mathbb{D}$ and $f \in L^2_a(\mathbb{U}_+)$, define V_a from $L^2_a(\mathbb{U}_+)$ into itself by $(V_af)(s) = (f \circ \tau_a)(s)l_a(s)$, where the functions $\tau_a(s)$ are automorphisms from $\mathbb{U}_+$ onto $\mathbb{U}_+$ given by $\tau_a(s) = \frac{(c-1)+sd}{(1+c)s-d}$, where $a = c + id \in \mathbb{D}$ and $s \in \mathbb{U}_+$ and $\tau_a'(s) = \frac{1-|a|^2}{[(1+c)s-d]^2}$ and $l_a(s) = \frac{|a|^2-1}{[(1+c)s-d]^2}$. It is not difficult to see that $\tau_a'(s) = -l_a(s)$.

For $s, w \in \mathbb{U}_+$, define $d_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{(-2i)Im\, w}{(s+w)^2}$. If $w = i\frac{1-\overline{a}}{1+\overline{a}} \in \mathbb{U}_+$, then $\overline{a} \in \mathbb{D}$ and $\overline{a} = \frac{i-w}{i+w} = Mw$. That is, $M^{-1}\overline{a} = w$. Then

$$\begin{aligned}
d_{\overline{w}}(-\overline{w}) &= \frac{1}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{(-2i)(Im\, w)}{(-\overline{w}+w)^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{M^{-1}\overline{a}+i}{\overline{M^{-1}\overline{a}}-i}\frac{Im\, w}{(w-\overline{w})^2} \\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{i\frac{1-\overline{a}}{1+\overline{a}}+i}{\left(\overline{i\frac{1-\overline{a}}{1+\overline{a}}}\right)-i}\frac{w-\overline{w}}{(2i)(w-\overline{w})^2} \\
&= -\frac{1}{\sqrt{\pi}}\frac{i\left[\frac{1-\overline{a}}{1+\overline{a}}+1\right]}{[-i\frac{1-a}{1+a}-i]}\frac{1}{w-\overline{w}} \\
&= \frac{1}{\sqrt{\pi}}\frac{2}{1+\overline{a}}\frac{1+a}{2}\frac{1}{i\frac{1-\overline{a}}{1+\overline{a}}+i\frac{1-a}{1+a}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi}}\frac{1+a}{(1+\overline{a})}\frac{(1+\overline{a})(1+a)}{i[(1-\overline{a})(1+a)+(1-a)(1+\overline{a})]}\\
&= \frac{1}{i\sqrt{\pi}}\frac{(1+a)^2}{[1+a-\overline{a}-|a|^2+1+\overline{a}-a-|a|^2]}\\
&= \frac{1}{i\sqrt{\pi}}\frac{(1+a)^2}{2(1-|a|^2)}\\
&= \frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{(1-|a|^2)}.
\end{aligned}
$$

Now

$$
\begin{aligned}
d_{\overline{w}}(s)d_{\overline{w}}(-\overline{w}) &= \frac{(-2i)}{\sqrt{\pi}}\frac{w+i}{\overline{w}-i}\frac{Im\, w}{(s+w)^2}\frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{1-|a|^2}\\
&= \frac{(-2i)}{\sqrt{\pi}}\left(\frac{i\frac{1-\overline{a}}{1+\overline{a}}+i}{-i\frac{1-a}{1+a}-i}\right)\frac{\left(\frac{w-\overline{w}}{2i}\right)}{(s+i\frac{1-\overline{a}}{1+\overline{a}})^2}\frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{1-|a|^2}\\
&= \frac{(-2i)}{\sqrt{\pi}}\frac{\left(\frac{1-\overline{a}}{1+\overline{a}}+1\right)}{-\left(\frac{1-a}{1+a}+1\right)}\frac{\left[\left(i\frac{1-\overline{a}}{1+\overline{a}}\right)-\left(-i\frac{1-a}{1+a}\right)\right](1+\overline{a})^2}{(2i)[s(1+\overline{a})+i(1-\overline{a})]^2}\frac{1}{(2i)\sqrt{\pi}}\frac{(1+a)^2}{1-|a|^2}\\
&= \frac{1}{(2i)\pi}\left(\frac{\frac{1-\overline{a}+1+\overline{a}}{1+\overline{a}}}{\frac{1-a+1+a}{1+a}}\right)\frac{i\left[\frac{1-\overline{a}}{1+\overline{a}}+\frac{1-a}{1+a}\right]}{[s(1+\overline{a})+i(1-\overline{a})]^2}\frac{(1+a)^2}{1-|a|^2}(1+\overline{a})^2\\
&= \frac{1}{2\pi}\frac{1+a}{1+\overline{a}}\frac{(1+a)^2}{(1-|a|^2)}\frac{2(1-|a|^2)}{(1+a)(1+\overline{a})}\frac{(1+\overline{a})^2}{[s(1+\overline{a})+i(1-\overline{a})]^2}\\
&= \frac{1}{\pi}\left(\frac{1+a}{1+\overline{a}}\right)^2\frac{(1+\overline{a})^2}{[i+s+\overline{a}(s-i)]^2}\\
&= \frac{1}{\pi}\left(\frac{1+a}{1+\overline{a}}\right)^2\frac{(1+\overline{a})^2}{[i+s-\overline{a}(i-s)]^2}\\
&= \frac{1}{\pi}\left(\frac{1+a}{1+\overline{a}}\right)^2\frac{(1+\overline{a})^2}{(i+s)^2\left[1-\overline{a}\left(\frac{i-s}{i+s}\right)\right]^2}\\
&= \frac{1}{\pi}\frac{(1+a)^2}{(i+s)^2}\frac{1}{(1-\overline{a}Ms)^2}\\
&= D(s,w)\\
&= D_{\overline{w}}(s).
\end{aligned}
$$

Hence

$$
d_{\overline{w}}(s) = \frac{D(s,w)}{d_{\overline{w}}(-\overline{w})} \text{ and } (d_{\overline{w}}(-\overline{w}))^2 = D(\overline{w},w).
$$

Now

$$
\begin{aligned}
\|D_{\overline{w}}\|^2 &= \langle D_{\overline{w}}, D_{\overline{w}}\rangle\\
&= \int_{\mathbb{U}_+}|D_{\overline{w}}(s)|^2 d\widetilde{A}(s)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{U}_+} |D(s,w)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{U}_+} |d_{\overline{w}}(-\overline{w})|^2 |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= |d_{\overline{w}}(-\overline{w})|^2 \int_{\mathbb{U}_+} |d_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= |d_{\overline{w}}(-\overline{w})|^2 \|d_{\overline{w}}\|_2^2 \\
&= |d_{\overline{w}}(-\overline{w})|^2 \ \text{Since } \|d_{\overline{w}}\|_2 = 1.
\end{aligned}
$$

Thus

$$\|D_{\overline{w}}\| = |d_{\overline{w}}(-\overline{w})| \text{ and } |d_{\overline{w}}(s)| \, \|D_{\overline{w}}\| = |D_{\overline{w}}(s)|. \text{ Further, } D_{\overline{w}} \in L^\infty(\mathbb{U}_+).$$

3. The symbol correspondence

In this section with the help of the unitary operators V_a, we describe the symbol correspondence between the Toeplitz operators and little Hankel operators defined on $L_a^2(\mathbb{D})$ and $L_a^2(\mathbb{U}_+)$. Further we show that $\hbar_{\overline{\theta_a}} = -c(d_w \otimes d_{\overline{w}})$, where $w = M^{-1}\overline{a}, \overline{a} \in \mathbb{D}$, $\theta_a = (M \circ \tau_a)(s)$.

Lemma 3.1. *For each $a \in \mathbb{D}$, $-JVV_{\overline{a}} = \beta(s)V_aVJ$, where β is a function defined on $\mathbb{U}_+$ with $|\beta(s)| = 1$, for all $s \in \mathbb{U}_+$.*

Proof. First we shall show that $U = W^{-1}VW$. Now we proceed to show that for $G \in L_a^2(\mathbb{U}_+)$, then

$$
\begin{aligned}
(WUW^{-1}G)(s) &= WU\left((2i)\sqrt{\pi}(G \circ M^{-1})(s)\frac{1}{(1+s)^2}\right) \\
&= W\left(\overline{s}(2i)\sqrt{\pi}(G \circ M^{-1})(\overline{s})\frac{1}{(1+\overline{s})^2}\right) \\
&= \frac{2i}{\sqrt{\pi}}\left(\overline{Ms}(2i)\sqrt{\pi}(G \circ M^{-1})(\overline{Ms})\frac{1}{(1+\overline{Ms})^2}\frac{1}{(i+s)^2}\right) \\
&= (-4)\left(\overline{Ms}(G \circ M^{-1})(\overline{Ms})\frac{1}{(1+\overline{Ms})^2}\frac{1}{(i+s)^2}\right) \\
&= (-4)\overline{Ms}G(M^{-1}(\overline{Ms}))\frac{(M^{-1})'(\overline{Ms})}{(-2i)}\frac{M'(s)}{(-2i)} \\
&= (-4)\overline{Ms}G(-\overline{s})\frac{(M^{-1})'(\overline{Ms})M'(s)}{(-4)} \\
&= \overline{Ms}G(-\overline{s})(M^{-1})'(\overline{Ms})M'(s). \qquad (1)
\end{aligned}
$$

Notice that $\overline{Ms} = \frac{\overline{i}-\overline{s}}{\overline{i}+\overline{s}} = \frac{-i-\overline{s}}{-i+\overline{s}} = \frac{-(i+\overline{s})}{-(i-\overline{s})} = \frac{i+\overline{s}}{i-\overline{s}} = \frac{1}{M\overline{s}}$ and

$$
\begin{aligned}
M^{-1}(\overline{Ms}) &= i\frac{1-\overline{Ms}}{1+\overline{Ms}} = i\frac{1-\overline{(\frac{i-s}{i+s})}}{1+\overline{(\frac{i-s}{i+s})}} = i\frac{1-\frac{\overline{i}-\overline{s}}{\overline{i}+\overline{s}}}{1+\frac{\overline{i}-\overline{s}}{\overline{i}+\overline{s}}} \\
&= i\frac{\frac{\overline{i}+\overline{s}-\overline{i}+\overline{s}}{\overline{i}+\overline{s}}}{\frac{\overline{i}+\overline{s}+\overline{i}-\overline{s}}{\overline{i}+\overline{s}}} = i\frac{2\overline{s}}{2\overline{i}} = i\frac{\overline{s}}{-i} = i^2\overline{s} = -\overline{s}.
\end{aligned}
$$

Again since $\left(M^{-1}\circ M\right)(s)=s$. Thus $(M^{-1})'(Ms)(M'(s))=1$, for all $s\in\mathbb{U}_+$. Now from (1) it follows that

$$\begin{aligned}\left(M^{-1}\right)'(\overline{Ms})M'(s)\overline{Ms} &= \frac{-2i}{(1+\overline{Ms})^2}\frac{-2i}{(i+s)^2}\frac{\bar{i}-\bar{s}}{\bar{i}+\bar{s}}\\ &= \frac{-2i}{(1+\frac{\bar{i}-\bar{s}}{\bar{i}+\bar{s}})^2}\frac{-2i}{(i+s)^2}\frac{\bar{i}-\bar{s}}{\bar{i}+\bar{s}}\\ &= (-4)\frac{(\bar{i}+\bar{s})^2}{(\bar{i}+\bar{s}+\bar{i}-\bar{s})^2}\frac{1}{(i+s)^2}\frac{\bar{i}-\bar{s}}{\bar{i}+\bar{s}}\\ &= (-4)\frac{(\bar{i}+\bar{s})}{(2\bar{i})^2}\frac{1}{(i+s)^2}(\bar{i}-\bar{s})\\ &= \left(\frac{\bar{i}+\bar{s}}{i+s}\right)\left(\frac{\bar{i}-\bar{s}}{i+s}\right).\end{aligned}$$

As $Ms+1=\frac{i-s}{i+s}+1=\frac{i-s+i+s}{i+s}=\frac{2i}{i+s}$. Hence $i+s=\frac{2i}{Ms+1}$ and as $\frac{1}{\overline{Ms}}=\frac{\bar{i}+\bar{s}}{\bar{i}-\bar{s}}$, so $\frac{1}{\overline{Ms}}+1=\frac{\bar{i}+\bar{s}}{\bar{i}-\bar{s}}+1=\frac{\bar{i}+\bar{s}+\bar{i}-\bar{s}}{\bar{i}-\bar{s}}$, that is $\frac{1+\overline{Ms}}{\overline{Ms}}=\frac{2\bar{i}}{\bar{i}-\bar{s}}$. Hence $\bar{i}-\bar{s}=2\bar{i}\frac{\overline{Ms}}{1+\overline{Ms}}=-2i\frac{\overline{Ms}}{1+\overline{Ms}}$.

Hence

$$\begin{aligned}\left(M^{-1}\right)'(\overline{Ms})M'(s)\overline{Ms} &= (-2i)\frac{\bar{i}+\bar{s}}{i+s}\frac{\overline{Ms}}{1+\overline{Ms}}\frac{Ms+1}{2i}\\ &= -\frac{\bar{i}+\bar{s}}{i+s}\frac{1+Ms}{1+\overline{Ms}}\overline{Ms}.\end{aligned}$$

Thus

$$\begin{aligned}(WUW^{-1}G)(s) &= -\frac{\bar{i}+\bar{s}}{i+s}\frac{1+Ms}{1+\overline{Ms}}\overline{Ms}G(-\bar{s})\\ &= -\alpha(s)\overline{Ms}G(-\bar{s})=(VG)(s),\end{aligned}$$

where α is a function defined on $\mathbb{U}_+$ with $|\alpha(s)|=1$, for all $s\in\mathbb{U}_+$.

Now

$$\begin{aligned}-(JVV_{\bar{a}}f)(s) &= -[V(V_{\bar{a}}f)(\bar{s})]\\ &= \frac{\bar{i}+s}{i+\bar{s}}\frac{1+M\bar{s}}{1+\overline{M\bar{s}}}\overline{M\bar{s}}(V_{\bar{a}}f)(-s)\\ &= \frac{\bar{i}+s}{i+\bar{s}}\frac{1+M\bar{s}}{1+\overline{M\bar{s}}}\overline{M\bar{s}}(f\circ\tau_{\bar{a}})(-s)l_{\bar{a}}(-s)\end{aligned}$$

and

$$\begin{aligned}(V_a(VJf)(s)) &= (V(Jf)\circ\tau_a)(s)l_a(s)\\ &= (Vf)(\overline{\tau_a(s)})l_a(s)\\ &= -\frac{\bar{i}+\tau_a(s)}{i+\overline{\tau_a(s)}}\frac{1+M(\overline{\tau_a(s)})}{1+\overline{M(\overline{\tau_a(s)})}}\overline{M(\overline{\tau_a(s)})}f(-\tau_a(s))l_a(s).\end{aligned}$$

Notice that $l_a(s) = \frac{|a|^2-1}{[(1+c)s-d]^2}$. So $l_{\overline{a}}(-s) = \frac{|a|^2-1}{[(1+c)(-s)+d]^2} = \frac{|a|^2-1}{[(1+c)(s)-d]^2}$. Further $-\tau_a(s) = -\frac{(c-1)+sd}{(1+c)s-d} = \frac{(c-1)+sd}{(1+c)(-s)+d}$, and $\tau_{\overline{a}}(-s) = \frac{(c-1)+sd}{(1+c)(-s)+d}$. Thus we obtain

$$-JVV_{\overline{a}} = \beta(s)V_aVJ, \text{where } |\beta(s)| = 1, \text{for all } s \in \mathbb{U}_+.$$

□

Lemma 3.2. *For each $w \in \mathbb{U}_+$, $(U_{\overline{a}}\hbar_{\overline{z}}U_af)(w) = -\hbar_{\overline{\varphi_a}}M_{\overline{b_a}}$, where $\overline{b_a} = \frac{\varphi_aJk_{\overline{a}}}{\varphi_0k_a}$ and $M_{\overline{b_a}}$ is the multiplication operator on $L^2_a(\mathbb{D})$ with symbol $\overline{b_a}$.*

Proof. Notice that

$$\begin{aligned}
(U_{\overline{a}}\hbar_{\overline{z}}U_af)(w) &= [U_{\overline{a}}\hbar_{\overline{z}}(f\circ\varphi_a)k_a](w)\\
&= U_{\overline{a}}PU(\overline{z}(f\circ\varphi_a)k_a)(w)\\
&= PU_{\overline{a}}(\overline{w}w(f\circ\varphi_a)(\overline{w})k_a(\overline{w}))\\
&= PU_{\overline{a}}(|w|^2(f\circ\varphi_a)(\overline{w})k_a(\overline{w}))\\
&= P(|\varphi_{\overline{a}}(w)|^2(f\circ\varphi_a)(\overline{\varphi_{\overline{a}}(w)})k_a(\overline{\varphi_{\overline{a}}(w)})k_{\overline{a}}(w))\\
&= P(|\varphi_{\overline{a}}(w)|^2(f\circ\varphi_a\circ\varphi_a)(\overline{w})(k_a\circ\varphi_a)(\overline{w})k_{\overline{a}}(w))\\
&= P(\varphi_{\overline{a}}(w)\overline{\varphi_{\overline{a}}(w)}f(\overline{w})(k_a\circ\varphi_a)(\overline{w})k_{\overline{a}}(w))\\
&= P(\varphi_{\overline{a}}(w)\varphi_a(\overline{w})f(\overline{w})(k_a\circ\varphi_a)(\overline{w})k_{\overline{a}}(w))\\
&= P(\overline{\varphi_a(\overline{w})}f(\overline{w})k_a(\varphi_a(\overline{w}))\varphi_a(\overline{w})k_{\overline{a}}(w))\\
&= P(\overline{\varphi_a(\overline{w})}f(\overline{w})\frac{\overline{w}}{\overline{w}}(k_a\circ\varphi_a)(\overline{w})\varphi_a(\overline{w})k_{\overline{a}}(w))\\
&= PU(\overline{\varphi_a(w)}f(w)\frac{1}{w}(k_a\circ\varphi_a)(w)\varphi_a(w)k_{\overline{a}}(\overline{w}))\\
&= PU(\overline{\varphi_a(w)}f(w)\frac{(k_a\circ\varphi_a)(w)k_a(w)}{wk_a(w)}\varphi_a(w)k_{\overline{a}}(\overline{w}))\\
&= -\hbar_{\overline{\varphi_a}}M_{\frac{\varphi_a(w)k_{\overline{a}}(\overline{w})}{\varphi_0(w)k_a(w)}}f \text{ (since } (k_a\circ\varphi_a)k_a = 1 \text{ for all } a\in\mathbb{D})\\
&= -\hbar_{\overline{\varphi_a}}M_{\frac{\varphi_aJk_{\overline{a}}}{\varphi_0k_a}}f.
\end{aligned}$$

Thus

$$(U_{\overline{a}}\hbar_{\overline{z}}U_af) = -\hbar_{\overline{\varphi_a}}M_{\overline{b_a}}.$$

□

Lemma 3.3. *(i) Let $G \in L^\infty(\mathbb{U}_+)$. The little Hankel operator $\hbar_G$ defined on $L^2_a(\mathbb{U}_+)$ is unitarily equivalent to the little Hankel operator $\hbar_\varphi$ defined on $L^2_a(\mathbb{D})$ with symbol φ given by $\varphi(z) = \alpha\left(M^{-1}(\overline{z})\right)\left(\frac{1+z}{1+\overline{z}}\right)^2\left(G\circ M^{-1}\right)(z)$ $= \gamma_s\left(G\circ M^{-1}\right)(z)$.*

(ii) If $G \in h^\infty(\mathbb{U}_+)$, the Toeplitz operator T_G defined on $L^2_a(\mathbb{U}_+)$ with symbol G is unitarily equivalent to the Toeplitz operator $\mathcal{T}_\varphi$ defined on $L^2_a(\mathbb{D})$ with symbol $\varphi(z) = G\left(i\frac{1-z}{1+z}\right) = (G\circ M^{-1})(z)$.

Proof. (i) Now the operator W maps $\sqrt{n+1}z^n$ to the function $\frac{2i}{\sqrt{\pi}}\sqrt{n+1}(Ms)^n\frac{1}{(i+s)^2} = \frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}$ which belongs to $L^2_a(\mathbb{U}_+)$. The Hankel operator $\hbar_G$ defined on $L^2_a(\mathbb{U}_+)$ maps this vector to $P_+\left(V\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}\right)\right)$

Now

$$\begin{aligned}
& P_+\left(V\left(G(s)\frac{2i}{\sqrt{\pi}}\left(\frac{i-s}{i+s}\right)^n\sqrt{n+1}\frac{1}{(i+s)^2}\right)\right)\\
= \ & -P_+\left(\alpha(s)\overline{Ms}G(-\bar{s})\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i+\bar{s}}{i-\bar{s}}\right)^n\frac{1}{(i-\bar{s})^2}\right)\\
= \ & -WPW^{-1}\left(\alpha(s)G(-\bar{s})\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\overline{\left(\frac{i-s}{i+s}\right)}\left(\frac{i+\bar{s}}{i-\bar{s}}\right)^n\frac{1}{(i-\bar{s})^2}\right)\\
= \ & -WP\left((2i)\sqrt{\pi}\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\alpha\left(M^{-1}(z)\right)G\left(-\overline{M^{-1}(z)}\right)\overline{\left(\frac{i-M^{-1}z}{i+M^{-1}z}\right)}\right.\\
& \left.\left(\frac{i+\overline{M^{-1}(z)}}{i-\overline{M^{-1}(z)}}\right)^n\frac{1}{(i-\overline{M^{-1}(z)})^2}\frac{1}{(1+z)^2}\right)\\
= \ & -WP\left((-4)\sqrt{n+1}\alpha\left(M^{-1}(z)\right)G\left(i\frac{1-\bar{z}}{1+\bar{z}}\right)\left(\frac{-i+i\frac{1-\bar{z}}{1+\bar{z}}}{-i-i\frac{1-\bar{z}}{1+\bar{z}}}\right)\left(\frac{i-i\frac{1-\bar{z}}{1+\bar{z}}}{i+i\frac{1-\bar{z}}{1+\bar{z}}}\right)^n\right.\\
& \left.\left(\frac{1}{i+i\frac{1-\bar{z}}{1+\bar{z}}}\right)^2\frac{1}{(1+z)^2}\right)\\
= \ & -WP\left((-4)\sqrt{n+1}\alpha\left(M^{-1}(z)\right)G\left(i\frac{1-\bar{z}}{1+\bar{z}}\right)\left(\frac{1-\frac{1-\bar{z}}{1+\bar{z}}}{1+\frac{1-\bar{z}}{1+\bar{z}}}\right)\left(\frac{1-\frac{1-\bar{z}}{1+\bar{z}}}{1+\frac{1-\bar{z}}{1+\bar{z}}}\right)^n\right.\\
& \left.\frac{-1}{\left(1+\frac{1-\bar{z}}{1+\bar{z}}\right)^2}\frac{1}{(1+z)^2}\right)\\
= \ & -WP\left((-4)\sqrt{n+1}\alpha\left(M^{-1}(z)\right)G\left(i\frac{1-\bar{z}}{1+\bar{z}}\right)\left(\frac{2\bar{z}}{2}\right)\left(\frac{2\bar{z}}{2}\right)^n(-1)\frac{(1+\bar{z})^2}{4}\frac{1}{(1+z)^2}\right)\\
= \ & WP\left(\bar{z}\mathcal{J}\left(-\alpha\left(M^{-1}(\bar{z})\right)G\left(i\frac{1-z}{1+z}\right)\right)\sqrt{n+1}z^n\frac{(1+z)^2}{(1+\bar{z})^2}\right)\\
= \ & W\hbar_{-\alpha\left(M^{-1}(\bar{z})\right)G\left(i\frac{1-z}{1+z}\right)\frac{(1+z)^2}{(1+\bar{z})^2}}(z^n\sqrt{n+1})\\
= \ & W\hbar_{-\alpha\left(M^{-1}(\bar{z})\right)\left(G\circ M^{-1}\right)(z)\left(\frac{1+z}{1+\bar{z}}\right)^2}(z^n\sqrt{n+1}).
\end{aligned}$$

Thus

$$\hbar_G W = W\hbar_{-\alpha\left(M^{-1}(\bar{z})\right)\left(G\circ M^{-1}\right)(z)\left(\frac{1+z}{1+\bar{z}}\right)^2}.$$

Thus $\hbar_G = W\hbar_{-\alpha\left(M^{-1}(\bar{z})\right)\left(G\circ M^{-1}\right)(z)\left(\frac{1+z}{1+\bar{z}}\right)^2}W^{-1}$.

(ii) The operator W maps $\sqrt{n+1}z^n$ to the function $\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}$ which belongs to $L_a^2(\mathbb{U}_+)$. The Toeplitz operator T_G maps this vector to $P_+\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}\right)$ which is equal to

$WPW^{-1}\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}\right)$. Now

$$\begin{aligned}
&WPW^{-1}\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}\right)\\
&= WP\left(W^{-1}\left(G(s)\frac{2i}{\sqrt{\pi}}\sqrt{n+1}\left(\frac{i-s}{i+s}\right)^n\frac{1}{(i+s)^2}\right)\right)\\
&= \frac{2i}{\sqrt{\pi}}\sqrt{n+1}WP\left(2i\sqrt{\pi}G(L^{-1}z)\frac{1}{(1+z)^2}\left(L(L^{-1}z)\right)^n\frac{1}{(i+L^{-1}z)^2}\right)\\
&= 2i\sqrt{\pi}\frac{2i}{\sqrt{\pi}}\sqrt{n+1}WP\left(G\left(i\frac{1-z}{1+z}\right)z^n\frac{1}{(1+z)^2}\frac{1}{\left(i+i\frac{1-z}{1+z}\right)^2}\right)\\
&= (-4)WP\left(G\left(i\frac{1-z}{1+z}\right)z^n\sqrt{n+1}\frac{1}{(1+z)^2}\frac{(1+z)^2}{(i(1+z)+i(1-z))^2}\right)\\
&= WP\left(G\left(i\frac{1-z}{1+z}\right)z^n\sqrt{n+1}\right)\\
&= W\mathcal{T}_{\varphi}\left(z^n\sqrt{n+1}\right),
\end{aligned}$$

where $\varphi(z)=G\left(i\frac{1-z}{1+z}\right)=\left(G\circ M^{-1}\right)(z)$.
□

Lemma 3.4. *Let $a\in\mathbb{D}$ and $\hbar_{\varphi}$ be the little Hankel operator on $L^2_a(\mathbb{U}_+)$ with symbol φ. Then*

$$V_{\overline{a}}\hbar_{\overline{M(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}}V_a=-\hbar_{\overline{(M\circ\tau_a)(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}}$$

.

Proof. From Lemma 3.3 it follows that $W\hbar_{\overline{z}}W^{-1}$ is a little Hankel operator $\hbar_G$ on $L^2_a(\mathbb{U}_+)$ and we shall calculate G by the formula established in Lemma 3.3. Observe that $\phi_a(z)=\frac{a-z}{1-\overline{a}z}$ and $\tau_a(s)=\frac{(c-1)+sd}{(1+c)s-d}$. Hence $\phi_0(z)=-z$ and $\tau_0(s)=-\frac{1}{s}$. Now $M^{-1}\circ\phi_a\circ M=\tau_a$. Hence $(M^{-1}\circ\phi_0\circ M)(s)=\tau_0(s)$. Thus $(M\circ\tau_0\circ M^{-1})(z)=\phi_0(z)=-z$. Thus $\overline{z}=-\overline{(M\circ\tau_0\circ M^{-1})}(z)$. Again $M(\tau_0(s))=M(-\frac{1}{s})=-M(s)$ and $\left(\frac{1+\overline{Ms}}{1+Ms}\right)^2=\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}$. Now $W\hbar_{\overline{z}}W^{-1}=\hbar_G$(let). That is, $W\hbar_{\overline{-(M\circ\tau_0\circ M^{-1})}}W^{-1}=\hbar_G$. Hence

$$-\overline{(M\circ\tau_0\circ M^{-1})}(z)=-\alpha\left(M^{-1}(\overline{z})\right)\left(\frac{1+z}{1+\overline{z}}\right)^2(G\circ M^{-1})(z)$$

and therefore it follows

$$\overline{(M\circ\tau_0\circ M^{-1})}(Ms)=\alpha\left(M^{-1}(\overline{Ms})\right)\left(\frac{1+Ms}{1+\overline{Ms}}\right)^2(G\circ M^{-1})(Ms).$$

That is,

$$\overline{(M\circ\tau_0)(s)}=\alpha(-\overline{s})\left(\frac{1+Ms}{1+\overline{Ms}}\right)^2G(s).$$

Thus

$$\begin{aligned}
G(s) &= \frac{1}{\alpha(-\overline{s})}\overline{(M\circ\tau_0)(s)}\left(\frac{1+\overline{Ms}}{1+Ms}\right)^2\\
&= \frac{1}{\alpha(-\overline{s})}\overline{(M\circ\tau_0)(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}
\end{aligned}$$

$$= \quad -\frac{1}{\alpha(-\bar{s})}\overline{M(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}$$

Thus

$$W\hbar_{\bar{z}}W^{-1} = \hbar_{-\frac{1}{\alpha(-\bar{s})}\overline{M(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}}.$$

Now again by Lemma 3.3, $W\hbar_{\overline{\phi_a}}W^{-1}$ is a little Hankel operator on $L^2_a(\mathbb{U}_+)$ and let $W\hbar_{\overline{\phi_a}}W^{-1} = \hbar_G$. Then $W\hbar_{\overline{(M\circ\tau_a\circ M^{-1})}}W^{-1} = \hbar_G$.
Hence

$$\overline{(M\circ\tau_a\circ M^{-1})}(z) = -\alpha\left(M^{-1}(\bar{z})\right)\left(\frac{1+z}{1+\bar{z}}\right)^2(G\circ M^{-1})(z)$$

and therefore it follows that

$$\overline{(M\circ\tau_a\circ M^{-1})}(Ms) = -\alpha\left(M^{-1}(\overline{Ms})\right)\left(\frac{1+Ms}{1+\overline{Ms}}\right)^2(G\circ M^{-1})(Ms).$$

Thus

$$\overline{(M\circ\tau_a)(s)} = -\alpha(-\bar{s})\left(\frac{1+Ms}{1+\overline{Ms}}\right)^2 G(s).$$

Thus

$$\begin{aligned} G(s) &= -\frac{1}{\alpha(-\bar{s})}\overline{(M\circ\tau_a)(s)}\left(\frac{1+\overline{Ms}}{1+Ms}\right)^2 \\ &= -\frac{1}{\alpha(-\bar{s})}\overline{(M\circ\tau_a)(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}} \end{aligned}$$

Thus

$$W\hbar_{\overline{\phi_a}}W^{-1} = \hbar_{-\frac{1}{\alpha(-\bar{s})}\overline{(M\circ\tau_a)(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}}.$$

Now it is not difficult to verify that

$$V_{\bar{a}}\hbar_{-\frac{1}{\alpha(-\bar{s})}\overline{M(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}}V_a = -\hbar_{-\frac{1}{\alpha(-\bar{s})}\overline{(M\circ\tau_a)(s)}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}}}. \quad (2)$$

□

Let x and y be two vectors in L^2. Define $x\otimes y$ to be the following operator of rank one: for $f\in L^2$,

$$(x\otimes y)(f) = \langle f, y\rangle x.$$

Theorem 3.5. *For fixed $s\in\mathbb{U}_+$,*

$$\hbar_{\overline{\theta_a}} = -c(d_w\otimes d_{\overline{w}}),$$

where $w = M^{-1}\bar{a}, \bar{a}\in\mathbb{D}$, $\theta_a = (M\circ\tau_a)(s)$.

Proof. The sequence of vectors $\{\sqrt{n+1}z^n\}_0^\infty$ forms an orthonormal basis for $L^2_a(\mathbb{D})$. For $n>1, n\in\mathbb{N}$,

$$\begin{aligned} \hbar_{\bar{z}}(\sqrt{n+1}z^n) &= PU(\bar{z}\sqrt{n+1}z^n) \\ &= P(\bar{z}z\sqrt{n+1}\bar{z}^n) \\ &= \sqrt{n+1}P(|z|^2\bar{z}^n) \\ &= 0. \end{aligned}$$

and

$$\hbar_{\bar{z}}1 = PU(\bar{z}) = P(\bar{z}z) = P(|z|^2) = \frac{1}{2}.$$

Hence

$$\hbar_{\bar{z}} = r(1 \otimes 1),$$

where r is a constant. We shall now find the adjoint of the operator W^{-1}. Notice that

$$\begin{aligned}
(W^{-1}G)(z) &= (2i)\sqrt{\pi}(G \circ M^{-1})(z)\frac{1}{(1+z)^2} \\
&= -\sqrt{\pi}(G \circ M^{-1})(z)\frac{-2i}{(1+z)^2} \\
&= -\sqrt{\pi}(G \circ M^{-1})(z)(M^{-1})'(z)
\end{aligned}$$

Now for $f \in L^2_a(\mathbb{D}), G \in L^2_a(\mathbb{U}_+)$,

$$\begin{aligned}
\langle G, (W^{-1})^* f\rangle &= \langle W^{-1}G, f\rangle \\
&= \langle -\sqrt{\pi}(G \circ M^{-1})(z)(M^{-1})'(z), f\rangle \\
&= -\sqrt{\pi}\langle (G \circ M^{-1})(z)(M^{-1})'(z), f\rangle \\
&= -\sqrt{\pi}\int_{\mathbb{D}} (G \circ M^{-1})(z)(M^{-1})'(z)\overline{f(z)}dA(z) \\
&= -\sqrt{\pi}\int_{\mathbb{D}} (G \circ M^{-1})(Ms)(M^{-1})'(Ms)\overline{f(M(s))}|M'(s)|^2 dA(s) \\
&= -\sqrt{\pi}\int_{\mathbb{D}} G(s)((M^{-1})' \circ M)(s)\overline{(f \circ M)(s)}|M'(s)|^2 dA(s) \\
&= -\sqrt{\pi}\int_{\mathbb{D}} G(s)\overline{\overline{((M^{-1})' \circ M)(s)}(f \circ M)(s)}|M'(s)|^2 dA(s) \\
&= -\sqrt{\pi}\langle G, \overline{((M^{-1})' \circ M)(s)}(f \circ M)(s)|M'(s)|^2\rangle \\
&= -\sqrt{\pi}\langle G, \overline{((M^{-1})' \circ M)(s)M'(s)}(f \circ M)(s)M'(s)\rangle \\
&= -\sqrt{\pi}\langle G, (f \circ M)(s)M'(s)\rangle, \qquad (3)
\end{aligned}$$

as $(M^{-1} \circ M)(s) = s$, and therefore $((M^{-1})' \circ M)(s)M'(s) = 1$. Hence it follows from (3) that, $(W^{-1})^* f = -\sqrt{\pi}(f \circ M)(s)M'(s)$, this implies $(W^{-1})^*1 = -\sqrt{\pi}(1 \circ M)(s)M'(s) = -\sqrt{\pi}M'(s)$.

From the proof of Lemma 3.4, we obtain

$$\begin{aligned}
\hbar_{-\frac{1}{\alpha(-\bar{s})}\overline{Ms}\frac{(M^{-1})'(Ms)}{(M^{-1})'(\overline{Ms})}} &= W\hbar_{\bar{z}}W^{-1}f \\
&= rW(1 \otimes 1)W^{-1}f \\
&= rW\langle W^{-1}f, 1\rangle 1 \\
&= r\langle f, (W^{-1})^*1\rangle W1 \\
&= r(W1 \otimes (W^{-1})^*1)f \\
&= r\langle f, (W^{-1})^*1\rangle W1 \\
&= r\langle f, -\sqrt{\pi}M'\rangle\left(\frac{-1}{\sqrt{\pi}}M'\right) \\
&= r(M' \otimes M')f,
\end{aligned}$$

where $(W1)(s) = \dfrac{2i}{\sqrt{\pi}(i+s)^2} = -\dfrac{1}{\sqrt{\pi}}M'(s)$. It is not difficult to verify that $\frac{1}{\alpha(-\bar{s})}\frac{(M^{-1})'(Ms)}{(M^{-1})'\overline{Ms}} = 1$ for all $s \in \mathbb{U}_+$. Thus from (2) it follows that

$$\begin{aligned}
\hbar_{\overline{\theta_a}} &= \hbar_{\overline{M\circ\tau_a}} \\
&= -\hbar_{\overline{-M\circ\tau_a}} \\
&= V_{\bar{a}}\hbar_{\overline{-Ms}}V_a \\
&= V_{\bar{a}}(W\hbar_{\bar{z}}W^{-1})V_a \\
&= rV_{\bar{a}}(W1\otimes(W^{-1})^*1)V_af \\
&= r\langle V_af,(W^{-1})^*1\rangle V_{\bar{a}}W1 \\
&= r\langle f,V_a(W^{-1})^*1\rangle V_{\bar{a}}W1 \\
&= r\langle f,d_{\overline{w}}\rangle d_w \\
&= r(d_w\otimes d_{\overline{w}}),
\end{aligned}$$

where $\theta_a = M\circ\tau_a$. Hence $\hbar_{\overline{\theta_a}} = -c(d_w\otimes d_{\overline{w}})$, where $c = -r$. □

To get the relationship between these two classes of operators, we consider the multiplication operator M_φ on $L^2_h(\mathbb{U}_+)$ for $\varphi \in h^\infty(\mathbb{U}_+)$, defined by $M_\varphi h = \phi h$, for $h \in L^2_h(\mathbb{U}_+)$. If M_φ is expressed as an operator matrix with respect to the decomposition $L^2_h = L^2_a(\mathbb{U}_+)\oplus(\overline{L^2_a(\mathbb{U}_+)})_0$, the result is of the form

$$M_\varphi = \begin{pmatrix} T_\varphi & \hbar_{J\varphi}U \\ U\hbar_\varphi & UT_{J\varphi}U \end{pmatrix}, \text{ where } Uh(w) = \overline{w}Jh(w).$$

If f and g are in $h^\infty(\mathbb{U}_+)$, then $M_{fg} = M_fM_g$, and therefore (multiply matrices and compare upper or lower left corners)

$$T_{fg} = T_fT_g + \hbar_{Jf}\hbar_g \tag{4}$$

and

$$\hbar_{(Jf)g} = T_f\hbar_g + \hbar_{Jf}T_g. \tag{5}$$

Now from (5) it follows that if $Jf \in H^\infty(\mathbb{U}_+)$, then for $g \in h^\infty(\mathbb{U}_+)$,

$$T_f\hbar_g = \hbar_gT_{Jf}, \text{ as } \hbar_{Jf} = 0. \tag{6}$$

4. Algebraic properties of Toeplitz and little Hankel operators

In this section we prove the main results of the paper. We establish certain algebraic properties of Toeplitz operators and little Hankel operators defined on $L^2_a(\mathbb{U}_+)$. We show that $T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\overline{\theta_a}} = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - c^2([\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}d_{\overline{w}}]\otimes d_{\overline{w}}) + c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}]\otimes[T_{\theta_a}\hbar^*_{\varphi\circ M}d_w])$ and $T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}} = T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} - c^2([T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}]\otimes d_{\overline{w}}) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}]\otimes[T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w])$ hold. Using these results we then show that if K is a compact operator on $L^2_a(\mathbb{U}_+)$ then $\|K - T_{J(\theta_a)}KT^*_{\overline{\theta_a}}\| \to 0$ as $|a| \to 1^-$. As an application of these results, we show that if $T \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ and assume T is a finite sum of finite products of Toeplitz operators with symbols $\varphi_{ij} \in L^\infty(\mathbb{U}_+)$. That is, $T = \sum_{i=1}^{n}\prod_{j=1}^{m_i}T_{\varphi_{ij}}$. If $\varphi_{ij}\circ M^{-1} \in C(\overline{\mathbb{D}})$ for all $i = 1,\cdots,n$, $j = 1,\cdots,m_i$ then $T = T_\varphi + K$ where $\varphi\circ M^{-1} \in C(\overline{\mathbb{D}})$ and $K \in \mathcal{L}(L^2_a(\mathbb{U}_+))$ is a compact operator. If in addition $\varphi \in V(\mathbb{U}_+)$ then $\lim_{|a|\to 1^-}\|T - T^*_{M\circ\tau_a}TT_{M\circ\tau_a}\| = 0$.

Theorem 4.1. *Suppose that $\varphi, \psi \in h^\infty(\mathbb{D})$. For each $a = Ms \in \mathbb{D}$, let $\varphi_a(w) = \dfrac{a-w}{1-\bar{a}w}, w \in \mathbb{D}$ and $\alpha_s = (\gamma_s \circ M) = \left(\alpha\left(M^{-1}(\overline{z})\right)\left(\frac{1+z}{1+\overline{z}}\right)^2 \circ M\right)$. Then*

$$T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\overline{\theta_a}} = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - c^2([\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}d_{\overline{w}}] \otimes d_{\overline{w}}) + c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\varphi\circ M}d_w]).$$

Proof. Since $J(\theta_a) \in \overline{H^\infty(\mathbb{U}_+)}$ for $a \in \mathbb{D}$, hence (6) implies $T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)} = \hbar_{\alpha_s(\psi\circ M)}T_{\theta_a}$. Hence $T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\overline{\theta_a}} = \hbar_{\alpha_s(\psi\circ M)}T_{\theta_a}T_{\varphi\circ M}T_{\overline{\theta_a}} = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\theta_a}T_{\overline{\theta_a}} - \hbar_{\alpha_s(\psi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M}T_{\overline{\theta_a}}$, since $T_{\theta_a}T_{\varphi\circ M} = T_{(\varphi\circ M)\theta_a} - \hbar_{J(\theta_a)}\hbar_{\varphi\circ M} = T_{\varphi\circ M}T_{\theta_a} - \hbar_{J(\theta_a)}\hbar_{\varphi\circ M}$ and $\theta_a \in H^\infty(\mathbb{U}_+)$. From (4), we obtain

$$T_{\theta_a}T_{\overline{\theta_a}} = T_{|\theta_a|^2} - \hbar_{J(\theta_a)}\hbar_{\overline{\theta_a}}.$$

But from Lemma 3.5, it follows that

$$\hbar_{\overline{\theta_a}} = -c(d_w \otimes d_{\overline{w}})$$

and

$$\hbar_{J(\theta_a)} = \hbar_{\overline{\theta_{\bar{a}}}} = -c(d_{\overline{w}} \otimes d_w).$$

Therefore

$$\begin{aligned}
&T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\overline{\theta_a}}\\
&= \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}\hbar_{J(\theta_a)}\hbar_{\overline{\theta_a}} - \hbar_{\alpha_s(\psi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M}T_{\overline{\theta_a}}\\
&= \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - c^2(\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}d_{\overline{w}} \otimes d_{\overline{w}}) + c\hbar_{\alpha_s(\psi\circ M)}(d_{\overline{w}} \otimes d_w)\hbar_{\varphi\circ M}T_{\overline{\theta_a}}\\
&= \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - c^2(\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}d_{\overline{w}} \otimes d_{\overline{w}}) + c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\varphi\circ M}d_w]).
\end{aligned}$$

□

Theorem 4.2. *Suppose that φ and ψ are in $h^\infty(\mathbb{U}_+)$. For each $a = Ms \in \mathbb{D}, \alpha_s = (\gamma_s \circ M) = \left(\alpha\left(M^{-1}(\overline{z})\right)\left(\frac{1+z}{1+\overline{z}}\right)^2 \circ M\right)$. Then*

$$T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}} = T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} - c^2([T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes d_{\overline{w}}) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w]).$$

Proof. From (4) and (5) it follows that

$$\begin{aligned}
T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}} &= T_{\varphi\circ M}T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}} + \hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}}\\
&= T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\theta_a}T_{\overline{\theta_a}} + \hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}}.
\end{aligned}$$

Thus we obtain (as in Theorem-4.1)

$$\begin{aligned}
&T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}}\\
&= T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}\hbar_{J(\theta_a)}\hbar_{\overline{\theta_a}} + \hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}}\\
&= T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} - c^2(T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}} \otimes d_{\overline{w}}) - c\hbar_{J(\varphi\circ M)}(d_{\overline{w}} \otimes d_w)\hbar_{\alpha_s(\psi\circ M)}T_{\overline{\theta_a}}\\
&= T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} - c^2(T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}} \otimes d_{\overline{w}}) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w]).
\end{aligned}$$

□

Theorem 4.3. *Suppose that φ and $\psi \in h^\infty(\mathbb{D})$. Let $a = Ms \in \mathbb{D}, \alpha_s = (\gamma_s \circ M) = \left(\alpha\left(M^{-1}(\overline{z})\right)\left(\frac{1+z}{1+\overline{z}}\right)^2 \circ M\right)$. Let $K = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}$. Then for each $a \in \mathbb{D}$,* *(i)* $KT_{\theta_a} = T_{J(\theta_a)}K - c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w])$.

(ii) *Let* $\lambda \neq 0$ *be a constant. For each* $a \in \mathbb{D}$, $\lambda KT_{\theta_a} = T_{J(\theta_a)}\lambda K + c([\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}d_w])$.

Proof. Since $\theta_a \in H^\infty(\mathbb{U}_+)$ and $J(\theta_a) \in \overline{H^\infty(\mathbb{U}_+)}$ for each $a \in \mathbb{D}$, we obtain the following identities:

$$T_{\varphi\circ M}T_{\theta_a} = T_{\theta_a}T_{\varphi\circ M} + \hbar_{J(\theta_a)}\hbar_{\varphi\circ M},$$

$$T_{\varphi\circ M}T_{J(\theta_a)} = T_{J(\theta_a)}T_{\varphi\circ M} - \hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}$$

and

$$T_{J(\theta_a)}\hbar_{\varphi\circ M} = \hbar_{\varphi\circ M}T_{\theta_a}.$$

Thus we have

$$\begin{aligned}
KT_{\theta_a} &= \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{\theta_a} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{\theta_a} \\
&= \hbar_{\alpha_s(\psi\circ M)}T_{\theta_a}T_{\varphi\circ M} + \hbar_{\alpha_s(\psi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M} - T_{\varphi\circ M}T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)} \\
&= T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} + \hbar_{\alpha_s(\psi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M} - T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)} \\
&\quad + \hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)} \\
&= T_{J(\theta_a)}K - c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) - c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w]).
\end{aligned}$$

This completes the proof of (i). Now since

$$\begin{aligned}
\hbar_{J(\varphi\circ M)(\varphi\circ M)} &= T_{\varphi\circ M}\hbar_{\varphi\circ M} + \hbar_{J(\varphi\circ M)}T_{\varphi\circ M} \\
&= \lambda[\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}] + T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} \\
&\quad + \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M},
\end{aligned}$$

hence we obtain

$$\lambda K = \hbar_{J(\varphi\circ M)(\varphi\circ M)} - T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} - \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}.$$

Thus

$$\begin{aligned}
\lambda KT_{\theta_a} &= [\hbar_{J(\varphi\circ M)(\varphi\circ M)} - T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} - \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}]T_{\theta_a} \\
&= [T_{\varphi\circ M}\hbar_{\varphi\circ M} + \hbar_{J(\varphi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} \\
&\quad - \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}]T_{\theta_a} \\
&= T_{\varphi\circ M}\hbar_{\varphi\circ M}T_{\theta_a} + \hbar_{J(\varphi\circ M)}T_{\varphi\circ M}T_{\theta_a} - T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}T_{\theta_a} \\
&\quad - \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}T_{\theta_a} \\
&= T_{\varphi\circ M}\hbar_{\varphi\circ M}T_{\theta_a} + \hbar_{J(\varphi\circ M)}T_{\varphi\circ M}T_{\theta_a} - T_{\varphi\circ M}\hbar_{\varphi\circ M}T_{\theta_a} \\
&\quad - T_{\varphi\circ M}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\theta_a} - \hbar_{J(\varphi\circ M)}T_{\varphi\circ M}T_{\theta_a} + \hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}T_{\theta_a} \\
&= -T_{\varphi\circ M}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\theta_a} + \hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}T_{\theta_a}.
\end{aligned}$$

Further for $f \in L^2_a(\mathbb{U}_+)$,

$$\begin{aligned}
([\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w])f &= \langle f, \hbar^*_{\varphi\circ M}d_w\rangle \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}d_{\overline{w}} \\
&= \langle \hbar_{\varphi\circ M}f, d_w\rangle \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}d_{\overline{w}} \\
&= \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}\langle \hbar_{\varphi\circ M}f, d_w\rangle d_{\overline{w}} \\
&= \hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}\,(d_{\overline{w}} \otimes d_w)\,\hbar_{\varphi\circ M}f \\
&= -\frac{1}{c}\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M}f.
\end{aligned}$$

Again

$$\begin{aligned} -(\hbar_{J(\varphi\circ M)}d_{\overline{w}}\otimes\hbar^*_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}d_w)f &= -\langle f,\hbar^*_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}d_w\rangle\hbar_{J(\varphi\circ M)}d_{\overline{w}} \\ &= -\hbar_{J(\varphi\circ M)}\langle\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}f,d_w\rangle d_{\overline{w}} \\ &= -\hbar_{J(\varphi\circ M)}(d_{\overline{w}}\otimes d_w)\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}f \\ &= \frac{1}{c}\hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}f, \end{aligned}$$

for $f\in L^2_a(\mathbb{U}_+)$. Thus

$$\begin{aligned} &T_{J(\theta_a)}\lambda K+c([\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}d_{\overline{w}}]\otimes[\hbar^*_{\varphi\circ M}d_w])-c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}]\otimes[\hbar^*_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}d_w]) \\ &= T_{J(\theta_a)}(\hbar_{J(\varphi\circ M)(\varphi\circ M)}-T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}-\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}) \\ &\quad -\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M}+\hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} \\ &= T_{J(\theta_a)}(T_{\varphi\circ M}\hbar_{\varphi\circ M}+\hbar_{J(\varphi\circ M)}T_{\varphi\circ M}-T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}-\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}) \\ &\quad -\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M}+\hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} \\ &= T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\varphi\circ M}+T_{J(\theta_a)}\hbar_{J(\varphi\circ M)}T_{\varphi\circ M}-T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))} \\ &\quad -T_{J(\theta_a)}\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}+\hbar_{\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M}+\hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\lambda(\alpha_s(\psi\circ M))} \\ &= -T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\lambda(\alpha_s(\psi\circ M))}+T_{J(\theta_a)}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}+\hbar_{\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M} \\ &\quad +\hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)}\hbar_{\lambda(\alpha_s(\psi\circ M))} \\ &= -(T_{J(\theta_a)}T_{\varphi\circ M}-\hbar_{J(\varphi\circ M)}\hbar_{J(\theta_a)})\hbar_{\lambda(\alpha_s(\psi\circ M))}+T_{J(\theta_a)}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M} \\ &\quad +\hbar_{\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M} \\ &= -T_{\varphi\circ M}T_{J(\theta_a)}\hbar_{\lambda(\alpha_s(\psi\circ M))}+T_{J(\theta_a)}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}+\hbar_{\lambda(\alpha_s(\psi\circ M))}\hbar_{J(\theta_a)}\hbar_{\varphi\circ M} \\ &= -T_{\varphi\circ M}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\theta_a}+\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\theta_a}T_{\varphi\circ M}+\hbar_{\lambda(\alpha_s(\psi\circ M))}(T_{\varphi\circ M}T_{\theta_a}-T_{\theta_a}T_{\varphi\circ M}) \\ &= -T_{\varphi\circ M}\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\theta_a}+\hbar_{\lambda(\alpha_s(\psi\circ M))}T_{\varphi\circ M}T_{\theta_a} \\ &= \lambda KT_{\theta_a}. \end{aligned}$$

Hence

$$\begin{aligned} \lambda KT_{\theta_a} = \quad & T_{J(\theta_a)}\lambda K+c([\hbar_{J(\varphi\circ M)-\lambda(\alpha_s(\psi\circ M))}d_{\overline{w}}]\otimes[\hbar^*_{\varphi\circ M}d_w]) \\ & -c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}]\otimes[\hbar^*_{\varphi\circ M+\lambda(\alpha_s(\psi\circ M))}d_w]). \end{aligned}$$

□

Theorem 4.4. *Let $K: L^2_a(\mathbb{U}_+)\to L^2_a(\mathbb{U}_+)$ is a compact operator. Then for $a\in\mathbb{D}$, $\lim_{|a|\to 1^-}\|K-T_{J(\theta_a)}KT^*_{\theta_a}\|=0$.*

Proof. Let $W^{-1}KW=K_1$. Then $K_1\in\mathcal{LC}(L^2_a(\mathbb{D}))$. Since $\mathcal{LF}(L^2_a(\mathbb{D}))$ is dense in $\mathcal{LC}(L^2_a(\mathbb{D}))$, hence given $\epsilon>0$, there exists vectors $f_1,\cdots,f_n$ and $g_1,\cdots,g_n$ in $L^2_a(\mathbb{D})$ such that $\|K_1-\sum_{i=1}^{n}(f_i\otimes g_i)\|<\epsilon$. Thus we shall prove the theorem only for operators of rank one. If $f\in L^2(\mathbb{D},dA)$ and $|a|\to 1^-$, we have $a-\varphi_a(z)=\frac{(1-|a|^2)z}{1-\overline{a}z}\to 0$ and $a-\mathcal{J}(\varphi_a(z))=\frac{(1-|a|^2)\overline{z}}{1-\overline{a}\overline{z}}$. Hence by the Lebesgue dominated convergence theorem, we get $\|af-\varphi_af\|_2\to 0$ and $\|af-\mathcal{J}(\varphi_a)f\|_2\to 0$, as $|a|\to 1^-$. Hence $\|\xi f-\varphi_af\|_2\to 0$ and $\|\xi f-\mathcal{J}(\varphi_a)f\|_2\to 0$ if $a\in\mathbb{D}$ tends to ξ. If $f\in L^2_a(\mathbb{D})$, then $\|\xi f-\mathcal{T}_{\varphi_a}f\|_2=\|\xi f-P(\varphi_af)\|_2\to 0$ and $\|\xi f-\mathcal{T}_{\mathcal{J}(\varphi_a)}f\|_2=\|\xi f-P(\mathcal{J}(\varphi_a)f)\|_2\to 0$ as $a\in\mathbb{D}$ tends to ξ. Here P is the Bergman projection from $L^2(\mathbb{D},dA)$ onto $L^2_a(\mathbb{D})$. Now for $f,g\in L^2_a(\mathbb{D})$, we have

$$\begin{aligned} &\|f\otimes g-\mathcal{T}_{\mathcal{J}(\varphi_a)}(f\otimes g)\mathcal{T}^*_{\varphi_a}\| \\ &= \|(\xi f)\otimes(\xi g)-(\mathcal{T}_{\mathcal{J}(\varphi_a)}f)\otimes(\mathcal{T}^*_{\overline{\varphi_a}}g)\| \\ &\le \|(\xi f-\mathcal{T}_{\mathcal{J}(\varphi_a)}f)\otimes(\xi g)\|+\|(\mathcal{T}_{\mathcal{J}(\varphi_a)}f)\otimes(\xi g-\mathcal{T}_{\varphi_a}g)\| \\ &\le \|\xi f-\mathcal{T}_{\mathcal{J}(\varphi_a)}f\|_2\|g\|_2+\|f\|_2\|\xi g-\mathcal{T}_{\varphi_a}g\|_2. \end{aligned}$$

Hence $\lim_{|a|\to 1^-} \|f \otimes g - \mathcal{T}_{\mathcal{J}(\varphi_a)}(f \otimes g)\mathcal{T}^*_{\varphi_a}\| = 0$. Therefore $\lim_{|a|\to 1^-} \|K_1 - \mathcal{T}_{\mathcal{J}(\varphi_a)} K_1 \mathcal{T}^*_{\varphi_a}\| = 0$ and $\lim_{|a|\to 1^-} \|WK_1W^{-1} - W\mathcal{T}_{\mathcal{J}(\varphi_a)}W^{-1}WK_1W^{-1}W\mathcal{T}^*_{\varphi_a}W^{-1}\| = 0$. Thus $\lim_{|a|\to 1^-} \|K - T_{J(\theta_a)}KT^*_{\theta_a}\| = 0$, where $K \in \mathcal{LC}(L^2_a(\mathbb{U}_+))$. □

Let $z \in \mathbb{D}$ and consider the Toeplitz operator $\mathcal{T}_z$ on the Bergman space $L^2_a(\mathbb{D})$ with symbol z. The operator $\mathcal{T}_z$ is called the Bergman shift operator. Notice that

$$I - \mathcal{T}^*_z\mathcal{T}_z = \operatorname{diag}\left(1 - \frac{n+1}{n+2}\right) = \operatorname{diag}\left(\frac{1}{n+2}\right)$$

and

$$I - \mathcal{T}_z\mathcal{T}^*_z = \operatorname{diag}\left(1 - \frac{n}{n+1}\right) = \operatorname{diag}\left(\frac{1}{n+1}\right).$$

Thus $I - \mathcal{T}^*_z\mathcal{T}_z = I - \mathcal{T}_{|z|^2}$ is a compact operator. Further, $I - \mathcal{T}_z\mathcal{T}^*_z$ is also a compact operator. Let $A = \mathcal{T}_{|z|^2}$. Since $\|I_{\mathcal{L}(L^2_a(\mathbb{D}))} - \mathcal{T}_{|z|^2}\| = \|\operatorname{diag}\left(\frac{1}{n+2}\right)\| < 1$, hence $\mathcal{T}_{|z|^2}$ is invertible. For details see [4]. Thus $U_a\mathcal{T}_{|z|^2}U_a = \mathcal{T}_{|\varphi_a(z)|^2}$ is invertible where $U_af = (f \circ \varphi_a)k_a, a \in \mathbb{D}$. This is so, since U_a is an involution, self-adjoint and unitary on $L^2_a(\mathbb{D})$. Hence $W\mathcal{T}_{|\varphi_a(z)|^2}W^{-1} = T_{|M\circ\tau_a|^2}$ is also invertible as W is invertible and $T_{|M\circ\tau_a|^2} = I_{\mathcal{L}(L^2_a(\mathbb{U}_+))} - D$ where D is a compact diagonal operator and $I_{\mathcal{L}(L^2_a(\mathbb{U}_+))}$ is the identity operator from $L^2_a(\mathbb{U}_+)$ into itself.

Theorem 4.5. *Suppose $\varphi, \psi \in h^\infty(\mathbb{D})$. If $\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}$ is compact, then*

$$\lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w])\| = 0.$$

Proof. Suppose that $K = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}$ is compact. Then by Theorem 4.4, we obtain $\lim_{|a|\to 1^-} \|K - T_{J(\theta_a)}KT^*_{\theta_a}\| = 0$. From Theorem 4.1, and Theorem 4.2, we obtain

$$\begin{aligned}
T_{J(\theta_a)}KT^*_{\theta_a} &= T_{J(\theta_a)}[\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}]T^*_{\theta_a} \\
&= T_{J(\theta_a)}\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T^*_{\theta_a} - T_{J(\theta_a)}T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T^*_{\theta_a} \\
&= \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}T_{|\theta_a|^2} - c^2([\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M}d_{\overline{w}}] \otimes d_{\overline{w}}) \\
&\quad + c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\varphi\circ M}d_w]) - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}T_{|\theta_a|^2} \\
&\quad + c^2([T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes d_{\overline{w}}) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w]) \\
&= KT_{|\theta_a|^2} - c^2([Kd_{\overline{w}}] \otimes d_{\overline{w}}) + c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\varphi\circ M}d_w]) \\
&\quad + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w]).
\end{aligned}$$

Now since $d_{\overline{w}}$ converges to zero weakly as $|a| \to 1^-$, we obtain $Kd_{\overline{w}} \to 0$. Hence

$$\lim_{|a|\to 1^-} \|c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\varphi\circ M}d_w]) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [T_{\theta_a}\hbar^*_{\alpha_s(\psi\circ M)}d_w])\| = 0.$$

Now

$$\begin{aligned}
&\lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w])\| \\
&= \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L^2_a(\mathbb{U}_+)\\ \|g\|=1}} \left\|\Big(c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w])\Big) g\right\| \\
&= \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L^2_a(\mathbb{U}_+)\\ \|g\|=1}} \left\|\Big(c([\hbar_{\alpha_s(\psi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\varphi\circ M}d_w]) + c([\hbar_{J(\varphi\circ M)}d_{\overline{w}}] \otimes [\hbar^*_{\alpha_s(\psi\circ M)}d_w])\Big)\Big(T_{|\theta_a|^2} + D\Big) g\right\|
\end{aligned}$$

$$\leq \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L_a^2(\mathbb{U}_+)\\ \|g\|=1}} \|c\langle T_{\theta_a}^* T_{\theta_a} g, \hbar_{\varphi\circ M}^* d_w\rangle \hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}$$
$$+c\langle T_{\theta_a}^* T_{\theta_a} g, \hbar_{\alpha_s(\psi\circ M)}^* d_w\rangle \hbar_{J(\varphi\circ M)} d_{\overline{w}}\|$$
$$+ \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L_a^2(\mathbb{U}_+)\\ \|g\|=1}} \|c\langle Dg, \hbar_{\varphi\circ M}^* d_w\rangle \hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}} + c\langle Dg, \hbar_{\alpha_s(\psi\circ M)}^* d_w\rangle \hbar_{J(\varphi\circ M)} d_{\overline{w}}\|$$
$$= \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L_a^2(\mathbb{U}_+)\\ \|g\|=1}} \|c\langle T_{\theta_a} g, T_{\theta_a}\hbar_{\varphi\circ M}^* d_w\rangle \hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}$$
$$+c\langle T_{\theta_a} g, T_{\theta_a}\hbar_{\alpha_s(\psi\circ M)}^* d_w\rangle \hbar_{J(\varphi\circ M)} d_{\overline{w}}\|$$
$$+ \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L_a^2(\mathbb{U}_+)\\ \|g\|=1}} \|c\langle g, D\hbar_{\varphi\circ M}^* d_w\rangle \hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}} + c\langle g, D\hbar_{\alpha_s(\psi\circ M)}^* d_w\rangle \hbar_{J(\varphi\circ M)} d_{\overline{w}}\|$$
$$= \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L_a^2(\mathbb{U}_+)\\ \|g\|=1}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\varphi\circ M}^* d_w]$$
$$+[\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\alpha_s(\psi\circ M)}^* d_w]) T_{\theta_a} g\|$$
$$+ \lim_{\substack{w=x+iy\\ y\to 0}} \sup_{\substack{g\in L_a^2(\mathbb{U}_+)\\ \|g\|=1}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [D\hbar_{\varphi\circ M}^* d_w] + [\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [D\hbar_{\alpha_s(\psi\circ M)}^* d_w]) g\|$$
$$= \lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\varphi\circ M}^* d_w]$$
$$+[\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\alpha_s(\psi\circ M)}^* d_w]) T_{\theta_a}\|$$
$$+ \lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [D\hbar_{\varphi\circ M}^* d_w]) + c([\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [D\hbar_{\alpha_s(\psi\circ M)}^* d_w])\|$$

Since D is compact, hence $D\hbar_{\varphi\circ M}^*$ is compact. Further $\{d_w\}$ converges to 0 weakly as $y \to 0$, hence the last limit is equal to 0. Thus we get

$$\lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [\hbar_{\varphi\circ M}^* d_w] + [\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [\hbar_{\alpha_s(\psi\circ M)}^* d_w])\|$$
$$\leq \lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\varphi\circ M}^* d_w] + [\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\alpha_s(\psi\circ M)}^* d_w]) T_{\theta_a}\|$$
$$\leq \lim_{\substack{w=x+iy\\ y\to 0}} \|c([\hbar_{\alpha_s(\psi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\varphi\circ M}^* d_w] + [\hbar_{J(\varphi\circ M)} d_{\overline{w}}] \otimes [T_{\theta_a}\hbar_{\alpha_s(\psi\circ M)}^* d_w])\| \, \|T_{\theta_a}\|$$
$$= 0,$$

since T_{θ_a} is bounded for $a \in \mathbb{D}$. □

Proposition 4.6. *Let $\hbar_\varphi$ be a bounded Hankel operator on $L_a^2(\mathbb{U}_+)$ with $\varphi \in L^\infty(\mathbb{U}_+)$ and let $g \in L_a^2(\mathbb{U}_+)$. Then $\hbar_\varphi^* g^+ = (\hbar_\varphi g)^+$ where $h^+(z) = \overline{h(\overline{z})}$.*

Proof. Notice that for all $g \in L_a^2(\mathbb{U}_+)$, we have $(Ug)(w) = \overline{w}Jg(w) = \overline{w}g(\overline{w}), w \in \mathbb{U}_+$. Hence $(Ug)^+(w) = \overline{(Ug)(\overline{w})} = \overline{wg(w)} = \overline{w}\overline{g(w)}$ and $(Ug^+)(w) = \overline{w}g^+(\overline{w}) = \overline{w}\overline{g(w)}$. Thus $(Ug)^+ = Ug^+$ and $P_+g^+ = (P_+g)^+$ where $P_+ : L^2(\mathbb{U}_+) \to L_a^2(\mathbb{U}_+)$ is the orthogonal projection. Thus

$$\hbar_\varphi^* g^+ = \hbar_{\varphi^+} g^+ = P_+U(\varphi^+ g^+) = P_+(U\varphi g)^+ = (P_+U\varphi g)^+ = (\hbar_\varphi g)^+.$$

Since $\|g^+\| = \|g\|$ for all $g \in L^2$, we obtain $\|\hbar_\varphi^* g^+\| = \|(\hbar_\varphi g)^+\| = \|\hbar_\varphi g\|$. □

Corollary 4.7. *Suppose that $\varphi \in L^\infty(\mathbb{U}_+)$. For each $w \in \mathbb{U}_+$,*

$$\|\hbar^*_\varphi d^+_{\overline{w}}\|_2 = \|\hbar_\varphi d_{\overline{w}}\|_2$$

.

Proposition 4.8. *Suppose that $\varphi, \psi \in h^\infty(\mathbb{U}_+)$. Let $K = \hbar_{\alpha_s(\psi\circ m)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ m)}$. Then K^*K is a finite sum of finite products of Toeplitz operators.*

Proof. Let $K = \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} - T_{\varphi\circ M}\hbar_{\alpha_s(\psi\circ M)}$. Then by (5),

$$K = -\hbar_{J(\varphi\circ M)\alpha_s(\psi\circ M)} + \hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} + \hbar_{J(\varphi\circ M)}T_{\alpha_s(\psi\circ M)}. \tag{7}$$

Taking adjoints of both sides of (7), we obtain

$$K^* = -\hbar_{(J(\varphi\circ M)\alpha_s(\psi\circ M))^+} + T^*_{\alpha_s(\psi\circ M)}\hbar_{(J(\varphi\circ M))^+} + T^*_{\varphi\circ M}\hbar_{[\alpha_s(\psi\circ M)]^+},$$

since $\hbar^*_f = \hbar_{f^+}$ where $f^+(w) = \overline{f(\overline{w})}$, $f \in L^\infty(\mathbb{U}_+)$. Thus

$$\begin{aligned} K^*K \;=\; & \hbar_{(J(\varphi\circ M)\alpha_s(\psi\circ M))^+}\hbar_{J(\varphi\circ M)\alpha_s(\psi\circ M)} - \hbar_{(J(\varphi\circ M)\alpha_s(\psi\circ M))^+} \\ & [\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} + \hbar_{J(\varphi\circ M)}T_{\alpha_s(\psi\circ M)}] - [T^*_{\alpha_s(\psi\circ M)}\hbar_{(J(\varphi\circ m))^+} \\ & + T^*_{\varphi\circ M}\hbar_{(\alpha_s(\psi\circ M))^+}]\hbar_{J(\varphi\circ M)\alpha_s(\psi\circ M)} + [T^*_{\alpha_s(\psi\circ M)}\hbar_{J(\varphi\circ M))^+} \\ & + T^*_{\varphi\circ M}\hbar_{(\alpha_s(\psi\circ M))^+}][\hbar_{\alpha_s(\psi\circ M)}T_{\varphi\circ M} + \hbar_{J(\varphi\circ M)}T_{\alpha_s(\psi\circ M)}]. \end{aligned} \tag{8}$$

The first form in the right hand side of (8) is a semi commutator of two Toeplitz operators since for two functions f and g in $h^\infty(\mathbb{U}_+)$, by (4) $\hbar_{f^+}\hbar_g = T_{J(f)g} - T_{Jf}T_g$; both the second and the third terms are products of a Toeplitz operator and a semi commutator of two Toeplitz operators; the fourth term is the product of two Toeplitz operators and a semi commutator of two Toeplitz operators. Therefore, it follows from (8) that K^*K is a finite sum of finite products of Toeplitz operators. □

Theorem 4.9. *Let $V(\mathbb{U}_+) = \{\varphi \in L^\infty(\mathbb{U}_+) : ess \lim_{\substack{s=x+iy \\ y\to 0}} \varphi(s) = 0\}$. If $\varphi \in V(\mathbb{U}_+)$ then $M_\varphi\big|_{L^2_a(\mathbb{U}_+)}$, T_φ and $\hbar_\varphi, S_\varphi$ are compact operators.*

Proof. Let $\varphi \in L^\infty(\mathbb{U}_+)$ and assume that the support supp φ is a compact subset of $\mathbb{U}_+$. Let $G =$ supp φ is a compact subset of $\mathbb{U}_+$. Let $L = \text{dist}(G, \mathbb{C} \setminus \mathbb{U}_+)$. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence in $L^2_a(\mathbb{U}_+)$ which converges weakly to zero. This implies the sequence $\{f_n\}$ must be bounded. Let $\|f_n\|_2 \le M$ for all $n \in \mathbb{N}$. Then $|f_n(w)| \le \|f_n\|_2\|b_w\|_2 \le \dfrac{M}{\sqrt{\pi L^2}}$ for all $w \in \mathbb{U}_+$, where b_w is the reproducing kernel of $L^2_a(\mathbb{U}_+)$ at $w \in \mathbb{U}_+$. Hence $|\varphi(w)f_n(w)| \le \dfrac{M\|\varphi\|_\infty}{\sqrt{\pi L^2}}$ for all $w \in \mathbb{U}_+$. Further, $\{f_n\}$ converges to 0 weakly implies $f_n(w) = \langle f_n, b_w\rangle \to 0$ for all $w \in \mathbb{U}_+$. Using Lebesgue dominated convergence theorem we obtain $\|\varphi f_n\|_2^2 = \int_{\mathbb{U}_+} |\varphi(w)f_n(w)|^2 dA(w) = \int_G |\varphi(w)f_n(w)|^2 dA(w) \to 0$ as n tends to infinity. Hence $M_\varphi\big|_{L^2_a(\mathbb{U}_+)}$ maps weakly convergent sequence into norm convergent sequence and $M_\varphi\big|_{L^2_a(\mathbb{U}_+)}$ is compact. Since $T_\varphi = P_+M_\varphi\big|_{L^2_a(\mathbb{U}_+)}$ and S_φ is unitarily equivalent to $h_\varphi f = \overline{P_+}(\varphi f) = \overline{P_+}M_\varphi f$ hence T_φ, S_φ are compact. Thus if $\varphi \in L^\infty(\mathbb{U}_+)$ and supp φ is a compact subset of $\mathbb{U}_+$, then $M_\varphi\big|_{L^2_a(\mathbb{U}_+)}$, T_φ and S_φ are compact operators. Now let $\varphi \in V(\mathbb{U}_+)$. Then there exist a sequence $\{\varphi_n\} \in V(\mathbb{U}_+)$ such that supp φ_n are compact subsets of $\mathbb{U}_+$ and $\{\varphi_n\}$ converges uniformly to φ. Hence $\{M_{\varphi_n}\}$ converges to M_φ in norm. From the first part of the proof it follows that M_{φ_n} are all compact. Hence M_φ is compact. Hence T_φ, S_φ are also compact. Since $\hbar_\varphi f = S_{w\varphi} f$ hence, $\hbar_\varphi$ is also a compact operator. □

Corollary 4.10. *Let $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ and assume T is a finite sum of finite products of Toeplitz operators with symbols $\varphi_{ij} \in L^\infty(\mathbb{U}_+)$. That is, $T = \sum_{i=1}^{n} \prod_{j=1}^{m_i} T_{\varphi_{ij}}$. If $\varphi_{ij} \circ M^{-1} \in C(\overline{\mathbb{D}})$ for all $i = 1, \cdots, n, j = 1, \cdots, m_i$ then $T = T_\varphi + K$ where $\varphi \circ M^{-1} \in C(\overline{\mathbb{D}})$ and $K \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ is a compact operator. If in addition $\varphi \in V(\mathbb{U}_+)$ then $\lim_{|a|\to 1^-} \|T - T^*_{M\circ\tau_a} T T_{M\circ\tau_a}\| = 0$.*

Proof. Let $T \in \mathcal{L}(L_a^2(\mathbb{U}_+))$ and $T = \sum_{i=1}^{n} \prod_{j=1}^{m_i} T_{\varphi_{ij}}$ where $\varphi_{ij} \in L^\infty(\mathbb{U}_+)$ and $\varphi_{ij} \circ M^{-1} \in C(\overline{\mathbb{D}}), i = 1, \cdots, n, j = 1, \cdots, m_i$. Then $W^{-1}TW \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $W^{-1}TW = \sum_{i=1}^{n} \prod_{j=1}^{m_i} W^{-1} T_{\varphi_{ij}} W = \sum_{i=1}^{n} \prod_{j=1}^{m_i} \mathcal{T}_{\varphi_{ij}\circ M^{-1}}$. Now since $\varphi_{ij} \circ M^{-1} \in C(\overline{\mathbb{D}})$ for all i and j, hence $W^{-1}TW$ belongs to the C^*-algebra generated by $\{\mathcal{T}_\psi : \psi \in C(\overline{\mathbb{D}})\}$. From [6], [10] it follows that $W^{-1}TW = \mathcal{T}_\Xi + K_1$ where $\Xi \in C(\overline{\mathbb{D}}), K_1 \in \mathcal{L}(L_a^2(\mathbb{D}))$ is compact. Thus

$$\begin{aligned} T = W\mathcal{T}_\Xi W^{-1} + K &= T_{\Xi\circ M} + K, \text{where } K \in \mathcal{LC}(L_a^2(\mathbb{U}_+)) \\ &= T_\varphi + K, \text{where } \Xi \circ M = \varphi \in L^\infty(\mathbb{U}_+) \text{ and } \varphi \circ M^{-1} \in C(\overline{\mathbb{D}}). \end{aligned}$$

If $\varphi \in V(\mathbb{U}_+)$, then by Theorem 4.9, T_φ is compact. Hence $T_\varphi + K = T$ is a compact operator in $\mathcal{L}(L_a^2(\mathbb{U}_+))$. Thus from Theorem 4.4, it follows that

$$\lim_{|a|\to 1^-} \|T - T^*_{M\circ\tau_a} T T_{M\circ\tau_a}\| = 0.$$

□

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