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On σ-Amenability of T_{σ_A,σ_B}

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Abstract. Let σ_A and σ_B be two homomorphisms on Banach algebras A and B, respectively. In this paper, we study σ-amenability, σ-weak amenability, σ-biflatness, and σ-biprojectivity of triangular Banach algebras of the form T_{σ_A,σ_B}, where $\sigma = \sigma_A \oplus \sigma_B$.

1. Introduction

Let A be a Banach algebra. The set of all continuous homomorphisms from A into A is denoted by $Hom(A)$. Suppose that $\sigma \in Hom(A)$, and that X is a Banach A-bimodule. A bounded linear map $D : A \longrightarrow X$ is a *σ-derivation* if $D(ab) = D(a).\sigma(b) + \sigma(a).D(b)$ for all $a, b \in A$. A σ-derivation D is *σ-inner derivation* if there exists $x \in X$ such that $D(a) = \sigma(a)\cdot x - x\cdot\sigma(a)$ for all $a \in A$. The set of all σ-derivation from A into X is denoted by $Z^1_\sigma(A, X)$, and the set of all σ-inner derivations from A into X by $N^1_\sigma(A, X)$. Then, we define the space $H^1_\sigma(A, X) = \frac{Z^1_\sigma(A,X)}{N^1_\sigma(A,X)}$. We say A is *σ-amenable* if $H^1_\sigma(A, X^*) = 0$ for every Banach A-bimodule X [11]. We call A is *σ-weakly amenable* if $H^1_\sigma(A, A^*) = 0$ [3, 13]. Note that the module version of such notions are available in [2].

For a Banach algebra A, the corresponding diagonal operator $\pi : A\widehat{\otimes}A \longrightarrow A$ is defined by $\pi(a \otimes b) = ab$. Let X and Y be Banach A-bimodules, and $\sigma \in Hom(A)$. A bounded linear map $T : X \longrightarrow Y$ is a σ-A-bimodule homomorphism if $T(a \cdot x) = \sigma(a) \cdot T(x)$ and $T(x \cdot a) = T(x) \cdot \sigma(a)$ for $a \in A, x \in X$. Then, A is *σ-biprojective* if there exists a σ-A-bimodule homomorphism $\rho : A \longrightarrow A\widehat{\otimes}A$ such that $\pi \circ \rho = \sigma$ [14]. Moreover, A is *σ-biflat* if there exists a bounded linear map $\rho : (A\widehat{\otimes}A)^* \longrightarrow A^*$ satisfying $\rho(\sigma(a) \cdot \lambda) = a \cdot \rho(\lambda)$ and $\rho(\lambda \cdot \sigma(a)) = \rho(\lambda) \cdot a$, such that $\rho \circ \pi^* = \sigma^*$ where $a \in A$, $\lambda \in (A\widehat{\otimes}A)^*$ [7].

Let A and B be Banach algebras, and X be a Banach A, B-module; that is, X is a left Banach A-module and is a right Banach B-module such that $\|a \cdot x \cdot b\| \leq \|a\|\,\|x\|\,\|b\|$, for $a \in A$, $x \in X$ and $b \in B$. We define the corresponding triangular Banach algebra $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ with the sum and product being given by the usual 2×2 matrix operations and obvious internal module actions along with the norm

$$\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|x\| + \|b\|, \quad (a \in A, b \in B, x \in X).$$

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For Banach A, B-module X, the first dual space of X, that is denoted by X^* is a Banach B, A-module with the following actions:

$$\langle b \cdot x^*, x\rangle = \langle x^*, x \cdot b\rangle \ \ and \ \ \langle x^* \cdot a, x\rangle = \langle x^*, a \cdot x\rangle$$

for all $a \in A, b \in B, x \in X and\ x^* \in X^*$. Moreover, for each $x \in X, x^* \in X^*$ we can consider $x \cdot x^* \in A^*$ *and* $x^* \cdot x \in B^*$ through

$$\langle x \cdot x^*, a\rangle = \langle x^*, a \cdot x\rangle, \ \langle x^* \cdot x, b\rangle = \langle x^*, x \cdot b\rangle \ \ (a \in A, b \in B).$$

Similarly for each $x \in X, a^{**} \in A^{**}$ and $b^{**} \in B^{**}$ we can consider $a^{**} \cdot x \in X^{**}$ and $x \cdot b^{**} \in X^{**}$ through

$$\langle a^{**} \cdot x, x^*\rangle = \langle a^{**}, x \cdot x^*\rangle, \ \langle x \cdot b^{**}, x^*\rangle = \langle b^{**}, x^* \cdot x\rangle,$$

for all $x^* \in X^*$. We may continue this process to higher order dual spaces of X ; that is, $X^{(2n)}$ is a Banach A, B-module $X^{(2n-1)}$ is a Banach B, A-module and $A^{(2n)} \cdot X \subseteq X^{(2n)}$, $X \cdot B^{(2n)} \subseteq X^{(2n)}$, $X \cdot X^{(2n-1)} \subseteq A^{(2n-1)}$, $X^{(2n-1)} \cdot X \subseteq B^{(2n-1)}$ for all $n \in \mathbb{N}$. Here, we recall that the n-weak amenability of Banach algebras based on homomorphisms were investigated in [4].

In [1], the authors introduced a new product on triangular Banach algebras as follows.

Definition 1.1. ([1, Definition 1.1]) *Let A and B be Banach algebras, X be a Banach A, B-module, $\sigma_A \in Hom(A)$ and $\sigma_B \in Hom(B)$. Let T_{σ_A,σ_B} denote the algebra whose underlying Banach space is T but whose multiplication is defined by*

$$\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & b_1b_2 \end{pmatrix},$$

for all $\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} \in T.$

Amenability, weak amenability, biflatness and biprojectivity of T_{σ_A,σ_B} have been studied in [1]. In this paper, motivated by [1–4], we shall study σ-amenability, σ-weak amenability, σ- biflatness and σ-biprojectivity of T_{σ_A,σ_B} for the homomorphism $\sigma = \sigma_A \oplus \sigma_B$.

The organization of the paper is as follows. In section 2, we prove that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-amenable if and only if A is σ_A-amenable and B is σ_B-amenable and $X = 0$ provided σ_A and σ_B are idempotents. Section 3 is devoted to $\sigma_A \oplus \sigma_B$-weak amenability of T_{σ_A,σ_B}. In other words, for unital Banach algebras A and B with idempotents σ_A and σ_B, we show that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-weakly amenable if and only if A is σ_A-weakly amenable and B is σ_B-weakly amenable. In section 4, under some mild conditions, we prove that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-biprojective (biflat) if and only if A is σ_A-biprojective (biflat) and B is σ_B-biprojective (biflat) and $X = 0$.

2. $\sigma_A \oplus \sigma_B$- amenability of T_{σ_A,σ_B}

Let A and B be Banach algebras and $\sigma_A \in Hom(A)$, $\sigma_B \in Hom(B)$. Let X be a Banach algebra A, B-module. We consider the map $\sigma_A \oplus \sigma_B : T_{\sigma_A,\sigma_B} \to T_{\sigma_A,\sigma_B}$ defined by

$$\sigma_A \oplus \sigma_B \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} \sigma_A(a) & x \\ 0 & \sigma_B(b) \end{pmatrix} \ (a \in A, b \in B, x \in X).$$

Proposition 2.1. *Let A and B be Banach algebras and $\sigma_A \in Hom(A)$, $\sigma_B \in Hom(B)$. Let X be a Banach algebra A, B-module. If $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$, then $\sigma_A \oplus \sigma_B \in Hom(T_{\sigma_A,\sigma_B})$.*

Proof. For $\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} \in T_{\sigma_A,\sigma_B}$,

$$\begin{aligned} \sigma_A \oplus \sigma_B(\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix}) &= \sigma_A \oplus \sigma_B \begin{pmatrix} a_1a_2 & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & b_1b_2 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_A(a_1a_2) & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & \sigma_B(b_1b_2) \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma_A \oplus \sigma_B \begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix} \sigma_A \oplus \sigma_B \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} &= \begin{pmatrix} \sigma_A(a_1) & x_1 \\ 0 & \sigma_B(b_1) \end{pmatrix} \begin{pmatrix} \sigma_A(a_2) & x_2 \\ 0 & \sigma_B(b_2) \end{pmatrix} \\
&= \begin{pmatrix} \sigma_A(a_1a_2) & \sigma_A^2(a_1) \cdot x_2 + x_1 \cdot \sigma_B^2(b_2) \\ 0 & \sigma_B(b_1b_2) \end{pmatrix} \\
&= \begin{pmatrix} \sigma_A(a_1a_2) & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & \sigma_B(b_1b_2) \end{pmatrix}.
\end{aligned}$$

Thus $\sigma_A \oplus \sigma_B$ is a homomorphism. □

Theorem 2.2. *Let A and B be Banach algebras, X be a Banach A,B-module, and $\sigma_A \in Hom(B)$, $\sigma_B \in Hom(A)$ such that $\sigma_A^2 = \sigma_A$, $\sigma_B^2 = \sigma_B$. Then T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-amenable if and only if A is σ_A-amenable and B is σ_B-amenable and $X = 0$.*

Proof. Suppose that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-amenable and $D : A \longrightarrow Y^*$ is a σ_A-derivation such that Y is a Banach A-bimodule. Define the map $P : T_{\sigma_A,\sigma_B} \longrightarrow A$ by $P\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = a$. It is obvious, that P is a homomorphism. Now we can consider Y as a T_{σ_A,σ_B}-bimodule via

$$\sigma_A \oplus \sigma_B(T') \cdot y = P(\sigma_A \oplus \sigma_B(T')) \cdot y \quad \textit{and} \quad y \cdot \sigma_A \oplus \sigma_B(T') = y \cdot P(\sigma_A \oplus \sigma_B(T')),$$

for $T' \in T_{\sigma_A,\sigma_B}$, $y \in Y$. Hence for each $T_1, T_2 \in T_{\sigma_A,\sigma_B}$, we have

$$\begin{aligned}
D \circ P(T_1T_2) = D(P(T_1)P(T_2)) &= D(P(T_1)) \cdot \sigma_A(P(T_2)) + \sigma_A(P(T_1)) \cdot D(P(T_2)) \\
&= D \circ P(T_1) \cdot P(\sigma_A \oplus \sigma_B(T_2)) + P(\sigma_A \oplus \sigma_B(T_1)) \cdot D \circ P(T_2) \\
&= D \circ P(T_1) \cdot \sigma_A \oplus \sigma_B(T_2) + \sigma_A \oplus \sigma_B(T_1) \cdot D \circ P(T_2).
\end{aligned}$$

Hence $D \circ P$ is a $\sigma_A \oplus \sigma_B$-derivation, so $D \circ P$ is $\sigma_A \oplus \sigma_B$-inner, thus there exists a $y^* \in Y^*$ such that for every $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T_{\sigma_A,\sigma_B}$, we have,

$D(a) = D \circ P\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \sigma_A \oplus \sigma_B \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot y^* - y^* \cdot \sigma_A \oplus \sigma_B \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \sigma_A(a) \cdot y^* - y^* \cdot \sigma_A(a).$

It implies that A is σ_A-amenable and similarly, B is σ_B-amenable. Now we prove that $X = 0$. Since $X \simeq \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, we get $X^{**} \simeq \begin{pmatrix} 0 & X^{**} \\ 0 & 0 \end{pmatrix}$. Hence for the map $D : T_{\sigma_A,\sigma_B} \longrightarrow X^{**}$ defined by $D\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, we have

$$\begin{aligned}
D(\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix}) = D\begin{pmatrix} a_1a_2 & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & b_1b_2 \end{pmatrix} &= \begin{pmatrix} 0 & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & x_1 \cdot \sigma_B(b_2) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_A(a_1) \cdot x_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_A(a_2) & x_2 \\ 0 & \sigma_B(b_2) \end{pmatrix} + \begin{pmatrix} \sigma_A(a_1) & x_1 \\ 0 & \sigma_B(b_1) \end{pmatrix}\begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix} \\
&= D\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix} \cdot \sigma_A \oplus \sigma_B \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} + \sigma_A \oplus \sigma_B \begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix} \cdot D\begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix}.
\end{aligned}$$

Therefore, D is a $\sigma_A \oplus \sigma_B$-derivation. Since T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-amenable, there exists an element $\begin{pmatrix} 0 & x^{**} \\ 0 & 0 \end{pmatrix} \in X^{**}$ such that,

$$\begin{aligned}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = D\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} &= \sigma_A \oplus \sigma_B \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 0 & x^{**} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x^{**} \\ 0 & 0 \end{pmatrix} \cdot \sigma_A \oplus \sigma_B \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_A(a) \cdot x^{**} - x^{**} \cdot \sigma_B(b) \\ 0 & 0 \end{pmatrix},\end{aligned}$$

set $a = b = 0$, as a result $x = 0$ and hence $X = 0$. Conversely, suppose that A is σ_A-amenable and B is σ_B-amenable and $X = 0$. Then T_{σ_A,σ_B} is the l^1-direct sum of A and B , that is, $T_{\sigma_A,\sigma_B} = A \oplus_1 B$. Since A and $B \simeq \frac{A \oplus_1 B}{A}$ are ideals in $A \oplus_1 B$, similar to the proof of [12, Theorem 2.3.10], we obtain that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-amenable. □

Let A and B be Banach algebras, and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A, \sigma_B^2 = \sigma_B$. By [10, Theorem 4.2] and [1, Theorem 2.3], amenability of two triangulars Banach algebras T and T_{σ_A,σ_B} are equivalent.

It is clear that amenability of T_{σ_A,σ_B} implies $\sigma_A \oplus \sigma_B$-amenability of T_{σ_A,σ_B}. However, we show that the converse is not true.

Example 2.3. *Suppose that A is a non-amenable Banach algebra with a right (or a left) approximate identity. Then $A^\sharp$ (the unitization of A) is not amenable [12, Corollary 2.3.11]. Define $\sigma_{A^\sharp} \in Hom(A^\sharp)$ by $\sigma_{A^\sharp}(a + \lambda) = \lambda$ for $a \in A$, $\lambda \in C$. Then $\sigma_{A^\sharp}^2 = \sigma_{A^\sharp}$. By [8, Corollary 3.2], $A^\sharp$ is $\sigma_{A^\sharp}$-amenable. Hence $T_{\sigma_{A^\sharp},\sigma_{A^\sharp}} = \begin{pmatrix} A^\sharp & 0 \\ 0 & A^\sharp \end{pmatrix}$ is $\sigma_A^\sharp \oplus \sigma_A^\sharp$-amenable by Theorem 2.2, however $T_{\sigma_{A^\sharp},\sigma_{A^\sharp}}$ is not amenable, since $A^\sharp$ is not amenable [1, Theorem 2.3]. Consequently T is not amenable.*

3. $\sigma_A \oplus \sigma_B$-weak amenability of T_{σ_A,σ_B}

Lemma 3.1. *Let A and B be Banach algebras, X be a Banach A, B-module and $\sigma_A \in Hom(A)$, $\sigma_B \in Hom(B)$. Then for $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T_{\sigma_A,\sigma_B}$ and $\begin{pmatrix} a^{(2n-1)} & x^{(2n-1)} \\ 0 & b^{(2n-1)} \end{pmatrix} \in T^{(2n-1)}_{\sigma_A,\sigma_B}$, the following statements hold;*

(i) $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} a^{(2n-1)} & x^{(2n-1)} \\ 0 & b^{(2n-1)} \end{pmatrix} = \begin{pmatrix} a \cdot a^{(2n-1)} + \sigma_A^{(2n-1)}(x \cdot x^{(2n-1)}) & \sigma_B(b).x^{(2n-1)} \\ 0 & b \cdot b^{(2n-1)} \end{pmatrix}$;

(ii) $\begin{pmatrix} a^{(2n-1)} & x^{(2n-1)} \\ 0 & b^{(2n-1)} \end{pmatrix}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^{(2n-1)} \cdot a & x^{(2n-1)} \cdot \sigma_A(a) \\ 0 & b^{(2n-1)} \cdot b + \sigma_B^{(2n-1)}(x^{(2n-1)} \cdot x) \end{pmatrix}$.

Proof. (i) It is easily seen that for each $\begin{pmatrix} a^{(2n-2)} & x^{(2n-2)} \\ 0 & b^{(2n-2)} \end{pmatrix} \in T^{(2n-2)}_{\sigma_A,\sigma_B}$ we have $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} a^{(2n-2)} & x^{(2n-2)} \\ 0 & b^{(2n-2)} \end{pmatrix} = \begin{pmatrix} a \cdot a^{(2n-2)} & \sigma_A(a) \cdot x^{(2n-2)} + x \cdot \sigma_B^{(2n-2)}(b^{(2n-2)}) \\ 0 & b \cdot b^{(2n-2)} \end{pmatrix}$,

and

$$\begin{pmatrix} a^{(2n-2)} & x^{(2n-2)} \\ 0 & b^{(2n-2)} \end{pmatrix}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^{(2n-2)} \cdot a & \sigma_A^{(2n-2)}(a^{(2n-2)}) \cdot x + x^{(2n-2)} \cdot \sigma_B(b) \\ 0 & b^{(2n-2)} \cdot b \end{pmatrix}.$$

So,

$$
\begin{aligned}
&\langle \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} a^{(2n-1)} & x^{(2n-1)} \\ 0 & b^{(2n-1)} \end{pmatrix}, \begin{pmatrix} a^{(2n-2)} & x^{(2n-2)} \\ 0 & b^{(2n-2)} \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} a^{(2n-1)} & x^{(2n-1)} \\ 0 & b^{(2n-1)} \end{pmatrix}, \begin{pmatrix} a^{(2n-2)} & x^{(2n-2)} \\ 0 & b^{(2n-2)} \end{pmatrix} \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} a^{(2n-1)} & x^{(2n-1)} \\ 0 & b^{(2n-1)} \end{pmatrix}, \begin{pmatrix} a^{(2n-2)} \cdot a & \sigma_A^{(2n-2)}(a^{(2n-2)}) \cdot x + x^{(2n-2)} \cdot \sigma_B(b) \\ 0 & b^{(2n-2)} \cdot b \end{pmatrix} \rangle \\
&= \langle a^{(2n-1)}, a^{(2n-2)} \cdot a \rangle + \langle x^{(2n-1)}, \sigma_A^{(2n-2)}(a^{(2n-2)}) \cdot x + x^{(2n-2)} \cdot \sigma_B(b) \rangle \\
&+ \langle b^{(2n-1)}, b^{(2n-2)} \cdot b \rangle \\
&= \langle a \cdot a^{(2n-1)}, a^{(2n-2)} \rangle + \langle \sigma_A^{(2n-1)}(x \cdot x^{(2n-1)}), a^{(2n-2)} \rangle \\
&+ \langle \sigma_B(b) \cdot x^{(2n-1)}, x^{(2n-2)} \rangle + \langle b \cdot b^{(2n-1)}), b^{(2n-2)} \rangle \\
&= \langle \begin{pmatrix} a \cdot a^{(2n-1)} + \sigma_A^{(2n-1)}(x \cdot x^{(2n-1)}) & \sigma_B(b) \cdot x^{(2n-1)} \\ 0 & b \cdot b^{(2n-1)} \end{pmatrix}, \begin{pmatrix} a^{(2n-2)} & x^{(2n-2)} \\ 0 & b^{(2n-2)} \end{pmatrix} \rangle.
\end{aligned}
$$

It implies (*i*). The proof of (*ii*) is similar. □

Suppose that A has a unit e_A and B has a unit e_B and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$, then X is unital, if $e_A \cdot x = x \cdot e_B = x$ for all $x \in X$. Moreover, X is said (σ_A, σ_B)-*unital* if $\sigma_A(e_A) \cdot x = x \cdot \sigma_B(e_B) = x$ for all $x \in X$.

Lemma 3.2. *Let A and B be unital Banach algebras and $\sigma_A \in Hom(A)$, $\sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$. Let X be a (σ_A, σ_B)-unital Banach A, B-module. Let $D : T_{\sigma_A,\sigma_B} \longrightarrow T_{\sigma_A,\sigma_B}^{(2n-1)}$ be a $\sigma_A \oplus \sigma_B$-derivation and $n \in N$. Then there exist σ_A-derivation $\delta_A : A \longrightarrow A^{(2n-1)}$ and σ_B-derivation $\delta_B : B \longrightarrow B^{(2n-1)}$ and $x_0^{(2n-1)} \in X^{(2n-1)}$, such that*

(*i*) $D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_A(a) & x_0^{(2n-1)} \cdot \sigma_A(a) \\ 0 & 0 \end{pmatrix}$;

(*ii*) $D\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_B(b) \cdot x_0^{(2n-1)} \\ 0 & \delta_B(b) \end{pmatrix}$;

(*iii*) $D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)}) & 0 \\ 0 & \sigma_B^{(2n-1)}(x_0^{(2n-1)} \cdot x) \end{pmatrix}$.

Proof. (i) Setting $D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_A(a) & s(a) \\ 0 & \theta(a) \end{pmatrix}$, we wish to find the maps $\delta_A(a), s(a)$ *and* $\theta(a)$. Since D is a $\sigma_A \oplus \sigma_B$-derivation, so

$$D\begin{pmatrix} aa' & 0 \\ 0 & 0 \end{pmatrix} = D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_A(a') & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_A(a) & 0 \\ 0 & 0 \end{pmatrix} D\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}$$

.
By lemma 3.1, $\begin{pmatrix} \delta_A(aa') & s(aa') \\ 0 & \theta(aa') \end{pmatrix} = \begin{pmatrix} \delta_A(a) \cdot \sigma_A(a') & s(a) \cdot \sigma_A(a) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_A(a) \cdot \delta_A(a') & 0 \\ 0 & 0 \end{pmatrix}$. As a result $\delta_A(aa') = \delta_A(a) \cdot \sigma_A(a') + \sigma_A(a) \cdot \delta_A(a')$, i.e, δ_A is a σ_A-derivation, and also $s : A \longrightarrow X^{(2n-1)}$ is a right σ_A-A-module homomorphism. Consider $s(e_A) = x_0^{(2n-1)} \in X^{(2n-1)}$, therefore $s(a) = s(e_A) \cdot \sigma_A(a) = x_0^{(2n-1)} \cdot \sigma_A(a)$. Moreover $\theta(aa') = 0$ for each $a, a' \in A$, hence $\theta(a) = 0$.
(*ii*) Suppose that $D\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \theta(b) & r(b) \\ 0 & \delta_B(b) \end{pmatrix}$. A calculation similar to (*i*) shows that $\delta_B : B \longrightarrow X^{(2n-1)}$ is a

σ_B-derivation and $\theta(b) = 0$. Furthermore from
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = D\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = D\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} \sigma_A(a) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_B(b) \end{pmatrix} \cdot D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$
we obtain $-\sigma_B(b) \cdot x_0^{(2n-1)} \cdot \sigma_A(a) = r(b) \cdot \sigma_A(a)$. Putting $a = e$, since X is (σ_A, σ_B)-unital, we conclude that $r(b) = -\sigma_B(b) \cdot x_0^{(2n-1)}$.
(iii) Suppose $D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \theta(x) & s(x) \\ 0 & r(x) \end{pmatrix}$. Since D is a $\sigma_A \oplus \sigma_B$-derivation,
$D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = D\left(\begin{pmatrix} e_A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = D\begin{pmatrix} e_A & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_A(e_A) & 0 \\ 0 & 0 \end{pmatrix} \cdot D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$.
Similar arguments as in (i) and (ii) shows that $r(x) = \sigma_B^{(2n-1)}(x_0^{(2n-1)} \cdot x)$ and $s(x) = 0$.
Moreover from,
$D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = D(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_B \end{pmatrix}) = D\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \sigma_B(e_B) \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot D\begin{pmatrix} 0 & 0 \\ 0 & e_B \end{pmatrix}$,
we get $\theta(x) = -\sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)})$. $\square$

Lemma 3.3. *Let A and B be Banach algebras, let $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$. Let X be a Banach A, B-module. If $\delta_A : A \longrightarrow A^{(2n-1)}$ is a σ_A-derivation, then the mapping $D_{\delta_A} : T_{\sigma_A,\sigma_B} \longrightarrow T_{\sigma_A,\sigma_B}^{(2n-1)}$ by $D_{\delta_A}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} \delta_A(a) & 0 \\ 0 & 0 \end{pmatrix}$ is a $\sigma_A \oplus \sigma_B$-derivation. Futhermore, δ_A is σ_A-inner if and only if D_{δ_A} is $\sigma_A \oplus \sigma_B$-inner.*

Proof. For each $\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix} \in T_{\sigma_A,\sigma_B}$ we have

$$\begin{aligned} D_{\delta_A}\left(\begin{pmatrix} a_1 & x_1 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ 0 & b_2 \end{pmatrix}\right) &= D_{\delta_A}\begin{pmatrix} a_1a_2 & \sigma_A(a_1) \cdot x_2 + x_1 \cdot \sigma_B(b_2) \\ 0 & b_1b_2 \end{pmatrix} = \begin{pmatrix} \delta_A(a_1a_2) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \delta_A(a_1) \cdot \sigma_A(a_2) + \sigma_A(a_1) \cdot \delta_A(a_2) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \delta_A(a_1) & 0 \\ 0 & 0 \end{pmatrix} \cdot \sigma_A \oplus \sigma_B\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} + \sigma_A \oplus \sigma_B\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta_A(a_2) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The relation above implies that D_{δ_A} is $\sigma_A \oplus \sigma_B$-derivation. Now, suppose that δ_A is σ_A-inner. Then there exists $a_0^{(2n-1)} \in A^{(2n-1)}$ such that $\delta_A(a) = \sigma_A(a) \cdot a_0^{(2n-1)} - a_0^{(2n-1)} \cdot \sigma_A(a)$ for all $a \in A$. Consider $\begin{pmatrix} a_0^{(2n-1)} & 0 \\ 0 & 0 \end{pmatrix} \in T_{\sigma_A,\sigma_B}^{(2n-1)}$, then

$$\begin{aligned} D_{\delta_A}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} &= \begin{pmatrix} \delta_A(a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_A(a) \cdot a_0^{(2n-1)} - a_0^{(2n-1)} \cdot \sigma_A(a) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_A(a) \cdot a_0^{(2n-1)} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a_0^{(2n-1)} \cdot \sigma_A(a) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_A(a) & x \\ 0 & \sigma_B(b) \end{pmatrix} \cdot \begin{pmatrix} a_0^{(2n-1)} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a_0^{(2n-1)} & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma_A(a) & x \\ 0 & \sigma_B(b) \end{pmatrix}. \end{aligned}$$

Thus, D_{δ_A} is $\sigma_A \oplus \sigma_B$-inner.

Conversely, suppose that D_{δ_A} is $\sigma_A \oplus \sigma_B$-inner. Then there exists $\begin{pmatrix} a_0^{(2n-1)} & x_0^{(2n-1)} \\ 0 & b_0^{(2n-1)} \end{pmatrix} \in T^{(2n-1)}_{\sigma_A,\sigma_B}$ such that

$$\begin{aligned}\begin{pmatrix} \delta_A(a) & 0 \\ 0 & 0 \end{pmatrix} = D_{\delta_A}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} &= \begin{pmatrix} \sigma_A(a) & 0 \\ 0 & \sigma_B(b) \end{pmatrix} \cdot \begin{pmatrix} a_0^{(2n-1)} & x_0^{(2n-1)} \\ 0 & b_0^{(2n-1)} \end{pmatrix} \\ &- \begin{pmatrix} a_0^{(2n-1)} & x_0^{(2n-1)} \\ 0 & b_0^{(2n-1)} \end{pmatrix} \cdot \begin{pmatrix} \sigma_A(a) & 0 \\ 0 & \sigma_B(b) \end{pmatrix} \\ &= \begin{pmatrix} \sigma_A(a)\cdot a_0^{(2n-1)} + \sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)}) & \sigma_B^2(b)\cdot x_0^{(2n-1)} \\ 0 & \sigma_B(b)\cdot b_0^{(2n-1)} \end{pmatrix} \\ &- \begin{pmatrix} a_0^{(2n-1)}\cdot \sigma_A(a) & x_0^{(2n-1)}\cdot \sigma_A^2(a) \\ 0 & b_0^{(2n-1)}\cdot \sigma_B(b) + \sigma_B^{(2n-1)}(x_0^{(2n-1)}\cdot x) \end{pmatrix}.\end{aligned}$$

It follows that $\delta_A(a) = \sigma_A(a)\cdot a_0^{(2n-1)} + \sigma_A^{(2n-1)}(x\cdot x_0^{(2n-1)}) - a_0^{(2n-1)}\cdot\sigma_A(a)$, for each $a \in A, x \in X$. Setting $x = 0$, we get $\delta_A(a) = \sigma_A(a)\cdot a_0^{(2n-1)} - a_0^{(2n-1)}\cdot \sigma_A(a)$, so δ_A is σ_A-inner, as required. □

Theorem 3.4. *Let A and B be unital Banach algebras and X be a (σ_A, σ_B)-unital Banach A, B-module. Let $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$. Then for each $n \in N$,*

$$H^1_{\sigma_A,\sigma_B}(T_{\sigma_A,\sigma_B}, T^{(2n-1)}_{\sigma_A,\sigma_B}) \simeq H^1_{\sigma_A}(A, A^{(2n-1)}) \oplus H^1_{\sigma_B}(B, B^{(2n-1)})$$

Proof. Suppose that $\delta : T_{\sigma_A,\sigma_B} \longrightarrow T^{(2n-1)}_{\sigma_A,\sigma_B}$ is a $\sigma_A \oplus \sigma_B$-derivation. By Lemma 3.2, there exist σ_A-derivation $\delta_A : A \longrightarrow A^{(2n-1)}$, and σ_B-derivation $\delta_B : B \longrightarrow B^{(2n-1)}$ and $x_0^{(2n-1)} \in X^{(2n-1)}$ such that

$$\delta\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} \delta_A(a) - \sigma_A^{(2n-1)}(x\cdot x_0^{(2n-1)}) & x_0^{(2n-1)}\cdot\sigma_A(a) - \sigma_B(b)\cdot x_0^{(2n-1)} \\ 0 & \sigma_B^{(2n-1)}(x_0^{(2n-1)}\cdot x) + \delta_B(b) \end{pmatrix}.$$

It is clear that the map $K : Z^1(T_{\sigma_A,\sigma_B}, T^{(2n-1)}_{\sigma_A,\sigma_B}) \longrightarrow H^1_{\sigma_A}(A, A^{(2n-1)}) \oplus H^1_{\sigma_B}(B, B^{(2n-1)})$,
defined by $K(\delta) = (\delta_A + N^1_{\delta_A}(A, A^{(2n-1)}), \delta_B + N^1_{\delta_B}(B, B^{(2n-1)}))$ is linear. Then Lemmas 3.2 and 3.3 together with the proof of [6, Theorem 3.4], show that the map K is onto and
$kerK = N^1(T_{\sigma_A,\sigma_B}, T^{(2n-1)}_{\sigma_A,\sigma_B})$. Thus,

$$H^1_{\sigma_A\oplus\sigma_B}(T_{\sigma_A,\sigma_B}, T^{(2n-1)}_{\sigma_A,\sigma_B}) \simeq H^1_{\sigma_A}(A, A^{(2n-1)}) \oplus H^1_{\sigma_B}(B, B^{(2n-1)})$$

□

Corollary 3.5. *Let A and B be unital Banach algebras and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$. Let X be a (σ_A, σ_B)-unital Banach A, B-module. Then T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-weakly amenable if and only if A is σ_A-weakly amenable and B is σ_B-weakly amenable.*

Let A, B be Banach algebras and X be a Banach A, B-module. Then, X is called essential, if $\overline{A\cdot X} = X = \overline{X\cdot B}$. A Banach A, B-module X is non-degenerate, if $A \cdot x = 0$ implies $x = 0$ and $x \cdot B = 0$ implies $x = 0$ for all $x \in X$. It is easily see that if X is essential then X^* is a non-degenerate Banach B, A-module. Moreover, for a Banach algebra A with a bounded approximate identity, A^* is non-degenerate.

Definition 3.6. *Let A, B be Banach algebras, X be a Banach A, B-module and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$. We say that X is (σ_A, σ_B)-essential, if $\overline{\sigma_A(A)\cdot X} = X = \overline{X\cdot\sigma_B(B)}$. Furthermore, X is (σ_A, σ_B)-non-degenerate, if $\sigma_A(A)\cdot x = 0$ implies $x = 0$ and $x\cdot\sigma_B(B) = 0$ implies $x = 0$.*

It is easily checked that if X is (σ_A, σ_B)-essential or (σ_A, σ_B)-non-degenerate then, it is essential or non-degenerate. The following lemma is easily proved.

Lemma 3.7. *Let A have a bounded approximate identity and let $S : A \longrightarrow X^*$ be a right(left) σ-A-module homomorphism. Then there is a $x_0^* \in X^*$ such that $S(a) = x_0^* \cdot \sigma(a)$ $(S(a) = \sigma(a) \cdot x_0^*)$ for all $a \in A$.*

Theorem 3.8. *Let A and B be Banach algebras and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$. Let A have a bounded approximate identity, let $A^{(2n-1)}$ be σ_A-non-degenerate, $B^{(2n-1)}$ be σ_B-non-degenerate and $X^{(2n-1)}$ be (σ_B, σ_A)-non-degenerate. Then for each $n \in \mathbb{N}$,*

$$H^1_{\sigma_A \oplus \sigma_B}(T_{\sigma_A,\sigma_B}, T^{(2n-1)}_{\sigma_A,\sigma_B}) \simeq H^1_{\sigma_A}(A, A^{(2n-1)}) \oplus H^1_{\sigma_B}(B, B^{(2n-1)}).$$

Proof. Suppose that $D : T_{\sigma_A,\sigma_B} \longrightarrow T^{(2n-1)}_{\sigma_A,\sigma_B}$ is a $\sigma_A \oplus \sigma_B$-derivation. By Lemmas 3.2 and 3.7, there exist σ_A-derivation $\delta_A : A \longrightarrow A^{(2n-1)}$, σ_B-derivation $\delta_B : B \longrightarrow B^{(2n-1)}$, and $x_0^{(2n-1)} \in X^{(2n-1)}$ such that $D\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_A(a) & x_0^{(2n-1)} \cdot \sigma_A(a) \\ 0 & 0 \end{pmatrix}$. Now set $D\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \theta(b) & r(b) \\ 0 & \delta_B(b) \end{pmatrix}$. By Lemma 3.2, we obtain $\delta_B : B \longrightarrow B^{(2n-1)}$ is a σ_B-derivation, $\theta(b) \cdot \sigma_A(a) = 0$ and $-\sigma_B(b) \cdot x_0^{(2n-1)} \cdot \sigma_A(a) = r(b) \cdot \sigma_A(a)$ for each $a \in A, b \in B$. Since $A^{(2n-1)}$ is σ_A-non-degenerate and $X^{(2n-1)}$ is (σ_B, σ_A)-non-degenerate, we have $\theta(b) = 0$ and $r(b) = -\sigma_B(b) \cdot x_0^{(2n-1)}$, hence $D(b) = \begin{pmatrix} 0 & -\sigma_B(b) \cdot x_0^{(2n-1)} \\ 0 & \delta_B(b) \end{pmatrix}$. For $D\begin{pmatrix} 0 & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} \theta(x) & r(x) \\ 0 & s(x) \end{pmatrix}$. From the equation

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= D\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} \theta(x) & r(x) \\ 0 & s(x) \end{pmatrix} \cdot \begin{pmatrix} \sigma_A(a) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta_A(a) & x_0^{(2n-1)}.\sigma_A(a) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We have $r(x) \cdot \sigma_A(a) = 0$ and $\theta(x) \cdot \sigma_A(a) + \sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)} \cdot \sigma_A(a)) = 0$, hence $(\theta(x) + \sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)})) \cdot \sigma_A(a) = 0$ because $\sigma_A^2 = \sigma_A$. Since $A^{(2n-1)}$ is σ_A-non-degenerate and $X^{(2n-1)}$ is (σ_B, σ_A)-non-degenerate, we conclude that $r(x) = 0$ and $\theta(x) = -\sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)})$. Similarly $s(x) = \sigma_A^{(2n-1)}(x_0^{(2n-1)} \cdot x)$. Consequently

$$D\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} \delta_A(a) - \sigma_A^{(2n-1)}(x \cdot x_0^{(2n-1)}) & x_0^{(2n-1)} \cdot \sigma_A(a) - \sigma_B(b) \cdot x_0^{(2n-1)} \\ 0 & \delta_B(b) + \sigma_A^{(2n-1)}(x_0^{(2n-1)} \cdot x) \end{pmatrix},$$

The rest of proof follows from Theorem 3.4. □

Definition 3.9. *Let A be a Banach algebra. We say that A has a σ-bounded approximate identity, if there exists a bounded net $(e_\alpha) \subseteq A$ such that*

$$\sigma(e_\alpha) \cdot a \to a \ , \ a \cdot \sigma(e_\alpha) \to a \quad (a \in A).$$

Corollary 3.10. *Let A and B be Banach algebras and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$ and $\sigma_B^2 = \sigma_B$. Let A have a σ_A-bounded approximate identity and B have a σ_B-bounded approximate identity, and X be a (σ_A, σ_B)-essential. Then T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-weakly amenable if and only if A is σ_A-weakly amenable and B is σ_B-weakly amenable.*

Proof. It is easy to show that A^* is σ_A-non-degenerate, B^* is σ_B-non-degenerate and X^* is (σ_A, σ_B)-non-degenerate. Thus, it is immediate by Theorem 3.8. □

4. $\sigma_A \oplus \sigma_B$-biflatness and biprojectivity of T_{σ_A,σ_B}

Suppose that $A_1, ..., A_n$ are Banach algebras. Then their direct sum $A = \oplus_{k=1}^n A_k$ with componentwise operations and l^1-norm is a Banach algebra. We write $\phi_k : A_k \longrightarrow A$ for the natural embedding A_k into A, $1 \le k \le n$. Take $\sigma_k \in Hom(A_k)$, $1 \le k \le n$, and define $\sigma := \oplus_{k=1}^n \sigma_k : A \longrightarrow A$ via $\sigma(a) = (\sigma_1(a_1), ..., \sigma_n(a_n))$ for every $a = (a_1, ..., a_n) \in A$. Then, it is easy to see that $\sigma \in Hom(A)$.

Theorem 4.1. *Let $A_1, A_2, ..., A_n$ be Banach algebras, and $A = \oplus_{k=1}^{n} A_k$. Then*
(i) A is σ-biflat if and only if every A_k is σ_k-biflat, $1 \leq k \leq n$;
(ii) A is σ-biprojective if and only if every A_k is σ_k-biprojective, $1 \leq k \leq n$.

Proof. We only prove (i). Suppose that A is σ-biflat, so there exists a bounded linear map $\rho : (A\widehat{\otimes}A)^* \longrightarrow A^*$ satisfying $\rho(\sigma(a)\cdot\lambda) = a\cdot\rho(\lambda)$ and $\rho(\lambda\cdot\sigma(a)) = \rho(\lambda)\cdot a$ for $a \in A$, $\lambda \in (A\widehat{\otimes}A)^*$ such that $\rho\circ\pi^* = \sigma^*$. Consider the bounded σ-A-bimodule homomorphism $s : A \longrightarrow A\widehat{\otimes}A$ such that $s^* = \rho$. Then $\pi\circ s = \sigma$. Take the projection $p_k : A \longrightarrow A_k$, and then define $\rho_k := \phi_k^*\circ\rho\circ(p_k\otimes p_k)^* : (A_k\widehat{\otimes}A_k)^* \longrightarrow (A_k)^*$, $1 \leq k \leq n$. So for each $a_k, b_k \in A_k$, $\lambda_k \in (A_k\widehat{\otimes}A_k)^*$,

$$\begin{aligned}
\langle\rho_k(\sigma_k(a_k)\cdot\lambda_k), b_k\rangle &= \langle\sigma_k(a_k)\cdot\lambda_k, (p_k\otimes p_k)\circ s\circ\phi_k(b_k)\rangle \\
&= \langle\lambda_k, (p_k\otimes p_k)\circ s\circ\phi_k(b_k)\cdot\sigma_k(a_k)\rangle \\
&= \langle\lambda_k, (p_k\otimes p_k)(s\circ\phi_k(b_k)\cdot(0,...,0,\sigma_k(a_k),0,...,0))\rangle \\
&= \langle\lambda_k, (p_k\otimes p_k)(s(\phi_k(b_k a_k)))\rangle \\
&= \langle a_k\cdot(\phi_k^*\circ\rho\circ(p_k\otimes p_k)^*)(\lambda_k), b_k\rangle = \langle a_k\cdot\rho_k(\lambda_k), b_k\rangle.
\end{aligned}$$

We get $\rho_k(\sigma_k(a_k)\cdot\lambda_k) = a_k\cdot\rho_k(\lambda_k)$ and similarly $\rho_k(\lambda_k\cdot\sigma_k(a_k)) = \rho_k(\lambda_k)\cdot a_k$. For the diagonal operator $\pi_k : A_k\widehat{\otimes}A_k \longrightarrow A_k$, because $(p_k\otimes p_k)^*\circ\pi_k^* = \pi^*\circ p_k^*$, we see that $\rho_k\circ\pi_k^* = \phi_k^*\circ\rho\circ(p_k\otimes p_k)^*\circ\pi_k^* = \phi_k^*\circ\rho\circ\pi^*\circ p_k^* = \phi_k^*\circ\sigma^*\circ p_k^* = \sigma_k^*$. Thus $\rho_k\circ\pi_k^* = \sigma_k^*$, which implies that A_k is σ_k-biflat, $1 \leq k \leq n$.

Conversely, suppose that A_k is σ_k-biflat for each $1 \leq k \leq n$. Hence there are bounded linear maps $\rho_k : (A_k\widehat{\otimes}A_k)^* \longrightarrow A_k^*$ with $\rho_k(\sigma_k(a_k)\cdot\lambda_k) = a_k\cdot\rho_k(\lambda_k)$ and $\rho_k(\lambda_k\cdot\sigma_k(a_k)) = \rho_k(\lambda_k)\cdot a_k$, where $a_k \in A_k$, $\lambda_k \in (A_k\widehat{\otimes}A_k)^*$ such that $\rho_k\circ\pi_k^* = \sigma_k^*$, $1 \leq k \leq n$. Define $\rho : (A\widehat{\otimes}A)^* \longrightarrow A^*$ by $\rho(\lambda) = \sum_{k=1}^{n} p_k^*\circ\rho_k\circ(\phi_k\otimes\phi_k)^*(\lambda)$. Therefore for each $a \in A$, $\lambda \in (A\widehat{\otimes}A)^*$, we have

$$\begin{aligned}
\rho(\sigma(a)\cdot\lambda) &= \sum_{k=1}^{n} p_k^*\circ\rho_k\circ(\phi_k\otimes\phi_k)^*(\sigma(a)\cdot\lambda) \\
&= \sum_{k=1}^{n} p_k^*\circ\rho_k(\sigma_k(a_k)\cdot(\phi_k\otimes\phi_k)^*(\lambda)) \\
&= \sum_{k=1}^{n} p_k^*(a_k\cdot(\rho_k\circ(\phi_k\otimes\phi_k)^*(\lambda))) \\
&= a\cdot(\sum_{k=1}^{n} p_k^*\circ\rho_k\circ(\phi_k\otimes\phi_k)^*(\lambda)) \\
&= a\cdot\rho(\lambda).
\end{aligned}$$

Similarly $\rho(\lambda\cdot\sigma(a)) = \rho(\lambda)\cdot a$. As $(\phi_k\otimes\phi_k)^*\circ\pi^* = \pi_k^*\circ\phi_k^*$, thus

$$\begin{aligned}
\rho\circ\pi^* &= \sum_{k=1}^{n} p_k^*\circ\rho_k\circ(\phi_k\otimes\phi_k)^*\circ\pi^* \\
&= \sum_{k=1}^{n} p_k^*\circ\rho_k\circ\pi_k^*\circ\phi_k^* \\
&= \sum_{k=1}^{n} p_k^*\circ\sigma_k^*\circ\phi_k^* \\
&= \sigma^*.
\end{aligned}$$

Therefore A is $\sigma-$biflat. $\square$

Lemma 4.2. *Let A be a Banach algebra and $\sigma \in Hom(A)$ such that $\sigma^2 = \sigma$. Let $N \subseteq A$ be a closed complemented ideal where $\sigma(N) = N$ and $N^2 = 0$. Then $\sigma(A)N \cap \overline{N\sigma(A)} = 0$*

Proof. Let $i : N \longrightarrow A$ be the inclusion map, $q : A \longrightarrow \frac{A}{N}$ be the quotient map, I_A, I_N and $I_{\frac{A}{N}}$ be the identity maps on A, N and $\frac{A}{N}$, respectively, and let $p : \frac{A}{N}\widehat{\otimes}N \longrightarrow N$ be the map defined by $p((a + N) \otimes c) = ac$ for each $a + N \in \frac{A}{N}$ and $c \in N$. Suppose to wards a contraction that $\sigma(A)N \cap \overline{N\sigma(A)} \neq 0$. Suppose that $0 \neq \sigma(a)c \in \sigma(A)N \cap \overline{N\sigma(A)}$ where $a \in A, c \in N$. Hence $\sigma(a)c \in \overline{N\sigma(A)}$, so there exists sequences $(\sigma(a_n)) \subseteq \sigma(A)$ and $(c_n) \subseteq N$ such that $\sigma(a)c = \lim_{n\to\infty} c_n\sigma(a_n)$. Since A is σ-biflat, then there is a σ-A-bimodule homomorphism $\rho : A \longrightarrow (A\widehat{\otimes}A)^{**}$ such that $\pi^{**} \circ \rho = \sigma$. For $b \in N$, let $R_b(L_b) : A \longrightarrow N$ be the map of right (resp.left) multiplication by b. Consider the operator $q \otimes R_c : A\widehat{\otimes}A \longrightarrow \frac{A}{N}\widehat{\otimes}N$ and let $d = ((q \otimes R_c)^{**} \circ \rho)\sigma(a)$. We have $p \circ (q \otimes R_c) = R_c \circ \pi$ and so $p^{**} \circ (q \otimes R_c)^{**} = R_c^{**} \circ \pi^{**}$. As a result
$p^{**}(d) = (p^{**} \circ (q \otimes R_c)^{**} \circ \rho)\sigma(a) = ((R_c^{**} \circ \pi^{**}) \circ \rho)(\sigma(a)) = R_c^{**}((\pi^{**} \circ \rho)(\sigma(a))) = R_c^{**}(\sigma^2(a)) = \sigma(a)c \neq 0$, thus $d \neq 0$. By the assumption there exists $c_1 \in N$ such that $\sigma(c_1) = c$. As the proof of [9, Lemma 2.3], we have

$$
\begin{aligned}
(I_{\frac{A}{N}} \otimes i)^{**}(d) &= (((I_{\frac{A}{N}} \otimes i)^{**} \circ (q \otimes R_c)^{**} \circ \rho)(\sigma(a)) \\
&= (((I_{\frac{A}{N}} \otimes i) \circ (q \otimes I_N) \circ (I_A \otimes R_c))^{**} \circ \rho)(\sigma(a)) \\
&= (((q \otimes I_A) \circ (I_A \otimes i) \circ (I_A \otimes R_c))^{**} \circ \rho)(\sigma(a)) \\
&= (q \otimes I_A)^{**}(\rho(\sigma(a)) \cdot c) \\
&= (q \otimes I_A)^{**}(\rho(\sigma(a)) \cdot \sigma^2(c_1)) \\
&= (q \otimes I_A)^{**}(\rho(\sigma(a)\sigma(c_1))) \\
&= (q \otimes I_A)^{**}(\rho(\sigma(a)c)) \\
&= (q \otimes I_A)^{**}(\lim_{n\to\infty} \sigma(c_n)\rho(\sigma(a_n))) \\
&= \lim_{n\to\infty}((q \otimes I_A)^{**} \circ ((i \circ L_{\sigma(c_n)} \otimes\ I_A)^{**}(\rho(\sigma(a_n))) \\
&= \lim_{n\to\infty}((q \otimes I_A)^{**} \circ ((i \otimes I_A) \circ (L_{\sigma(c_n)} \otimes I_A))^{**}(\rho(\sigma(a_n)) \\
&= \lim_{n\to\infty}((q \circ i \circ L_{\sigma(c_n)}) \otimes I_A)^{**}(\rho(\sigma(a_n)) = 0
\end{aligned}
$$

The last equality is hold because of $q \circ i \circ L_\sigma(c_n) = 0$. Since N is a complemented closed ideal in A , then the map $I_{\frac{A}{N}} \otimes i$ is injective and has closed range and hence $(I_{\frac{A}{N}} \otimes i)^{**}$ is injective by [5, A.3.48]. This contradicts $d = 0$. Therefore $\sigma(A)N \cap \overline{N\sigma(A)} = 0$. □

Theorem 4.3. *Let A be a Banach algebra and $\sigma \in Hom(A)$ where $\sigma^2 = \sigma$. Let $N \subseteq A$ be a closed σ-essential ideal, that is, $\overline{\sigma(A)N} = \overline{N\sigma(A)} = N$. Let $\sigma(N) = N$ and $N^2 = 0$. If A is σ-biflat, then A is not complement.*

Proof. Since $\sigma(A)N \subseteq N \subseteq \overline{N\sigma(A)}$. According to Lemma 4.2, $N = 0$, this is a contradiction. □

Theorem 4.4. *Let A and B be Banach algebras and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A, \sigma_B^2 = \sigma_B$. Let X be a (σ_A, σ_B)-essential module. Then the triangular Banach algebra $T_{\sigma_A,\sigma_B} = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ is $\sigma_A \oplus \sigma_B$-biflat if and only if A is σ_A-biflat and B is σ_B-biflat and $X = 0$.*

Proof. Suppose A is σ_A-biflat and B is σ_B-biflat and $X = 0$. Then T_{σ_A,σ_B} is the l^1-direct sum of A and B, thus by Theorem 4.1, it is $\sigma_A \oplus \sigma_B$-biflat.

Conversely, suppose that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-biflat. The closed ideal $N = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ of T_{σ_A,σ_B} is complemented closed ideal of T_{σ_A,σ_B} such that$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}^2 = 0$ and $\sigma_A \oplus \sigma_B(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\sigma_A(A)N} \\ 0 & 0 \end{pmatrix} = \overline{(\sigma_A \oplus \sigma_B(\begin{pmatrix} A & X \\ 0 & B \end{pmatrix})\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix})}^{-},$$
$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{X\sigma_B(B)} \\ 0 & 0 \end{pmatrix} = \overline{(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\sigma_A \oplus \sigma_B(\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}))}^{-}.$$

Hence by Theorem 4.3, we conclude $X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = 0$. Therefore T_{σ_A,σ_B} is the l^1-direct sum of A and B. By Theorem 4.1, A is σ_A-biflat and B is σ_B-biflat. □

Theorem 4.5. *Let A and B be Banach algebras and $\sigma_A \in Hom(A), \sigma_B \in Hom(B)$ such that $\sigma_A^2 = \sigma_A$. Let X be a (σ_A, σ_B)-essential module. Then the triangular Banach algebra $T_{\sigma_A,\sigma_B} = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ is $\sigma_A \oplus \sigma_B$-biprojective if and only if A is σ_A-biprojective and B is σ_B-biprojective and $X = 0$.*

Proof. Suppose that T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-biprojective, so T_{σ_A,σ_B} is $\sigma_A \oplus \sigma_B$-biflat. By Theorem 4.4, $X = 0$. Hence T_{σ_A,σ_B} is the l^1-direct sum A and B, thus by Theorem 4.1, A is σ_A-biprojective and B is σ_B-biprojective. Conversely, if $X = 0$ and A is σ_A-biprojective and B is σ_B-biprojective then T_{σ_A,σ_B} is the l^1-direct sum A and B, Thus by Theorem 4.1, it is $\sigma_A \oplus \sigma_B$-biprojective. □

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