Filomat 36:13 (2022), 4339–4346
https://doi.org/10.2298/FIL2213339C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sequences of Nonlinear Quasi Contractions and Fixed Points

Kieu Phuong Chi[a], Erdal Karapinar[b,c], Tran Duc Thanh[d]

[a]*Department of Mathematics and Applications, Saigon University, Ho Chi Minh City, Vietnam*
[b]*Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey.*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan*
[d]*Department of Mathematics and Applications, Saigon University, Ho Chi Minh City, Vietnam*

Abstract. In this paper, we state some results on the relationship between the convergence of the nonlinear quasi-contractions and the convergence of their fixed point. The observed results certainly extend some existing results on the topic in the literature, including the results of Nadler and Park. We also furnish an illustrative example to demonstrate the validity of the results expressed.

1. Introduction

Let (f_n) be a sequence of self-mappings over a metric space (X, d). Suppose that this sequence (f_n) converges to a self-mapping f, defined on a metric space (X, d), in some sense. It is a quite natural to ask the relationship between the convergence of the sequence (f_n) and the convergence of their fixed point. This question was considered and discussed first, by Nadler in [11]. Roughly speaking, Nadler considered two distinct results: a sequence contraction mappings which converges uniformly and a sequence contraction mappings that converges pointwise. This idea has been appreciated by a number of authors, see e.g.[5, 12]. In particular, Park [12] studied on the sequences in the form of quasi contractions. For some other approaches and an extensions can be found in the survey of Rus ([15]).

In this paper, we aim to extend the existing results by considering the relationship between the convergence of nonlinear of quasi contractions, almost contractions together with their fixed points. More precisely, we combine the outstanding results of Park [12] with the result of Pacurar [10].

2. Preliminaries

Before stating the main results, we give some useful definitions, preliminaries which will be used in the sequel. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. We shall introduce the class of the control functions (auxiliary functions) which have significant roles in the extension of the fixed point theory:

We say that $\Phi = \{\varphi : [0, +\infty) \to [0, +\infty)\}$ forms a the class of the control functions if the given properties are fulfilled:

2020 *Mathematics Subject Classification*. 47H09, 47H10
Keywords. Fixed point theorems, sequence of contraction, quasi contraction
Received: 02 May 2021; Accepted: 21 February 2022
Communicated by Dragan S. Djordjević
Corresponding Authors: Kieu Phuong Chi
Email addresses: kieuphuongchi@sgu.edu.vn (Kieu Phuong Chi), erdalkarapinar@yahoo.com (Erdal Karapinar), tr.ducthanh@sgu.edu.vn (Tran Duc Thanh)

(i) each φ is upper semicontinuous;
(ii) each φ is monotone non-decreasing;
(iii) $\lim_{t\to\infty}(t-\varphi(t)) = +\infty$;
(iv) $0 < \varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

For the immediate elementary examples of such control function, consider the following:

$$\varphi_1(t) = kt,\ (0 < k < 1),\ \varphi_2(t) = \ln(1+t),\ \text{with } t \geq 0.$$

After a few simple steps, we conclude easily that $\varphi_i \in \Phi$ for every $i = 1, 2, 3$.

Definition 2.1. ([2]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *nonlinear quasi-contractive* on X if there exists $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq \varphi\Big(M(x, y)\Big)$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

If $\varphi(t) = ht$ with $0 \leq h < 1$ then we arrive at a general type of contractions. It is usual to call quasi contraction or Ćirić's contraction. Due to Rhoades [13], we can conclude that most of the well-known contractions are followed from quasi contraction.

The following theorem were considered by two research teams, e.g. Arandelovic, Rajovic and Kilibarda [2] and Di Bari and Vetro [4].

Theorem 2.2. ([2], [4]) *Let (X, d) be a complete metric space. If T is a nonlinear quasi contraction then T has unique fixed point.*

In the corresponding literature, a number of extension has been expressed, in particular, Boyd and Wong [3], Ćirić [5], Ivanov [8], Jeong [9].

Definition 2.3. ([1]) Let (X, d) be a metric space. A map $T : X \to X$ is called to be *almost* if there are $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta.d(x, y) + Ld(y, Tx),\ \text{for all } x, y \in X. \tag{1}$$

Theorem 2.4. ([1]) *Let (X, d) be a complete space and $T : X \to X$ be an almost contraction with $\delta \in (0, 1)$ and $L \geq 0$. Then $F(T) = \{x \in X : Tx = x\} \neq \emptyset$, where $F(T)$ is the set of fixed points of T.*

Theorem 2.5. ([1]) *Let (X, d) be a complete space and $T : X \to X$ be an almost contraction. Suppose that there exist $\theta \in (0, 1)$ and $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta.d(x, y) + L_1.d(x, Tx),\ \text{for all } x, y \in X. \tag{2}$$

Then T has unique fixed point.

We recall the concepts of convergence.

Definition 2.6. ([11]) Let (X, d) be a metric space and maps $g_n : X \to X$, $n \in \mathbb{N}$, and $g : X \to X$.

1) The sequence (g_n) converges pointwise to g as $n \to \infty$ if for each $x \in X$ then $\lim\limits_{n\to\infty} d(g_n x, gx) = 0$.

2) The sequence (g_n) converges uniformly to g as $n \to \infty$ if $\lim\limits_{n\to\infty}\Big(\sup_{x\in X} d(g_n x, gx)\Big) = 0$.

It is easy to see that uniform convergence implies pointwise convergence. We need the following fact for the class of functions Φ. not origin.

Lemma 2.7. *Let $(\varphi_n)_{n=1}^{\infty} \subset \Phi$, $\varphi \in \Phi$. If φ_n converges pointwise to φ then $\lim\limits_{n\to\infty} \varphi_n(t_n) \leq \varphi(t)$ for every $(t_n) \subset [0,\infty)$ and $t_n \to t$.*

Proof. Since $\varphi_n \in \Phi$ for each n, we have that φ_n is monotone non-decreasing and upper semi-continuous for each n. Hence, for each $c \geq 0$, we have

$$\limsup_{t\to c^+} \varphi_n(t) \leq \varphi_n(c) \tag{3}$$

for each $n \in \mathbb{N}$. By the other hand, since φ_n is monotone non-decreasing, we can deduce that

$$\limsup_{t\to c^-} \varphi_n(t) \leq \varphi_n(c) \tag{4}$$

for each $n \in \mathbb{N}$. Combining (3) and 4, we arrive at

$$\limsup_{t\to c} \varphi_n(t) \leq \varphi_n(c) \tag{5}$$

for each n. Suppose that $(t_k) \subset [0,\infty)$ and $t_k \to t$. In view (5) we have

$$\limsup_{k\to\infty} \varphi_n(t_k) \leq \varphi_n(t)$$

for each $n \in \mathbb{N}$. Since φ_n converges pointwise to φ, we can conclude that

$$\lim_{n\to\infty} \varphi_n(t_n) \leq \limsup \varphi_n(t_n) \leq \limsup_{k\to\infty} \varphi_n(t_k) \leq \varphi_n(t) = \varphi(t)$$

for each $t \geq 0$. □

3. Sequence of nonlinear quasi contractions

We begin this section at stating the main result of the work.

Theorem 3.1. *Let d_n be a metric on a set X for each $n \in \mathbb{N}_0$ and $\{d_n\}_{n=1}^{\infty}$ converges uniformly to $d = d_0$. Suppose that g_n is a nonlinear quasi contraction of (X, d_n) with the control function φ_n for each $n \in \mathbb{N}$ and φ_n converges pointwise to φ. Then, if $g : (X,d) \longrightarrow (X,d)$ is pointwise limit of $\{g_n\}$ by the metric d then g is a nonlinear quasi contraction with the control function φ.*

Moreover, if each g_n has a fixed point $u_n (n \in \mathbb{N})$ and g has a fixed point u then $\{u_n\}_{n=1}^{\infty}$ converges to u.

Proof. For each $x, y \in X$, we have

$$d(gx, gy) \leq d(gx, g_nx) + d\left(g_nx, g_ny\right) + d\left(g_ny, gy\right).$$

Since d_n converges uniformly to d, for any $\varepsilon > 0$ there exists $N > 0$ such that for $n \geq N$, we have

$$|d_n\left(x,y\right) - d\left(x,y\right)| < \varepsilon \tag{6}$$

for every $x, y \in X$. Hence, for each $x, y \in X$ and $n \geq N$, we have

$$\begin{aligned}
d(gx,gy) &\leqslant d(gx,g_nx)+d_n(g_nx,g_ny)+\varepsilon+d(g_ny,gy)\\
&\leqslant d(gx,g_nx)+d(g_ny,gy)+\varepsilon\\
&+\max\Big\{\varphi_n\Big(d_n(x,y)\Big),\varphi_n\Big(d_n(x,g_nx)\Big),\varphi_n\Big(d_n(y,g_ny)\Big)\\
&,\varphi_n\Big(d_n(x,g_ny)\Big),\varphi_n\Big(d_n(y,g_nx)\Big)\Big\}\\
&\le d(gx,g_nx)+d(g_ny,gy)+\varepsilon\\
&+\max\Big\{\varphi_n\Big(d_n(x,y)\Big),\varphi_n\Big(d_n(x,g_nx)\Big),\varphi_n\Big(d_n(y,g_ny)\Big),\\
&\varphi_n\Big(d_n(x,g_ny)\Big),\varphi_n\Big(d_n(y,g_nx)\Big)\Big\}\\
&\le d(gx,g_nx)+d(g_ny,gy)+\varepsilon\\
&+\max\Big\{\varphi_n\Big(d(x,y)+\varepsilon\Big),\varphi_n\Big(d(x,g_nx)+\varepsilon\Big),\\
&\varphi_n\Big(d(y,g_ny)+\varepsilon\Big),\varphi_n\Big(d(x,g_ny)+\varepsilon\Big),\varphi_n\Big(d(y,g_nx)+\varepsilon\Big)\Big\}.
\end{aligned} \tag{7}$$

Letting $n\to\infty$. It follows from the Lemma 2.7 and the pointwise convergence of g_n to g by the metric d, that

$$\begin{aligned}
d(gx,gy) \leqslant \varepsilon+\max\Big\{&\varphi\Big(d(x,y)\Big),\varphi(d(x,gx)+\varepsilon\Big),\varphi\Big(d(y,gy)+\varepsilon\Big)),\\
&\varphi\Big(d(x,gy)+\varepsilon\Big),\varphi\Big(d(y,gx)+\varepsilon\Big)\Big\}.
\end{aligned} \tag{8}$$

Since $\varepsilon>0$ is arbitrary, we obtain

$$\begin{aligned}
d(gx,gy) \leqslant \max\Big\{&\varphi\Big(d(x,y)\Big),\varphi(d(x,gx)\Big),\varphi\Big(d(y,gy)\Big)),\\
&\varphi\Big(d(x,gy)\Big),\varphi\Big(d(y,gx)\Big)\Big\}.
\end{aligned} \tag{9}$$

This show that g is the nonlinear quasi contraction of (X,d_n) with the control function φ.

Now, suppose that g_n has a fixed point u_n for each $n\in\mathbb{N}$ and g has a fixed point u. For each $n\ge N$, we have

$$\begin{aligned}
d(u_n,u) &\leqslant d(u_n,g_nu)+d(g_nu,u)\\
&= d(gu_n,g_nu)+d(g_nu,u)\\
&= d_n(g_nu_n,g_nu)+d(g_nu,u)+\varepsilon\\
&\leqslant d(g_nu,u)+\varepsilon\\
&\quad+\max\Big\{\varphi_n\Big(d_n(u_n,u)\Big),\varphi_n\Big(d_n(u,g_nu)\Big),\varphi_n\Big(d_n(g_nu_n,u_n)\Big),\\
&\quad\varphi_n\Big(d_n(g_nu,u_n)\Big),\varphi_n\Big(d_n(g_nu_n,u)\Big)\Big\}\\
&= d(g_nu,u)+\varepsilon+\max\Big\{\varphi_n\Big(d_n(u_n,u)\Big),\varphi_n\Big(d_n(u,g_nu)\Big),\varphi_n\Big(d_n(u_n,g_nu)\Big)\Big\}\\
&\leqslant d(g_nu,u)+\varepsilon\\
&\quad+\max\Big\{\varphi_n\Big(d(u_n,u)+\varepsilon\Big),\varphi_n\Big(\varepsilon\Big),\varphi_n\Big(d(u,g_nu)+\varepsilon\Big),\\
&\quad\varphi_n\Big(d(u_n,g_nu)+\varepsilon\Big)\Big\}\\
&\leqslant d(g_nu,u)+\varepsilon\\
&\quad+\max\Big\{\varphi_n\Big(d(u_n,u)+\varepsilon\Big),\varphi_n\Big(\varepsilon\Big),\varphi_n\Big(d(u,g_nu)+\varepsilon\Big),\\
&\quad\varphi_n\Big(d(u_n,u)+d(u,g_nu)+\varepsilon\Big)\Big\}.\\
&= d(g_nu,u)+\varepsilon+\varphi_n\Big(d(u_n,u)+d(u,g_nu)+\varepsilon\Big).
\end{aligned}$$

It implies that

$$d(u_n, u) + d_n(g_n u, u) + \varepsilon - \varphi_n\Big(d\,(u_n, u) + d(u, g_n u) + \varepsilon\Big) < 2(d(g_n u, u) + \varepsilon). \tag{10}$$

If $\big(d(u_n, u)\big)$ is unbounded then $\limsup_{n\to\infty} d(u_n, u) = \infty$. Invoking the condition $\lim_{t\to\infty}\big(t - \varphi_n(t)\big) = \infty$ for each n and (10), we arrive at a contradiction if n sufficiently large. Hence $\big(d(u_n, u)\big)$ a bounded sequence. Set $t_0 = \limsup_{n\to\infty} d(u_n, u)$. Since φ_n is nondecreasing and 2.7, we infer from $\lim_{n\to\infty} d(gu_n, u) = 0$ and (10) that

$$t_0 \leq \varepsilon + \varphi(t_0 + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we can deduce that

$$t_0 \leq \varphi(t_0).$$

It follows that $t_0 = 0$. This proves that $u_n \to u$ by the metric d. □

We obtain the following corollaries.

Corollary 3.2. *Let (X, d) be a complete metric space. Suppose g_n be a nonlinear quasi contraction of (X, d) with the control function φ_n for each $n = 0, 1, 2, ...$ Then, if $g : (X, d) \longrightarrow (X, d)$ is pointwise limit of $\{g_n\}$ then g is a nonlinear quasi contraction with the control function φ.*

Moreover, g_n has unique fixed point u_n for each n, g has unique fixed point u and $\{u_n\}_{n=1}^{\infty}$ converges to u.

Proof. Since $d_n = d$ for all n, the uniformly convergence of (d_n) is trivial. The result are derived from Theorem 3.1. □

In particular, we have got a version for nonlinear contractions.

Corollary 3.3. *Let (X, d) be a complete metric space. Suppose g_n be a nonlinear contraction of (X, d) with the control function φ_n for each $n = 0, 1, 2, ...$ Then, if $g : (X, d) \longrightarrow (X, d)$ is pointwise limit of $\{g_n\}$ then g is a nonlinear contraction with the control function φ.*

Moreover, g_n has unique fixed point u_n for each n, g has unique fixed point u and $\{u_n\}_{n=1}^{\infty}$ converges to u.

In the Corollary 3.2 if we fix $\varphi_n(t) = \varphi(t) = qt$ with $0 < q < 1$ then we arrive at the result of Ivanov (see [8]). We also have got the main result of Park ([12]). We would like to emphasize that Park's result seem to be the best in "linear" contractions.

Corollary 3.4. ([12]) *Let d_n be a metric on a set X for each $n \in \mathbb{N}_0$ and $\{d_n\}_{n=1}^{\infty}$ converges uniformly to $d = d_0$. Suppose g_n be a quasi contraction of (X, d_n) with constant control function α_n for each $n = 0, 1, 2, ...$ and α_n converges to $\alpha \in (0, 1)$. Then, if $g : (X, d) \longrightarrow (X, d)$ is pointwise limit of $\{g_n\}$ by the metric d then g is a quasi contraction with the constant control function α. Moreover, if each g_n has a fixed point u_n and g has a fixed point u then $\{u_n\}_{n=1}^{\infty}$ converges to u.*

The following example state that our results are certainly extension of Park's and some of the result are previously mentioned.

Example 3.5. Let $X = [0, \infty)$ and d_n be the usual metric on X for all n. Let $f_n : X \to X$ define by

$$f_n(x) = \begin{cases} \dfrac{n}{n+1}\ln(1 + \dfrac{1}{n}) & \text{if } 0 \leq x \leq \frac{1}{n} \\ \dfrac{n}{n+1}\ln(1 + x) & \text{if } x \geq \dfrac{1}{n}. \end{cases}$$

It is easy to check that (f_n) converges pointwise to

$$f(x) = \ln(1 + x), x \in [0, \infty)$$

and f_n have a unique fixed point $u_n = \dfrac{n}{n+1}\ln(1+\dfrac{1}{n})$ on X for each $n \in \mathbb{N}$ and f has unique fixed point $u = 0$. On the other hand $u_n \to u$ as $n \to \infty$.

Suppose that there exists $\alpha \in (0,1)$ such that

$$d(fx, fy) = \alpha M(x,y)$$

for all $x, y \in X$. We have

$$M(x,y) = \max\{|x-y|, |x-fx|, |y-fy|, |x-fy|, |y-fx|\}.$$

If $0 < x < 1$ and $y = 0$ then we obtain

$$M(x,0) = \max\{|x|, |x-\ln(1+x)|, |0-0|, |x|, |\ln(1+x)|\} = |x|$$

and $d(fx, f0) = |\ln(1+x)|$. Hence

$$\ln(1+x)| \leq \alpha x$$

for all $x \geq 0$ and we arrive at a contradiction. This shows that we can not apply Corollary 3.4 for the sequence (f_n).

Now, we shall show that (f_n) are satisfied our theorem. More precisely, we can apply Corollary 3.3 with suitable nonlinear control functions (φ_n). Indeed, let

$$\varphi_n(t) = \begin{cases} \dfrac{n}{n+1}t & \text{if } 0 \leq t \leq \frac{1}{n} \\ \ln(1+t) & \text{if } t \geq \dfrac{1}{n}. \end{cases}$$

For each $n \in \mathbb{N}$ and for every $x, y \in X$, we can reduce to consider cases:

i) Cases 1: If $x, y \leq \dfrac{1}{n}$ then $d(f_n x, f_n y) = 0$ for all n. It follows that

$$d(f_n x, f_n y) \leq \varphi_n(d(x,y))$$

for all n.

ii) Cases 2: $x \geq y > \dfrac{1}{n}$. We have

$$d(f_n x, f_n y) = \frac{n}{n+1}|\ln(1+x) - \ln(1+y)| = \frac{n}{n+1}\ln(1+\frac{|x-y|}{1+y})$$

If$|x-y| > \frac{1}{n}$ then

$$\varphi_n(d(x,y)) = \ln(1+|x-y|) \geq \ln(1+\frac{x-y}{1+y}) > d(f_n x, f_n y)$$

for every $x \geq y \in X$.

If $|x-y| < \frac{1}{n}$ then

$$\varphi_n(d(x,y)) = \frac{n}{n+1}|x-y| \geq \frac{n}{n+1}\ln(1+|x-y|) \geq \frac{n}{n+1}\ln(1+\frac{|x-y|}{1+y}) = d(f_n x, f_n y).$$

iii) Cases 3: $x > \dfrac{1}{n} \geq y$. We have

$$d(f_n x, f_n y) = |\frac{n}{n+1}\ln(1+x) - \ln(1+\frac{1}{n})| = \frac{n}{n+1}\ln(1+\frac{x-\frac{1}{n}}{1+\frac{1}{n}}) \leq \frac{n}{n+1}\ln(1+\frac{x-\frac{1}{n}}{)}.$$

It easy to see that $d(x,y) = |x-y| \geq |x-\dfrac{1}{n}|$. Since

$$\varphi_n(t) = \begin{cases} \dfrac{n}{n+1}t & \text{if } 0 \leq t \leq \frac{1}{n} \\ \ln(1+t) & \text{if } x \geq \dfrac{1}{n}. \end{cases}$$

and the fact $\ln(1+|x-\dfrac{1}{n}|) \leq |x-\dfrac{1}{n}|)$ we arrive at

$$d(f_n x, f_n y) \leq \varphi_n(d(x,y))$$

for all n.

Hence, all conditions of Corollary 3.3 are fulfilled and the conclusion are derived.

4. Sequences of almost contractions

In this section, inspired Pacurar's Theorem [10], we shall express a general theorem for sequences of almost contractions.

Theorem 4.1. *Let (X,d) be a metric space. Suppose that $g_n, g: X \to X$, $\quad n \in \mathbb{N}$ are satisfied:*
1) For each $n \in \mathbb{N}$, g_n are (a_n, L_n)-almost contradictions with $a_n \in (0,1), L_n \geq 0$.
2) There exists, $b_n \in (0,1), K_n \geq 0$ such that

$$d(g_n x, g_n y) \leq b_n d(x,y) + K_n d(x, g_n x)), x, y \in X$$

and $n \in \mathbb{N}$
3) $a_n \to a \in (0,1)$, $b_n \to b$, $L_n \to L$ and $K_n \to K$ as $n \to \infty$.
4) g_n converges pointwise to g on X.
Then g_n has a unique fixed point $x_n (n \in \mathbb{N})$, g has a unique fixed point x^* and $x_n \to x^*$ as $n \to \infty$.

Proof. Letting $n \to \infty$, it follows from 3) 4) and the continuity of the metric d that g is a (a,L)- almost contradiction, and

$$d(gx, gy) \leq ad(x,y) + Ld(y, gx),$$

and

$$d(gx, gy) \leq bd(x,y) + Kd(x, gx),$$

for every $x, y \in X$. Hence, g are satisfied the Theorem 2.5 and g has a unique fixed point x^*. Moreover, g_n has a unique fixed point $\{x_n^*\}$ for each $n \in \mathbb{N}$. For each $n = 1, 2, ..$ we have

$$\begin{aligned} d(x_n^*, x^*) &= d(g_n x_n^*, gx^*) \\ &\leq d(g_n x_n^*, gx_n^*) + d(g_n x^*, gx^*) \\ &\leq b_n d(x_n^*, x^*) + K_n d(x_n^*, g_n x_n^*) + d(g_n x^*, gx^*) \\ &= b_n d(x_n^*, x^*) + d(g_n x^*, gx^*). \end{aligned}$$

Since $b_n < 1$ and $\lim\limits_{n\to\infty} b_n = b < 1$ we obtain $\sup b_n = \beta < 1$. It follows from

$$d(x_n^*, x^*) \leq b_n d(x_n^*, x^*) + d(g_n x^*, gx^*)$$

that

$$(1-\beta) d(x_n^*, x^*) \leq d(g_n x^*, gx^*)$$

for each $n \in \mathbb{N}$. Since (g_n) converges pointwise to g, we can deduce that $\lim\limits_{n\to\infty} d(x_n^*, x^*) = 0$. This means that $x_n \to x^*$. $\square$

Remark 4.2. If we choose $(a_n), (b_n), (L_n)$ and (K_n) are constant sequences then we get the main result of [10] (Theorem 2.5).

We give some examples that to illustrate the previous theorem.

Example 4.3. Consider $\mathbb{R}$ endow with the usual metric and the sequence of maps $g_n : \mathbb{R} \to \mathbb{R}$ defined by

$$g_n x = \frac{n}{2n+1}x + \frac{n}{n+1}, n \in \mathbb{N}, \forall x \in \mathbb{R}.$$

It is easy to see that g_n converges pointwise to $gx = \frac{x}{2} + 1$ and g has a unique fixed point $x = 2$. Moreover g_n has a unique fixed point $x_n = \frac{2n^2+n}{(n+1)^2}$. Clearly x_n converges to $x = 2$. We can not apply the classical Nadler's theorem (see [11]) because that g_n does not converges uniformly to g. Indeed,

$$\sup_{x\in\mathbb{R}} |g_n x - gx| = \sup_{x\in\mathbb{R}} |\frac{x}{4n+2} + \frac{n}{n+1} - 1| \geq |\frac{4n+2}{4n+2} + \frac{n}{n+1} - 1| = \frac{n}{n+1} \to 1$$

as $n \to \infty$. On the other hand, since $a_n = \frac{n}{2n+1}$ is not a constant sequence, we can not apply the all results of [10].

It is easy to check that g_n are satisfied the Theorem 4.1 with $a_n = \frac{n}{2n+1} \to \frac{1}{2}$, $L_n = 0$, $b_n = \frac{n}{2n+1} \to \frac{1}{2}$ and $K_n = 0$.

The following example shows that we can not assume that $a_n \to 1$ in the Theorem 4.1.

Example 4.4. Consider $\mathbb{R}$ endow with the usual metric and the sequence of maps $g_n : \mathbb{R} \to \mathbb{R}$ defined by

$$g_n x = \frac{n}{n+1}x + 1, n \in \mathbb{N}$$

It is easy to see that g_n converges pointwise to $gx = x + 1$ and g has not any fixed point. On the other hand g_n has the unique fixed $x_n = n+1$ and x_n tend to ∞. Since $a_n = \frac{n}{n+1} \to 1$, we can deduce g_n are not satisfied Theorem 4.1. Moreover, it is not satisfied Nadler's theorem. In fact g_n does not converges uniformly g. Indeed

$$\sup_{x\in\mathbb{R}} |g_n x - gx| = \sup_{x\in\mathbb{R}} |\frac{x}{4n+2} + \frac{n}{n+1} - 1| \geq |\frac{4n+2}{4n+2} + \frac{n}{n+1} - 1| = \frac{n}{n+1} \to 1$$

as $n \to \infty$.

References

[1] Berinde V., Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum, 9 (2004), no. 1, 43-53.
[2] Arandelovic I. D., Rajovic M. and Kilibarda V., On nonlinear quasi-contractions, Fixed Point Theory 9 (2008), no.2, 387-394.
[3] Boyd D. W. and Wong J. S. W. , On nonlinear contractions, Proc. Amer. Math. Soc., 20(1959), 458-464.
[4] Di Bari C. and Vetro P., Nonlinear quasi-contractions of Ćirić type Fixed Point Theory 13 (2012), no. 2, 453–459
[5] Ćirić L. B. On a family of contractive maps and fixed points. Publ. Inst. Math. (Beograd) (N.S.) 17(31) (1974), 45-51.
[6] Ok, E. A.Real Analysis with Economic applications, Princeton University Press (2007).
[7] Fraser R. and Nadler S. B. Sequences of contractive maps and fixed points, Pacific J. Math., 31(1969), 659-667.
[8] Ivanov A. A. Fixed points of mappings of metric spaces, (Russian) Studies in topology, II. Mat. Inst. Steklov. (LOMI) 66.
[9] Jeong S. U., Fixed point theorems related to Ćirić's contraction principle, J. Math. Anal. Appl. 225 (1998), no. 2, 630-640.
[10] Pacurar, M.. Sequences of almost contractions and fixed points, Carpathian J. Math., 24 (2008), no. 2, 101-109.
[11] Nadler, S. B.,Sequences of contractions and fixed points, Pacific J. Math., 27(1968), 579-585.
[12] Park, S., Sequences of quasi-contractions and fixed points. J. Korean Math. Soc. 14 (1977), no. 1, 99-103.
[13] Rhoades B. E., A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc., 226 (1977), 257-290.
[14] Reich S., Some remarks concerning contraction mappings, Canad. Math. Bull., 14 (1971), 121-124.
[15] Rus I. A.Sequences of operators and fixed points, Fixed Point Theory 5 (2004), no. 2, 349-36.