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L-weakly and M-weakly Demicompact Operators On Banach Lattices

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Abstract. In this paper, we introduce and investigate new concepts of L-weakly and M-weakly demicompact operators. Let E be a Banach lattice. An operator $T : E \longrightarrow E$ is called L-weakly demicompact, if for every norm bounded sequence (x_n) in $\mathcal{B}_E$ such that $\{x_n - Tx_n, \ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E, we have $\{x_n, \ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Additionally, an operator $T : E \longrightarrow E$ is called M-weakly demicompact if for every norm bounded disjoint sequence (x_n) in E such that $\|x_n - Tx_n\| \to 0$, we have $\|x_n\| \to 0$. L-weakly (resp. M-weakly) demicompact operators generalize known classes of operators which are L-weakly (resp. M-weakly) compact operators. We also elaborate some properties of these classes of operators.

1. Introduction

Throughout this paper X and Y will denote real Banach spaces, E and F will denote real Banach lattices. $\mathcal{B}_X$ is the closed unit ball of X and $Sol(A)$ denotes the solid hull of a subset A of a Banach lattice. The positive cone of E will be denoted by $E_+ = \{x \in E; \ \ 0 \leq x\}$.

Let A be a non-empty bounded subset of a Banach lattice E. A is said to be L-weakly compact if $\lim \|x_n\| = 0$ for every disjoint sequence (x_n) contained in the solid hull of A. The classes of L-weakly and M-weakly compact operators were introduced by Meyer-Nieberg [10]. An operator T from X into F is called L-weakly compact if $T(\mathcal{B}_X)$ is an L-weakly compact subset of F. An operator T from E into Y is called M-weakly compact if $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint sequence (x_n) in E.

Recall from [12] that an operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$, where $\mathcal{D}(T)$ is a subspace of X, is said to be demicompact if, for every bounded sequence (x_n) in the domain $\mathcal{D}(T)$ such that $(x_n - Tx_n)$ converges to $x \in X$, there is a convergent subsequence of (x_n). Note that each compact operator is demicompact, but the opposite is not always true. In fact, let $Id_X : X \longrightarrow X$ be the identity operator of a Banach space X of infinite dimension. It is clear that $-Id_X$ is demicompact but it is not compact. The concept of demicompactness has emerged in literature since 1966 in order to address fixed points. It was introduced by Petryshyn [12]. Jeribi [7] used the class of demicompact operators to obtain some results on Fredholm and spectral theories.

Next, in [8] some Fredholm and perturbation results including the class of weakly demicompact operators. Moreover, they explored the relationship between this class and measures of weak noncompactness

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of operators with respect to an axiomatic one. Let us recall that an operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be weakly demicompact if, every bounded sequence (x_n) in $\mathcal{D}(T)$ such that $(x_n - Tx_n)$ weakly converges in X, has a weakly convergent subsequence. As an example of a weakly demicompact operator, we mention weakly compact operators. Ferjani et al. confirmed in [5] that each demicompact operator on a Banach space X is weakly demicompact.

Recently, Benkhaled et al. [2] have introduced the class of order weakly demicompact operators on Banach lattices. An operator T from E into E is said to be order weakly demicompact if, for every order bounded sequence (x_n) in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$, we have $\|x_n\| \to 0$. This class includes both the order weakly compact and weakly demicompact operators.

The basic objective of this work lies in defining L-weakly and M-weakly demicompact operators. This paper is organized in the following way. In Section 2, we shall introduce a new concept of L-weakly demicompact operators (see Definition 2.1). Note that the class of L-weakly demicompact operators involves that of L-weakly compact operators (see Proposition 2.2). Subsequently, we shall illustrate our analysis by some outstanding properties (see Proposition 2.7 and Example 2.9). In Section 3, a characterization of L-weakly demicompact operators is displayed. The main result of this section is Theorem 3.3. In Section 4, the relationship between L-weakly demicompact operators and order weakly demicompact operators is enacted (see Proposition 4.1). In Section 5, the notion of M-weakly demicompact operators is introduced (see Definition 5.1). Note that the class of M-weakly demicompact operators includes that of M-weakly compact operators (see Proposition 5.2). Next, some interesting results are drawn (see Proposition 5.5, Example 5.7 and Theorem 5.9).

To state our results, we need to fix some notations and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

We use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \in F_+$ whenever $x \in E_+$. The operator T is regular if $T = T_1 - T_2$, where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous.

We refer the reader to the monographs [1, 11] for ambiguous terminology from Banach lattices and positive operators theory.

2. L-weakly Demicompact Operators

We start by the following definition.

Definition 2.1. *Let E be a Banach lattice. An operator $T : E \longrightarrow E$ is called L-weakly demicompact if, for every norm bounded sequence (x_n) in $\mathcal{B}_E$ such that $\{x_n - Tx_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E, we have $\{x_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E.*

Proposition 2.2. *Let E be a Banach lattice. Every L-weakly compact operator $T : E \longrightarrow E$ is L-weakly demicompact.*

Proof. Let (x_n) be a norm bounded sequence in $\mathcal{B}_E$ such that $\{x_n - Tx_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. We claim that $\{x_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Assuming that (w_k) is a disjoint sequence in the solid hull of $\{x_n,\ \ n \in \mathbb{N}\}$, we need to show that $\|w_k\| \to 0$. Consider (x_{n_k}) a subsequence of (x_n) such that $|w_k| \leq |x_{n_k}|$ for each $k \in \mathbb{N}$. Since

$$|w_k| \leq |x_{n_k} - Tx_{n_k}| + |Tx_{n_k}| \text{ for each } k \in \mathbb{N},$$

it follows, from the Riesz decomposition property (see [1, Theorem 1.13]), that for each k there exists $w_{k_1}, w_{k_2} \in E$ such that $w_k = w_{k_1} + w_{k_2}$ with

$$|w_{k_1}| \leq |x_{n_k} - Tx_{n_k}| \ \ (1)$$

and

$$|w_{k_2}| \leq |Tx_{n_k}| \ \ (2).$$

Inequality (1) yields that (w_{k_1}) is a disjoint sequence in $\mathrm{Sol}(\{x_{n_k} - Tx_{n_k},\ n \in \mathbb{N}\})$. Since $\{x_n - Tx_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E, then $\|w_{k_1}\|$ converges to 0. Furthermore, inequality (2) yields that (w_{k_2}) is a disjoint sequence in the solid hull of $\{Tx_{n_k},\ n \in \mathbb{N}\}$. Thus, the L-weak compactness of T indicates that $\|w_{k_2}\| \to 0$. Since we can write

$$\|w_k\| \leq \|w_{k_1}\| + \|w_{k_2}\|$$

for each k, then $\|w_k\| \to 0$, and the proof holds. □

Remark 2.3. *Note that the converse of Proposition 2.2 is not true in general. For instance, consider the identity operator $Id_{l^1} : l^1 \longrightarrow l^1$. It is clear that αId_{l^1}, for $\alpha \neq 1$, is L-weakly demicompact. On the other side, $\mathcal{B}_{l^1}$ is not relatively weakly compact and therefore is not L-weakly compact (see* [11, Proposition 3.6.5]*). Hence, αId_{l^1} is not L-weakly compact.*

Recall that an operator $T : E \longrightarrow E$ is said to be power L-weakly compact if there exists $m \in \mathbb{N}^*$ satisfying T^m which is L-weakly compact. To establish a sufficient condition for which each power L-weakly compact regular operator is L-weakly demicompact, we need to give the following Lemma.

Lemma 2.4. [13, Lemma 1.4.3] *Let T be a regular operator from a Banach lattice E into a Banach lattice F with an order continuous norm. If $A \subset E$ is L-weakly compact, then $T(A)$ is L-weakly compact.*

Proposition 2.5. *If E has an order continuous norm, then each power L-weakly compact regular operator $T : E \longrightarrow E$ is L-weakly demicompact.*

Proof. Assume that the norm of E is an order continuous norm. Let $T : E \longrightarrow E$ be a regular operator and (x_n) be a norm bounded sequence in $\mathcal{B}_E$ such that $\{x_n - Tx_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. The power L-weak compactness of T implies that there exists $m \in \mathbb{N}^\star$ such that T^m is L-weakly compact and therefore $\{T^m x_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Moreover, since T^{m-1} which is regular, it follows from Lemma 2.4 that $\{T^{m-1}x_n - T^m x_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. From the following inclusion

$$\{T^{m-1}x_n,\ n \in \mathbb{N}\} \subset \{T^{m-1}x_n - T^m x_n,\ n \in \mathbb{N}\} + \{T^m x_n,\ n \in \mathbb{N}\},$$

we obtain that $\{T^{m-1}x_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Repeating this process for $i = m-2, ...,$ 1. We have T^i which is regular. By using Lemma 2.4, we infer that $\{T^i x_n - T^{i+1}x_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Since we can write

$$\{T^i x_n,\ n \in \mathbb{N}\} \subset \{T^i x_n - T^{i+1}x_n,\ n \in \mathbb{N}\} + \{T^{i+1}x_n,\ n \in \mathbb{N}\},$$

then $\{T^i x_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Thus, for $i = 1$, we obtain $\{Tx_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. This implies that T is L-weakly compact. By using Proposition 2.2, we deduce that T is L-weakly demicompact. □

In what follows, we examine the L-weak demicompactness of the matrix operator $\mathcal{T}$ from $\mathcal{E}$ into $\mathcal{E}$ expressed by

$$\mathcal{T} = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m,1} & T_{m,2} & \cdots & T_{m,m} \end{pmatrix}$$

where $\mathcal{E} = \prod_{i=1}^{m} E_i$ is a direct sum of a family of Banach lattices $(E_i)_{1\leq i\leq m}$ and $T_{i,j} : E_j \longrightarrow E_i$ is an operator for all $1 \leq i, j \leq m$.

Proposition 2.6. *Let $\mathcal{E} = \prod_{i=1}^{m} E_i$ be a direct sum of a family of Banach lattices $(E_i)_{1\leq i\leq m}$ and let $T_{i,i} : E_i \longrightarrow E_i$ be an L-weakly demicompact operator for all $1 \leq i \leq m$. Then, the matrix operator $\mathcal{T}_1 : \mathcal{E} \longrightarrow \mathcal{E}$ defined by*

$$\mathcal{T}_1 = \begin{pmatrix} T_{1,1} & 0 & \cdots & 0 \\ 0 & T_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{m,m} \end{pmatrix}$$

is L-weakly demicompact.

Proof. Let $\{X_n = (x_n^1, x_n^2, ..., x_n^m),\ n \in \mathbb{N}\}$ be a norm bounded sequence in $\mathcal{B}_{\mathcal{E}}$ such that $\{X_n - \mathcal{T}_1 X_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of $\mathcal{E}$. We have to demonstrate that $\{X_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of $\mathcal{E}$. Since

$$X_n - \mathcal{T}_1 X_n = \begin{pmatrix} x_n^1 - T_{1,1} x_n^1 \\ x_n^2 - T_{2,2} x_n^2 \\ \vdots \\ x_n^m - T_{m,m} x_k^m \end{pmatrix},$$

for each $n \in \mathbb{N}$, then $\{x_n^i - T_{i,i} x_n^i,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i for each $i \in \{1, 2, ..., m\}$. Thus, the L-weak demicompactness of $T_{i,i}$ entails that $\{x_n^i,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i for each $1 \leq i \leq m$. Therefore, $\{X_n = (x_n^1, x_n^2, ..., x_n^m), n \in \mathbb{N}\}$ is an L-weakly compact subset of $\mathcal{E}$. □

Proposition 2.7. *Let $\mathcal{E} = \prod_{i=1}^{m} E_i$ be a direct sum of a family of Banach lattices $(E_i)_{1\leq i\leq m}$ and let $T_{i,j} : E_j \longrightarrow E_i$ be an operator for all $1 \leq i, j \leq m$. If the following conditions hold:*
(i) $T_{i,i} : E_i \longrightarrow E_i$ is L-weakly demicompact for all $1 \leq i \leq m$.
(ii) $T_{i,j} : E_j \longrightarrow E_i$ is L-weakly compact for all $1 \leq i < j \leq m$.
(iii) $T_{i,j} : E_j \longrightarrow E_i$ is regular for all $1 \leq j < i \leq m$.
(iv) The norm of $\mathcal{E}$ is order continuous.
Then, the matrix operator $\mathcal{T}_2 : \mathcal{E} \longrightarrow \mathcal{E}$ determined by

$$\mathcal{T}_2 = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m,1} & T_{m,2} & \cdots & T_{m,m} \end{pmatrix}$$

is L-weakly demicompact.

Proof. Since $\mathcal{E}$ has an order continuous norm, the norm of E_i is an order continuous norm for all $1 \leq i \leq m$. Let $\{X_n = (x_n^1, x_n^2, ..., x_n^m),\ n \in \mathbb{N}\}$ be a norm bounded sequence in $\mathcal{E}$ such that $\{X_n - \mathcal{T}_2 X_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of $\mathcal{E}$. We have to demonstrate that $\{X_n,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of $\mathcal{E}$. To this end, it is sufficient to confirm that $\{x_n^i,\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i for each

$1 \leq i \leq m$. Accordingly, for every n, we get

$$X_n - \mathcal{T}_2 X_n = \begin{pmatrix} x_n^1 - T_{1,1}x_n^1 - T_{1,2}x_n^2 - ... - T_{1,m}x_n^m \\ x_n^2 - T_{2,1}x_n^1 - T_{2,2}x_n^2 - ... - T_{2,m}x_n^m \\ \vdots \\ x_n^m - T_{m,1}x_n^1 - T_{m,2}x_n^2 - ... - T_{m,m}x_n^m \end{pmatrix}.$$

Then we have $\{x_n^i - T_{i,1}x_n^1 - T_{i,2}x_n^2 - ... - T_{i,m}x_n^m, \ n \in \mathbb{N}\}$ which is an L-weakly compact subset of E_i for all $1 \leq i \leq m$. First, let $i = 1$. Since $T_{1,j}$ is L-weakly compact for all $1 < j \leq m$, this means that $\{T_{1,2}x_n^1 + T_{1,3}x_n^3 + ... + T_{1,m}x_n^m, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_1. Therefore, departing from the relation

$$\{x_n^1 - T_{1,1}x_n^1, \ n \in \mathbb{N}\} \subset \{x_n^1 - T_{1,1}x_n^1 - T_{1,2}x_n^2 - ... - T_{1,m}x_n^m, \ n \in \mathbb{N}\} + \{T_{1,2}x_n^1 + ... + T_{1,m}x_n^m, \ n \in \mathbb{N}\},$$

it follows that $\{x_n^1 - T_{1,1}x_n^1, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_1. Thus, the L-weak demicompactness of $T_{1,1}$ yields that $\{x_n^1, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_1. Next, consider $i = 2, ..., m-1$. Since $T_{i,j}$ is regular for each $1 \leq j < i$. It follows from Lemma 2.4 that $\{T_{i,j}x_n^i, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i for each $1 \leq j < i$. Moreover, the L-weak demicompactness of $T_{i,j}$ for each $i < j \leq m$ implies that $\{T_{i,i+1}x_n^{i+1} + T_{i,i+2}x_n^{i+2} + ... + T_{i,m}x_n^m, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i. From the following inclusion

$$\begin{aligned}\{x_n^i - T_{i,i}x_n^i, \ n \in \mathbb{N}\} \subset &\{x_n^i - T_{i,1}x_n^1 - ... - T_{i,i}x_n^i - .... - T_{i,m}x_n^m, \ n \in \mathbb{N}\} \\ &+ \{T_{i,1}x_n^1, \ n \in \mathbb{N}\} + ... + \{T_{i,i-1}x_n^{i-1}, \ n \in \mathbb{N}\} \\ &+ \{T_{i,i+1}x_n^{i+1} + ... + T_{i,m}x_n^m, \ n \in \mathbb{N}\},\end{aligned}$$

it follows that $\{x_n^i - T_{i,i}x_n^i, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i. Thus, the L-weak demicompactness of $T_{i,i}$ implies that $\{x_n^i, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_i. Finally, for i=m. We have $\{T_{m,j}x_n^1, \ n \in \mathbb{N}\}$ which is an L-weakly compact subset of E_m for each $1 \leq j < m$. Since we can state that

$$\begin{aligned}\{x_n^m - T_{m,m}x_n^m, \ n \in \mathbb{N}\} \subset &\{T_{m,1}x_n^1, \ n \in \mathbb{N}\} + ... + \{T_{m,m-1}x_n^{m-1}, \ n \in \mathbb{N}\} \\ &+ \{x_n^m - T_{m,1}x_n^1 - ... - T_{m,m-1}x_n^{m-1} - T_{m,m}x_n^m, \ n \in \mathbb{N}\},\end{aligned}$$

we get $\{x_n^m - T_{m,m}x_n^m, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_m. The use of L-weak demicompactness of $T_{m,m}$ implies that $\{x_n^m, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E_m, and the proof of the proposition is finished. □

Corollary 2.8. *Let us assume that the following conditions hold:*
(*i*) *$T_{i,i} : E_i \longrightarrow E_i$ is L-weakly demicompact for all $1 \leq i \leq m$.*
(*ii*) *$T_{i,j} : E_j \longrightarrow E_i$ is regular for all $1 \leq i < j \leq m$.*
(*iii*) *$T_{i,j} : E_j \longrightarrow E_i$ is L-weakly compact for all $1 \leq j < i \leq m$.*
(*iv*) *The norm of $\mathcal{E}$ is order continuous.*
Then, $\mathcal{T}_2$ is L-weakly demicompact.

Proof. We invest the same reasoning as with the proof of Proposition 2.7. □

It is worth noting that the class of L-weakly demicompact operators lacks the vector space structure with the sum and with the external product. To illustrate this point, we provide the following example.

Example 2.9. *Let $J : L^2[0,1] \longrightarrow L^1[0,1]$ be the canonical injection. Consider $\mathcal{E} = L^2[0,1] \oplus L^1[0,1]$ and let the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ be defined via the matrix:*

$$\mathcal{T}_1 = \begin{pmatrix} 2Id_{L^2[0,1]} & J \\ 0 & 0 \end{pmatrix} \text{ and } \mathcal{T}_2 = \begin{pmatrix} -Id_{L^2[0,1]} & J \\ 0 & 0 \end{pmatrix}.$$

The canonical injection J is L-weakly compact. Relying on Proposition 2.7, it follows that $\mathcal{T}_1$ and $\mathcal{T}_2$ are L-weakly demicompact, but the sum $\mathcal{T}_1 + \mathcal{T}_2$ expressed by

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 = \begin{pmatrix} Id_{L^2[0,1]} & 2J \\ 0 & 0 \end{pmatrix}$$

is not. In fact, consider $\tilde{x}_n = (e_n, 0)$ for every n, where e_n is the sequence with the nth entry which equals 1 and others are zero. It is obvious that $(\tilde{x}_n)$ is a norm bounded sequence of $\mathcal{B}_{\mathcal{E}}$. Moreover, $\{\tilde{x}_n - \mathcal{T}\tilde{x}_n,\ \ n \in \mathbb{N}\}$ is an L-weakly compact subset of $\mathcal{E}$. Note that $\|\tilde{x}_n\|_{\mathcal{E}} = \|e_n\|_{L^2[0,1]} = 1$. This proves that $\{\tilde{x}_n,\ \ n \in \mathbb{N}\}$ is not an L-weakly compact subset of $\mathcal{E}$ and hence $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ is not an L-weakly demicompact.
If we follow the same reasoning as previously mentioned, we get $-\mathcal{T}_2$ which is not L-weakly demicompact.

3. Characterization of L-weakly demicompact opertors

The main target of this section is to exhibit a characterization of L-weakly demicompact operators. For this reason, we need to prove the following lemma:

Lemma 3.1. *Let E, F be two Banach lattices such that F has an order continuous norm, and let A, $B : E \longrightarrow F$ be operators with B is regular. Suppose that there are two sequences (f_n) and (g_n) in E such that:*

1. $\{g_n : \in \mathbb{N}\}$ *is an L-weakly compact subset of E.*
2. $\|Af_n - Bg_n\| \to 0$.

Then, $\{Af_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of F.

Proof. Let A, $B : E \longrightarrow F$ be two operators with B is regular. Let (f_n) and (g_n) be two sequences in E such that $\|Af_n - Bg_n\| \to 0$ and $\{g_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. We have to demonstrate that $\{Af_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Resting upon Proposition 3.6.2 in [11], it is sufficient to confirm that for every $\epsilon > 0$ there exists $0 \leq v_\epsilon \in F$ such that:

$$\{Af_n :\ n \in \mathbb{N}\} \subset [-v_\epsilon, v_\epsilon] + \epsilon B_F.$$

To this end, let $\epsilon > 0$. Taking into account that $\|Af_n - Bg_n\| \to 0$, we deduce that there exists $N \in \mathbb{N}$ such that $Af_n - Bg_n \in \frac{\epsilon}{2} B_E$ for each $n \geq \mathbb{N}$. Since $\{g_n :\ n \in \mathbb{N}\}$ is an L-weakly compact, we infer from Lemma 2.4 that $\{Bg_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of F. Using Proposition 3.6.2 in [11], we have

$$\{Bg_n : \in \mathbb{N}\} \subset [-u_\epsilon, u_\epsilon] + \frac{\epsilon}{2} B_E,$$

for some $0 \leq u_\epsilon \in F$. Put $v_\epsilon := u_\epsilon + \vee_{i=1}^{N} |Af_i|$. It is easy to infer that

$$\{Af_n :\ n \in \mathbb{N}\} \subset [-v_\epsilon, v_\epsilon] + \epsilon B_F,$$

and the proof of the lemma is finished. □

As a consequence, we obtain:

Corollary 3.2. *Let E be a Banach lattice with an order continuous norm, and let $T, S : E \longrightarrow E$ be two operators such that T which is L-weakly demicompact. If for every sequence (f_n) of E such that $\{Tf_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E and $\|Tf_n - Sf_n\| \to 0$, then S is L-weakly demicompact.*

Proof. Let (f_n) be a norm bounded sequence in $\mathcal{B}_E$ such that $\{f_n - Sf_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. We show that $\{f_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Since $\|Tf_n - Sf_n\| \to 0$, it follows from Lemma 3.1 (with $A = I - S$ and $B = I - T$) that $\{f_n - Tf_n :\ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Thus, the L-weak demicompactness of T implies that $\{f_n :\ n \in \mathbb{N}\}$ is L-weakly compact subset of E. □

Recall that a Banach lattice an with order continuous norm E has the subsequence splitting property [6], if for every bounded sequence (f_n), there exist a disjoint sequence (h_k) an equi-integrable sequence (g_k) and a subsequence (f_{n_k}) such that $f_{n_k} = g_k + h_k$ with g_k and h_k are disjoint for all k. Now, we are in a position to establish our theorem.

Theorem 3.3. *Let E be a Banach lattice with an order continuous norm and satsfies the subsequence splitting property. Let $T : E \longrightarrow E$ be a regular operator, then the following assertions are equivalent:*

1. *T is an L-weakly demicompact operator.*
2. $\|w_n\| \to 0$ *for each norm bounded disjoint sequence (w_n) in $\mathcal{B}_E$ such that $\{w_n - Tw_n : n \in \mathbb{N}\}$ is an L-weakly compact subset of E.*

Proof. (1) $\Longrightarrow$ (2) Consider that $T : E \longrightarrow E$ is a regular operator. Assume that T is an L-weakly demicompact. Let (w_n) be a norm bounded disjoint sequence in $\mathcal{B}_E$ such that $\{w_n - Tw_n : n \in \mathbb{N}\}$ is an L-weakly compact subset of E. The L-weak demicompactness of T implies that $\{w_n, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E and so $\|w_n\| \to 0$.
(2) $\Longrightarrow$ (1) Let (x_n) be a norm bounded sequence in $\mathcal{B}_E$ such that $\{x_n - Tx_n : n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Passing to a subsequence, we can assume that $x_n = w_n + y_n$, where (y_n) is an L-weakly compact sequence, (w_n) is a disjoint sequence and $|w_n| \wedge |y_n| = 0$ for each $n \in \mathbb{N}$. This implies that $Tx_n = Tw_n + Ty_n$ for each $n \in \mathbb{N}$. Since T is regular, it follows from Lemma 2.4 that $\{Ty_n : n \in \mathbb{N}\}$ is an L-weakly compact set. From the following inclusion

$$\{w_n - Tw_n : n \in \mathbb{N}\} \subset \{x_n - Tx_n : n \in \mathbb{N}\} + \{-y_n : n \in \mathbb{N}\} + \{Ty_n : n \in \mathbb{N}\},$$

we obtain that $\{w_n - Tw_n, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Applying the assertion (2), $\|w_n\| \to 0$. Therefore, $\|x_n - y_n\| \to 0$. According to Lemma 3.1 with $A = B = I$, we have $\{x_n : n \in \mathbb{N}\}$ is an L-weakly compact subset of E. □

4. Relationship between L-weakly demicompact and order weakly demicompact operators

Recall that an element $e \in E$ is said to be a weak unit if for $h \in E$, $e \wedge h = 0$ implies $h = 0$. Every separable Banach lattice has a weak unit. Note that an order continuous Banach lattice with a weak unit can be assumed to be included in $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ (see Theorem 1.b.14 in [9]). Hence, we denote this inclusion by $j : E \hookrightarrow L_1(\Omega, \Sigma, \mu)$. Let us remark that if X is a separable subspace of an order continuous Banach lattice E, it follows from Proposition 1.a.9 in [9] that E_X (E_X the ideal generated by X) has a weak unit.

The following result proves that the class of order weakly demicompact operators bigger than that of L-weakly demicompact operators.

Proposition 4.1. *Let E be a Banach lattice. If $T : E \longrightarrow E$ is an L-weakly demicompact operator, then T is order weakly demicompact.*

Proof. Let (x_n) be an order bounded sequence in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$. We need to demonstrate that $\|x_n\| \to 0$. It is clear that $\{x_n - Tx_n, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Thus, the L-weak demicompactness of T indicates that $\{x_n : n \in \mathbb{N}\}$ is an L-weakly compact subset of E. Now, let E_A be the ideal generated by $A := \{x_n, n \in \mathbb{N}\}$ in E. It remains to show that $x_n \to 0$. For this reason, we need to prove first that $\overline{E_A}$ is order continuous. Referring to Theorems 4.13 and 4.11 in [1], it is sufficient to show that every order bounded disjoint sequence in E_A is norm convergent to zero. Indeed, let (y_n) be a disjoint sequence with $0 \le x_n \le y$ for all n and some $y \in E_A$. Then there exist $x_{n_1}, ..., x_{n_k}$ and $\lambda > 0$ with $y \le \lambda \sum_{i=1}^{k} x_{n_i}$. Departing from the Riesz decomposition property, there exist $y_1^n, ..., y_k^n$ in E_+ such that

$$y_n = y_1^n + ... + y_k^n, \ \ y_i^n \le \lambda x_{n_i} \ \text{ and } \ y_i^n \le y_n,$$

for all $n \in \mathbb{N}$ and $i \in \{1, ..., k\}$. Clearly, for each i the sequence (y_i^n) is disjoint in Sol$\{x_n, n \in \mathbb{N}\}$. The L-weak compactness of $\{x_n, n \in \mathbb{N}\}$ implies that y_n converges to 0. Hence, $\overline{E_A}$ is order continuous.

Now, since $X := [x_n]$ is a separable subspace of E_A, it follows from Proposition 1.a.9 in [9] that $\overline{E_X}$ is an order ideal with a weak order unit. Therefore, it can be represented as a dense order ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ, such that the formal inclusion $j : \overline{E_X} \to L_1(\Omega, \Sigma, \mu)$ is continuous (see [9, Proposition 1.b.14]). Grounded on the Kadec-Pelczynski disjointification method in [9, Proposition 1.c.10]), we have one of the following statements is valid:
(i) $\|jx_n\|_1 \geq \delta\|x_n\|$ for some $\delta > 0$. Since $x_n \to 0$ in $\sigma(E, E')$, therefore $jx_n \to 0$ in $\sigma(L_1(\Omega, \Sigma, \mu), L_\infty(\Omega, \Sigma, \mu))$. Thus, the positive Schur property of $L_1(\Omega, \Sigma, \mu)$ implies that $\|jx_n\| \longrightarrow 0$ in $L_1(\Omega, \Sigma, \mu)$. Hence, $x_n \longrightarrow 0$ holds in E ($j_{|X}$ is an isomorphism).
(ii) There exist a subsequence (x_n) of (x_n) and a disjoint sequence (w_n) of $Sol\{x_n, n \in \mathbb{N}\}$ such that $\|x_n - w_n\| \longrightarrow 0$. Since $\{x_n, n \in \mathbb{N}\}$ is an L-weakly compact subset of E, it follows that $x_n \longrightarrow 0$. □

5. M-weakly Demicompact Operators

We start this section by the following definition.

Definition 5.1. *Let E be a Banach lattice. An operator $T : E \longrightarrow E$ is called M-weakly demicompact if for every norm bounded disjoint sequence (x_n) in E such that $\|x_n - Tx_n\| \to 0$, we have $\|x_n\| \to 0$.*

Proposition 5.2. *Let E be a Banach lattice. If $T : E \longrightarrow E$ is an M-weakly compact operator, then T is M-weakly demicompact.*

Proof. Let (x_n) be a norm bounded disjoint sequence in E such that $\|x_n - Tx_n\|$ converges to 0. The fact that T is an M-weakly compact operator, we obtain that $\|Tx_n\| \to 0$. Since we can state that

$$\|x_n\| \leq \|x_n - Tx_n\| + \|Tx_n\|$$

for each n, then $\|x_n\| \to 0$. □

Remark 5.3. *Note that an M-weakly demicompact operator is not necessarily M-weakly compact. Inded, let $\alpha \neq 1$ and $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ be the identity operator. It is clear that αId_{l^∞} is M-weakly demicompact. But, since αId_{l^1} is not L-weakly compact, $\alpha Id_{l^\infty} = (\alpha Id_{l^1})'$ is not M-weakly compact as well (see* [11, Proposition 3.6.17]*).*

Recall that an operator $T : E \longrightarrow E$ is said to be power M-weakly compact if there exists $m \in \mathbb{N}^*$ satisfying T^m which is M-weakly compact. The following result is a generalization of Proposition 5.2.

Proposition 5.4. *Let E be a Banach lattice. Then every power M-weakly compact operator $T : E \longrightarrow E$ is M-weakly demicompact.*

Proof. Let (x_n) be a norm bounded disjoint sequence in E such that $\|x_n - Tx_n\|$ converges to 0. We need to demonstrate have to show that $\|x_n\| \to 0$. Since T is power M-weakly compact, then there exists $m \in \mathbb{N}^\star$ such that T^m is M-weakly compact. Thus, $\|T^m x_n\| \to 0$. From the following inequalities

$$\begin{aligned}\|x_n\| &\leq \|x_n - T^m x_n\| + \|T^m x_n\| \\ &= \|\Big(I + T + ... + T^{m-1}\Big)\Big(x_n - Tx_n\Big)\| + \|T^m x_n\| \\ &\leq \|I + T + ... + T^{m-1}\|\|x_n - Tx_n\| + \|T^m x_n\|,\end{aligned}$$

for each n, it follows that $\|x_n\| \to 0$. □

Proposition 5.5. *Let $\mathcal{E} = \prod_{i=1}^m E_i$ be a direct sum of a family of Banach lattices $(E_i)_{1\leq i\leq m}$ and let $T_{i,j} : E_j \longrightarrow E_i$ be an operator for all $1 \leq i, j \leq m$. If the following conditions hold:*
(i) $T_{i,i} : E_i \longrightarrow E_i$ is M-weakly demicompact for all $1 \leq i \leq m$.

(ii) $T_{i,j} : E_j \longrightarrow E_i$ *is M-weakly compact for all* $1 \leq i < j \leq m$.
Then, the matrix operator $\mathcal{T}_2 : \mathcal{E} \longrightarrow \mathcal{E}$ *is provided by*

$$\mathcal{T}_2 = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m,1} & T_{m,2} & \cdots & T_{m,m} \end{pmatrix}$$

is M-weakly demicompact.

Proof. Let $\{X_n = (x_n^1, x_n^2, ..., x_n^m), n \in \mathbb{N}\}$ be a norm bounded disjoint sequence in $\mathcal{E}$ such that $\|X_n - \mathcal{T}_2 X_n\|_{\mathcal{E}} \to 0$. We have to confirm that $\|X_n\|_{\mathcal{E}} \to 0$. Since $\|X_n\|_{\mathcal{E}} = \|x_n^1\|_{E_1} + ... + \|x_n^m\|_{E_m}$, it is sufficient to prove that $\|x_n^i\|_{E_i} \to 0$ for every $1 \leq i \leq m$. Accordingly, for every n, we have

$$X_n - \mathcal{T}_2 X_n = \begin{pmatrix} x_n^1 - T_{1,1}x_n^1 - T_{1,2}x_n^2 - ... - T_{1,m}x_n^m \\ \\ x_n^2 - T_{2,1}x_n^1 - T_{2,2}x_n^2 - ... - T_{2,m}x_n^2 \\ \\ \vdots \\ \\ x_n^m - T_{m,1}x_n^1 - T_{m,2}x_n^2 - ... - T_{m,m}x_n^m \end{pmatrix},$$

then $\|x_n^i - T_{i,1}x_n^1 - T_{i,2}x_n^2 - ... - T_{i,m}x_n^m\|_{E_i} \to 0$ for all $1 \leq i \leq m$. First, let $i = 1$. Since (x_n^j) is disjoint, it follows from the M-weak compactness of $T_{1,j}$ that $\|T_{1,j}x_n^j\|_{E_1} \to 0$ for all $1 < j \leq m$. From the following inequality

$$\|x_n^1 - T_{1,1}x_n^1\|_{E_1} \leq \|x_n^1 - T_{1,1}x_n^1 - T_{1,2}x_n^2 - ... - T_{1,m}x_n^m\|_{E_1} + \|T_{1,2}x_n^2\|_{E_1} + ... + \|T_{1,m}x_n^m\|_{E_1}$$

for each n, we get $\|x_n^1 - T_{1,1}x_n^1\|_{E_1} \to 0$. Thus, the M-weak demicompactness of $T_{1,1}$ asserts that $\|x_n^1\|_{E_1} \to 0$. Next, consider $i = 2, ..., m-1$. Since $\|x_n^j\|_{E_j} \to 0$ for every $1 \leq j < i$, it follows that $\|T_{i,j}x_n^j\|_{E_i} \to 0$ for every $1 \leq j < i$. Moreover, the M-weakly compactness of $T_{i,j}$ for each $i < j \leq m$ implies that $\|T_{i,j}x_n^j\|_{E_i} \to 0$ for each $i < j \leq m$. Hence, from the relation

$$\begin{aligned} \|x_n^i - T_{ii}x_n^i\|_{E_i} \leq & \|T_{i,1}x_n^1\|_{E_i} + ... + \|T_{i,i-1}x_n^{i-1}\|_{E_i} + \|x_n^i - T_{i1}x_n^1 - ... - T_{i,i}x_n^i - ... - T_{im}x_n^m\|_{E_i} \\ & + \|T_{i,i+1}x_n^{i+1}\|_{E_i} + \|T_{i,i+2}x_n^{i+2}\|_{E_i} + ... + \|T_{im}x_n^m\|_{E_i}, \end{aligned}$$

it follows that $\|x_n^i - T_{i,i}x_n^i\|_{E_i} \to 0$. Since $T_{i,i}$ is M-weakly demicompact, we infer that $\|x_n^i\|_{E_i} \to 0$. Finally, let $i = m$. We have $\|T_{m,j}x_n^j\|_{E_m} \to 0$ for all $1 \leq j < m$. From the following inequality

$$\begin{aligned} \|x_n^m - T_{m,m}x_n^m\|_{E_m} \leq & \|T_{m,1}x_n^1\|_{E_m} + \|T_{m,2}x_n^2\|_{E_m} + ... + \|T_{m,m-1}x_n^{m-1}\|_{E_m} \\ & + \|x_n^m - T_{m,1}x_n^1 - T_{m,2}x_n^2 - ... - T_{m,m-1}x_n^{m-1} - T_{m,m}x_n^m\|_{E_m}, \end{aligned}$$

for each n, we get $\|x_n^m - T_{m,m}x_n^m\|_{E_m} \to 0$. Thus, the M-weak demicompactness of $T_{m,m}$ implies that $\|x_n^m\|_{E_m} \to 0$. This completes the proof. □

Corollary 5.6. *Let us assume that the following conditions hold:*
(i) $T_{i,i} : E_i \longrightarrow E_i$ *is M-weakly demicompact for all* $1 \leq i \leq m$.
(ii) $T_{i,j} : E_j \longrightarrow E_i$ *is M-weakly compact for all* $1 \leq j < i \leq m$.
Then, $\mathcal{T}_2$ *is M-weakly demicompact.*

Proof. We invest the same reasoning provided in the proof of Proposition 5.5. □

Example 5.7. *Note that the canonical injection $J : L^2[0,1] \longrightarrow L^1[0,1]$ is M-weakly compact. If we follow the same reasoning as reported in Example 2.9, then we can prove that the sum of M-weakly demicompact operators and the product of a complex number by a M-weakly demicompact operator are not necessarily M-weakly demicompact.*

The next result presents a relationship between L-weakly demicompact operators and M-weakly demicompact operators.

Proposition 5.8. *Let E be a Banach lattice. Every L-weakly demicompact operator $T : E \longrightarrow E$ is M-weakly demicompact.*

Proof. Let (x_n) be a norm bounded disjoint sequence in E such that $\|x_n - Tx_n\|$ converges to 0. We have to corroborate that $\|x_n\| \to 0$. Relying on Lemma 2.4 in [3], it follows that $\{x_n - Tx_n, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E. The L-weak demicompactness of T implies that $\{x_n, \ n \in \mathbb{N}\}$ is an L-weakly compact subset of E and hence $x_n \to 0$. □

The following theorem provides a characterization of M-weakly demicompact operators in terms of positive weak null sequences.

Theorem 5.9. *Let E be a Banach lattice such that both E and E' have order continuous norms, and $T : E \longrightarrow E$ be an operator. Then, the following assertions are equivalent:*
(1) *T is M-weakly demicompact.*
(2) *For each norm bounded sequence (x_n) in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$, we have $\|x_n\| \to 0$.*

Proof. (1) $\Longrightarrow$ (2) Let $(x_n)_n$ be a norm bounded sequence in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$. By investing the same reasoning as used in the proof of Proposition 4.1, one of the following statements is valid:
(*i*) $\|x_n\| \to 0$. This completes the proof.
(*ii*) For each subsequence (y_n) of (x_n), there exist a subsequence (z_n) of (y_n) and a disjoint sequence $(w_n) \subset \text{Sol}\{z_n, n \in \mathbb{N}\}$ such that $\|z_n - w_n\| \longrightarrow 0$. Since T is M-weakly demicompact and from the following inequalities

$$\begin{aligned} \|w_n - Tw_n\| &\le \|w_n - Tz_n + Tz_n - Tw_n\| \le \|w_n - Tz_n\| + \|Tz_n - Tw_n\| \\ &= \|w_n - z_n + z_n - Tz_n\| + \|Tz_n - Tw_n\| \\ &\le \|w_n - z_n\| + \|z_n - Tz_n\| + \|Tz_n - Tw_n\| \to 0, \end{aligned}$$

we get $\|w_n\| \to 0$. Using $\|z_n\| \le \|z_n - w_n\| + \|w_n\|$, we infer that $\|z_n\| \longrightarrow 0$. As (y_n) is arbitrary, we deduce that $\|x_n\| \longrightarrow 0$.

(2) $\Longrightarrow$ (1) Let (x_n) be a norm bounded disjoint sequence of E such that $\|x_n - Tx_n\| \to 0$. We need to demonstrate that $\|x_n\| \to 0$. Since (x_n) is disjoint and the norm on E' is order continuous, it follows from Theorem 2.4.14 in [11] that $x_n \to 0$ in $\sigma(E, E')$. Applying the assertion (2), $\|x_n\| \to 0$. Hence, T is M-weakly demicompact. □

We end this paper by an open question:
Question. Does the L-weakly and M-weakly demicompactness concept provide a characterization of the upper semi-Fredholm operators?

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