Filomat 36:13 (2022), 4293–4317
https://doi.org/10.2298/FIL2213293D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Class of Integral Operators Induced by Harmonic Bergman-Besov Kernels on Lebesgue Classes

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Abstract. We provide a full characterization in terms of the six parameters involved the boundedness of all standard weighted integral operators induced by harmonic Bergman-Besov kernels acting between different Lebesgue classes with standard weights on the unit ball of $\mathbb{R}^n$. These operators in some sense generalize the harmonic Bergman-Besov projections. To obtain the necessity conditions, we use a technique that heavily depends on the precise inclusion relations between harmonic Bergman-Besov and weighted Bloch spaces on the unit ball. This fruitful technique is new. It has been used first with holomorphic Bergman-Besov kernels by Kaptanoğlu and Üreyen. Methods of the sufficiency proofs we employ are Schur tests or Hölder or Minkowski type inequalities which also make use of estimates of Forelli-Rudin type integrals.

1. Introduction

Let $n \geq 2$ be an integer, $\mathbb{B}$ be the unit ball and $\mathbb{S}$ be the unit sphere of $\mathbb{R}^n$. Let ν and σ be the volume and surface measures on $\mathbb{B}$ and $\mathbb{S}$ normalized so that $\nu(\mathbb{B}) = 1$ and $\sigma(\mathbb{S}) = 1$. For $\alpha \in \mathbb{R}$, define the weighted volume measures ν_α on $\mathbb{B}$ by

$$d\nu_\alpha(x) = \frac{1}{V_\alpha}(1 - |x|^2)^\alpha d\nu(x).$$

These measures are finite when $\alpha > -1$ and in this case we choose V_α so that $\nu_\alpha(\mathbb{B}) = 1$. When $\alpha \leq -1$, we set $V_\alpha = 1$. For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_α by L^p_α and the corresponding norms by $\|\cdot\|_{L^p_\alpha}$.

Let $h(\mathbb{B})$ be the space of all complex-valued harmonic functions on $\mathbb{B}$ with the topology of uniform convergence on compact subsets. The space of bounded harmonic functions on $\mathbb{B}$ is denoted by h^∞. For $0 < p < \infty$ and $\alpha > -1$, the weighted harmonic Bergman space b^p_α is defined by $b^p_\alpha = L^p_\alpha \cap h(\mathbb{B})$ endowed with the norm $\|\cdot\|_{L^p_\alpha}$. When $p = 2$, the space b^2_α is a Hilbert space with respect to the inner product $[f, g]_{b^2_\alpha} = \int_{\mathbb{B}} f\overline{g}\, d\nu_\alpha(x)$ and for each $x \in \mathbb{B}$, the point evaluation functional $f \to f(x)$ is bounded on b^2_α. Thus, by the Riesz representation theorem, there exists the reproducing kernel $R_\alpha(x, \cdot)$ such that $f(x) = [f, R_\alpha(x, \cdot)]_{b^2_\alpha}$

2020 *Mathematics Subject Classification*. Primary 47B34, 47G10; Secondary 31B05, 31B10, 42B35, 45P05, 47B32, 46E20, 46E15
Keywords. Integral operator, Harmonic Bergman-Besov kernel, Harmonic Bergman-Besov space, Weighted harmonic Bloch space, Harmonic Bergman-Besov projection, Schur test, Forelli-Rudin estimate, Inclusion relation
Received: 20 April 2021; Accepted: 25 April 2022
Communicated by Miodrag Mateljević
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for every $f \in b^2_\alpha$ and $x \in \mathbb{B}$. The homogeneous expansion of R_α is given in the $\alpha > -1$ part of the formulas (2) and (3) below (see [12], [5]).

The orthogonal projection $Q_\alpha : L^2_\alpha \to b^2_\alpha$ is given by the integral operator

$$Q_\alpha f(x) = \frac{1}{V_\alpha} \int_{\mathbb{B}} R_\alpha(x,y) f(y)(1-|y|^2)^\alpha d\nu(y) \quad (f \in L^2_\alpha). \tag{1}$$

The above integral operator plays a major role in the theory of weighted harmonic Bergman spaces and the question when $Q_\alpha : L^p_\beta \to L^p_\beta$ is bounded is studied in many sources such as ([4, Lemma 3.3], [5, Theorem 7.3], [14, Theorem 3.1], [23, Lemma 2.4], [18, Propositions 3.5 and 3.6]).

The main purpose of this paper is to determine precisely when the integral operator in (1) is bounded with considering all possible generalizations. First we allow for the exponents and the weights to be different and consider the operator in (1) from L^p_α to L^q_β. Next we allow the parameters in the integrand in (1) to be different.

In addition we also remove the restriction $\alpha > -1$ and allow it to be any real number. The weighted harmonic Bergman spaces b^p_α initially defined for $\alpha > -1$ can be extended to the whole range $\alpha \in \mathbb{R}$. This is studied in [12] and will be briefly reviewed in Section 2. We call the extended family b^p_α ($\alpha \in \mathbb{R}$) as harmonic Bergman-Besov spaces and the corresponding reproducing kernels $R_\alpha(x,y)$ ($\alpha \in \mathbb{R}$) as harmonic Bergman-Besov kernels. The homogeneous expansion of R_α in terms of zonal harmonics have the form

$$R_\alpha(x,y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x,y) \quad (\alpha \in \mathbb{R},\ x,y \in \mathbb{B}), \tag{2}$$

where (see [11, Theorem 3.7], [12, Theorem 1.3])

$$\gamma_k(\alpha) := \begin{cases} \dfrac{(1+n/2+\alpha)_k}{(n/2)_k}, & \text{if } \alpha > -(1+n/2); \\[2ex] \dfrac{(k!)^2}{(1-(n/2+\alpha))_k (n/2)_k}, & \text{if } \alpha \le -(1+n/2), \end{cases} \tag{3}$$

and $(a)_b$ is the Pochhammer symbol. For definition and details about $Z_k(x,y)$, see [2, Chapter 5].

Finally, we allow the exponents p, q to be ∞. Let $L^\infty = L^\infty(\nu)$ be the Lebesgue class of all essentially bounded functions on $\mathbb{B}$ with respect to ν. In this case we have $L^\infty(d\nu_\alpha) = L^\infty$ for every $\alpha \in \mathbb{R}$ and because of this we need to use a different weighted class. For $\alpha \in \mathbb{R}$, we define

$$\mathcal{L}^\infty_\alpha := \{\varphi \text{ is measurable on } \mathbb{B} : (1-|x|^2)^\alpha \varphi(x) \in L^\infty\},$$

so that $\mathcal{L}^\infty_0 = L^\infty$. The norm on $\mathcal{L}^\infty_\alpha$ is

$$\|\varphi\|_{\mathcal{L}^\infty_\alpha} = \|(1-|x|^2)^\alpha \varphi(x)\|_{L^\infty}.$$

We are now ready to state our results. For $b, c \in \mathbb{R}$ define the integral operators T_{bc} and S_{bc} by

$$T_{bc} f(x) = \int_{\mathbb{B}} R_c(x,y) f(y)(1-|y|^2)^b d\nu(y)$$

and

$$S_{bc} f(x) = \int_{\mathbb{B}} \left|R_c(x,y)\right| f(y)(1-|y|^2)^b d\nu(y).$$

We are interested in determining exactly when the above operators are bounded from L^p_α to L^q_β. Our main results are the following seven theorems that describe their boundedness in terms of the six parameters

$(b, c, \alpha, \beta, p, q)$ involved. We include the operator S_{bc} because we need operators with positive kernels to apply Schur tests.

The holomorphic Bergman projections on the ball have been studied for some time; see [9, 19], for example. The corresponding integral operators between different Lebesgue classes on the unit ball of $\mathbb{C}^n$ in the holomorphic case are considered in several publications, such as [25]. But a complete investigation of the weighted integral operators arising from holomorphic Bergman-Besov kernels between different weighted Lebesgue classes is recently concluded in [16]. We do not attempt to survey the wide literature on different spaces or on more general domains or on more general weights.

We first consider the case $1 \le p \le q < \infty$. The special case $1 \le p = q < \infty$ and $\alpha = \beta$ with $\alpha \in \mathbb{R}$ is considered earlier in [12]. Notice that, in [12] they used the operators which contain an extra factor $(1 - |x|^2)^a$ (clearly it does not change anything on the boundedness of operators) and an extra constraints that $c = b + a$. When $\alpha > -n$, the kernel $R_\alpha(x, y)$ is dominated by $1/[x, y]^{\alpha+n}$ (see Lemma 2.6 below). Here and subsequently, $[x, y]$ denotes $[x, y] = \sqrt{1 - 2x \cdot y + |x|^2|y|^2}$ for $x, y \in \mathbb{B}$. For the boundedness and the norm of the integral operators which contain these dominating terms instead of the kernels and an extra factor $(1 - |x|^2)^a$ but only for the restricted case $1 \le p = q < \infty$ and $\alpha = \beta > -1$, see [26]. We do not try to estimate the norms of the main operators. The holomorphic counterparts of our two results below on the boundedness of integral operators induced by holomorphic Bergman-Besov kernels appear in [16]; also the more restricted kernels and case with $\alpha, \beta > -1$ in [15].

Theorem 1.1. *Let b and c be real numbers. Let $1 < p \le q < \infty$ and $\alpha, \beta \in \mathbb{R}$ with $\beta > -1$. The following are equivalent:*

(i) T_{bc} is bounded from L^p_α to L^q_β.

(ii) S_{bc} is bounded from L^p_α to L^q_β.

(iii) $\alpha + 1 < p(b + 1)$ and $c \le b + \dfrac{n+\beta}{q} - \dfrac{n+\alpha}{p}$.

We have to treat the case $p = 1$ separately.

Theorem 1.2. *Let b and c be real numbers. Let $1 = p \le q < \infty$ and $\alpha, \beta \in \mathbb{R}$ with $\beta > -1$. The following are equivalent:*

(i) T_{bc} is bounded from L^1_α to L^q_β.

(ii) S_{bc} is bounded from L^1_α to L^q_β.

(iii) $\alpha < b$ and $c \le b + \dfrac{n+\beta}{q} - (n + \alpha)$ or $\alpha \le b$ and $c < b + \dfrac{n+\beta}{q} - (n + \alpha)$

Now, we consider the case $1 \le q < p < \infty$.

Theorem 1.3. *Let b and c be real numbers. Let $1 \le q < p < \infty$ and $\alpha, \beta \in \mathbb{R}$ with $\beta > -1$. The following are equivalent:*

(i) T_{bc} is bounded from L^p_α to L^q_β.

(ii) S_{bc} is bounded from L^p_α to L^q_β.

(iii) $\alpha + 1 < p(b + 1)$ and $c < b + \dfrac{1+\beta}{q} - \dfrac{1+\alpha}{p}$.

We consider the case when either p or q is ∞ in the following four theorems. The special case $p = q = \infty$ and $\alpha = \beta$ is considered earlier in [7] where they used the operators which contain an extra factor $(1 - |x|^2)^a$ and an extra constraints that $c = b + a$.

Theorem 1.4. *Let b and c be real numbers. Let $1 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$ with $\beta \ge 0$. The following are equivalent:*

(i) T_{bc} is bounded from L^p_α to $\mathcal{L}^\infty_\beta$.

(ii) S_{bc} *is bounded from* L^p_α *to* $\mathcal{L}^\infty_\beta$.

(iii) $\alpha + 1 < p(b+1)$ *and* $c \leq b + \beta - \dfrac{n+\alpha}{p}$, *and the strict inequality holds when* $\beta = 0$

Again, we have to treat the case $p = 1$ separately.

Theorem 1.5. *Let b and c be real numbers. Let* $\alpha, \beta \in \mathbb{R}$ *with* $\beta \geq 0$. *The following are equivalent:*

(i) T_{bc} *is bounded from* L^1_α *to* $\mathcal{L}^\infty_\beta$.

(ii) S_{bc} *is bounded from* L^1_α *to* $\mathcal{L}^\infty_\beta$.

(iii) $\alpha < b$ *and* $c \leq b + \beta - (n+\alpha)$ *or* $\alpha \leq b$ *and* $c < b + \beta - (n+\alpha)$

Theorem 1.6. *Let b and c be real numbers. Let* $1 \leq q < \infty$ *and* $\alpha, \beta \in \mathbb{R}$ *with* $\beta > -1$. *The following are equivalent:*

(i) T_{bc} *is bounded from* $\mathcal{L}^\infty_\alpha$ *to* L^q_β.

(ii) S_{bc} *is bounded from* $\mathcal{L}^\infty_\alpha$ *to* L^q_β.

(iii) $\alpha - 1 < b$ *and* $c < b + \dfrac{\beta+1}{q} - \alpha$.

Theorem 1.7. *Let b and c be real numbers. Let* $\alpha, \beta \in \mathbb{R}$ *with* $\beta \geq 0$. *The following are equivalent:*

(i) T_{bc} *is bounded from* $\mathcal{L}^\infty_\alpha$ *to* $\mathcal{L}^\infty_\beta$.

(ii) S_{bc} *is bounded from* $\mathcal{L}^\infty_\alpha$ *to* $\mathcal{L}^\infty_\beta$.

(iii) $\alpha - 1 < b$ *and* $c \leq b + \beta - \alpha$, *and the strict inequality holds when* $\beta = 0$.

It is clear that

$$|T_{bc}f(x)| \leq S_{bc}(|f|)(x),$$

so the boundedness of S_{bc} implies the boundedness of T_{bc}. Thus it is obvious that (ii) implies (i) in all of our theorems above. So "Necessity" and "Sufficiency" in the proofs refers to the implications (i)→ (iii) and (ii)→ (i), respectively.

Remark 1.8. *Note the difference in the conditions on c in parts (iii) of Theorems 1.1–1.7. These conditions are connected with the inclusion relations between harmonic Bergman-Besov and weighted Bloch spaces (see Theorems 4.10, 4.11 and 4.12 below).*

Remark 1.9. *The conditions* $\beta > -1$ *when* $q < \infty$ *and* $\beta \geq 0$ *when* $q = \infty$ *in our theorems cannot be removed as we clarify later in Corollary 3.6. These constraints are consequences of the fact that* $T_{bc}f$ *is harmonic on B and* $|T_{bc}f|^q$ *is subharmonic when* $q < \infty$, *and by the maximum principle for harmonic functions when* $q = \infty$.

The sufficiency proofs for all seven theorems are either by Schur tests or by direct Hölder or Minkowski type inequalities which also make use of growth rate estimates of Forelli-Rudin type integrals. The necessity proofs are by an original technique that most heavily depends on the precise inclusion relations between harmonic Bergman-Besov and weighted Bloch spaces on $\mathbb{B}$. This technique as many others borrowed from [16] and have been modified to our kernels and spaces. We give all the sufficiency and necessity proofs in detail and it makes this paper more self-contained.

This paper is organized as follows. In Section 2, we collect some known facts about the harmonic Bergman-Besov and weighted Bloch spaces. In Section 3, we insert the main operators in context and derive their basic properties which we will need in the sequel. The corollary about the conditions $\beta > -1$ when $q < \infty$ and $\beta \geq 0$ when $q = \infty$ is also here. In Section 4, we list the some important results that we apply in the proofs. As indicated before, the proofs of Theorems 1.1–1.7 contain different methods which are interesting enough to be stated separately. Thus we prove necessity parts of these theorems in Section 5 and the sufficiency parts of these theorems in Section 6.

2. Preliminaries

In multi-index notation, $m = (m_1, \dots, m_n)$ is an n-tuple of non-negative integers $m_1, \dots, m_n$ and

$$\partial^m f = \frac{\partial^{|m|} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}$$

is the usual partial derivative for smooth f, where $|m| = m_1 + \cdots + m_n$.

For $1 \le p \le \infty$, we denote the conjugate exponent of p by p'. That is, if $1 < p < \infty$, then $\frac{1}{p} + \frac{1}{p'} = 1$; if $p = 1$, then $p' = \infty$ and if $p = \infty$, then $p' = 1$.

For two positive expressions X and Y, we write $X \sim Y$ if X/Y is bounded above and below by some positive constants. We will denote these constants whose exact values are inessential by a generic upper case C. We will also write $X \lesssim Y$ to mean $X \le CY$.

Now we clarify the notation used in (3) at the beginning. The Pochhammer symbol $(a)_b$ is given by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)},$$

when a and $a + b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. By the Stirling formula

$$\frac{(a)_c}{(b)_c} \sim c^{a-b}, \quad c \to \infty. \tag{4}$$

Let $f \in L^1_0$. The polar coordinates formula is

$$\int_{\mathbb{B}} f(x)\, d\nu(x) = n \int_0^1 \epsilon^{n-1} \int_{\mathbb{S}} f(\epsilon\zeta)\, d\sigma(\zeta)\, d\epsilon,$$

in which $x = \epsilon\zeta$ with $\epsilon > 0$ and $\zeta \in \mathbb{S}$.

As mentioned in the introduction, for $x, y \in \mathbb{B}$, we will use the notation

$$[x, y] = \sqrt{1 - 2x \cdot y + |x|^2|y|^2},$$

where $x \cdot y$ denotes the inner product of x and y in $\mathbb{R}^n$. It is elementary to show that the equalities

$$[x, y] = \left| |y|x - \frac{y}{|y|} \right| = \left| |x|y - \frac{x}{|x|} \right|,$$

hold for every nonzero x, y. Note that $0 < 1 - |x||y| \le [x, y] \le 1 + |x||y| < 2$ for $x, y \in \mathbb{B}$. Further, we have $[x, \zeta] = |x - \zeta|$ when $y = \zeta \in \mathbb{S}$.

We show an integral inner product on a function space A by $[\cdot, \cdot]_A$.

2.1. Harmonic Bergman-Besov and Weighted Bloch Spaces

It is well-known that $f \in h(\mathbb{B})$ has a homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$, where f_k is a homogeneous harmonic polynomial of degree k, the series absolutely and uniformly converges on compact subsets of $\mathbb{B}$ (see [2]).

The weighted harmonic Bergman spaces $b^p_\alpha (\alpha > -1)$ can be extended to all $\alpha \in \mathbb{R}$. Thus, we resort to derivatives. For $\alpha \in \mathbb{R}$ and $0 < p < \infty$, let N be a non-negative integer such that $\alpha + pN > -1$. The harmonic Bergman-Besov space b^p_α consists of all $f \in h(\mathbb{B})$ such that

$$(1 - |x|^2)^N \partial^m f \in L^p_\alpha,$$

for every multi-index m with $|m| = N$.

Roughly speaking the "$p = \infty$" case of Bergman-Besov spaces b^p_α is the family of weighted Bloch spaces b^∞_α. Let $\alpha \in \mathbb{R}$. Pick a non-negative integer N such that $\alpha + N > 0$. The weighted harmonic Bloch space b^∞_α consists of all $f \in h(\mathbb{B})$ such that

$$(1-|x|^2)^N \partial^m f \in \mathcal{L}^\infty_\alpha,$$

for every multi-index m with $|m| = N$. We mention one special case. When $\alpha = 0$ taking $N = 1$ shows

$$b^\infty_0 = \Big\{ f \in h(\mathbb{B}) : \sup_{x\in\mathbb{B}} (1-|x|^2)|\nabla f(x)| < \infty \Big\}.$$

This is the most studied member of the family.

Partial derivatives are not convenient in studying the spaces of interest in this work and it is more advantageous to use certain radial differential operators $D^t_s : h(\mathbb{B}) \to h(\mathbb{B})$, $(s, t \in \mathbb{R})$ introduced in [11] and [12] that are compatible with the kernels.

Before going to the definition, note that for every $\alpha \in \mathbb{R}$ we have $\gamma_0(\alpha) = 1$, and therefore

$$R_\alpha(x, 0) = R_\alpha(0, y) = 1, \quad (x, y \in \mathbb{B}, \alpha \in \mathbb{R}). \tag{5}$$

Checking the two cases in (3), we have by (4)

$$\gamma_k(\alpha) \sim k^{1+\alpha} \quad (k \to \infty). \tag{6}$$

Definition 2.1. *Let $f = \sum_{k=0}^\infty f_k \in h(\mathbb{B})$ be given by its homogeneous expansion. For $s, t \in \mathbb{R}$ we define $D^t_s : h(\mathbb{B}) \to h(\mathbb{B})$ by*

$$D^t_s f := \sum_{k=0}^\infty \frac{\gamma_k(s+t)}{\gamma_k(s)} f_k. \tag{7}$$

By (6), $\gamma_k(s+t)/\gamma_k(s) \sim k^t$ for any s, t. So D^t_s is a differential operator of order t. For every $s \in \mathbb{R}$, $D^0_s = I$, the identity. An important property of D^t_s is that it is invertible with two-sided inverse D^{-t}_{s+t}:

$$D^{-t}_{s+t} D^t_s = D^t_s D^{-t}_{s+t} = I, \tag{8}$$

which follows from the additive property

$$D^z_{s+t} D^t_s = D^{z+t}_s. \tag{9}$$

Thus any D^t_s maps $h(\mathbb{B})$ onto itself. Then for every $s, t \in \mathbb{R}$, the map $D^t_s : h(\mathbb{B}) \to h(\mathbb{B})$ is continuous. For a proof see [12, Theorem 3.2]

The parameter s plays a minor role. It is used to have the precise relation

$$D^t_s R_s(x, y) = R_{s+t}(x, y), \tag{10}$$

where differentiation is performed on either of the variables x or y and by symmetry it does not matter which.

One of the most important properties about the operators D^t_s is that it allows us to pass from one Bergman-Besov (or Bloch) space to another. More precisely, we have the following results.

Lemma 2.2. *Let $0 < p < \infty$ and $\alpha, s, t \in \mathbb{R}$.*

(i) The map $D^t_s : b^p_\alpha \to b^p_{\alpha+pt}$ is an isomorphism.

(ii) The map $D^t_s : b^\infty_\alpha \to b^\infty_{\alpha+t}$ is an isomorphism.

For a proof of part (i) of the above lemma see [12, Corollary 9.2] when $1 \le p < \infty$ and [6, Proposition 4.7] when $0 < p < 1$. For part (ii) see [7, Proposition 4.6].

Consider the linear transformation I_s^t defined for $f \in h(\mathbb{B})$ by

$$I_s^t f(x) := (1-|x|^2)^t D_s^t f(x).$$

The harmonic Bergman-Besov space and Bloch space can equivalently be defined by using the operators D_s^t.

Definition 2.3. *For $0 < p < \infty$ and $\alpha \in \mathbb{R}$, we define the harmonic Bergman-Besov space b_α^p to consists of all $f \in h(\mathbb{B})$ for which $I_s^t f$ belongs to L_α^p for some s, t satisfying (see [12] when $1 \le p < \infty$, and [6] when $0 < p < 1$)*

$$\alpha + pt > -1. \tag{11}$$

The quantity

$$\|f\|_{b_\alpha^p}^p = \|I_s^t f\|_{L_\alpha^p}^p = c_\alpha \int_{\mathbb{B}} |D_s^t f(x)|^p (1-|x|^2)^{\alpha+pt} d\nu(x) < \infty$$

defines a norm (quasinorm when $0 < p < 1$) on b_α^p for any such s, t.

When $\alpha > -1$, one can choose $N = 0$ and the resulting space is weighted harmonic Bergman space. when $\alpha = -n$, the measure $d\nu_{-n}$ is Möbius invariant and the spaces b_{-n}^p are called harmonic Besov spaces by many authors. In particular, the b_{-1}^2 is the harmonic Hardy space and b_{-n}^2 is the harmonic Dirichlet space.

Definition 2.4. *For $\alpha \in \mathbb{R}$, we define the harmonic Bloch space b_α^∞ to consists of all $f \in h(\mathbb{B})$ for which $I_s^t f$ belongs to $\mathcal{L}_\alpha^\infty$ for some s, t satisfying (see [7])*

$$\alpha + t > 0. \tag{12}$$

The quantity

$$\|f\|_{b_\alpha^\infty} = \|I_s^t f\|_{L_\alpha^\infty}^p = \sup_{x \in \mathbb{B}} (1-|x|^2)^{\alpha+t} |D_s^t f(x)| < \infty.$$

defines a norm on b_α^∞ for any such s, t.

Remark 2.5. *By now, it is well-known that Definitions 2.3 and 2.4 are independent of s, t under (11) and (12), respectively. Moreover, the norm (quasinorm when $0 < p < 1$) on a given space depends on s and t but this is not mentioned as it is known that every choice of the pair (s, t) leads to an equivalent norm. Thus for a given pair s, t, I_s^t isometrically imbeds b_α^p into L_α^p if and only if (11) holds, and I_s^t isometrically imbeds b_α^∞ into L_α^∞ if and only if (12) holds.*

We turn to properties and estimates of reproducing kernels. For every $\alpha \in \mathbb{R}$, the series in (2) absolutely and uniformly converges on $K \times \mathbb{B}$, for any compact subset K of $\mathbb{B}$. Furthermore $R_\alpha(x, y)$ is real-valued, symmetric in the variables x and y and harmonic with respect to each variable.

The $\alpha \ge -1$ part of the the following pointwise estimates for $R_\alpha(x, y)$ and its partial derivatives are proved in many places including [3, 13, 24]. For a proof when $\alpha \in \mathbb{R}$ we refer to [12, Corollary 7.1].

Lemma 2.6. *Let $\alpha \in \mathbb{R}$ and m be a multi-index. Then for every $x \in \mathbb{B}$, $y \in \overline{\mathbb{B}}$,*

$$\left|(\partial^m R_\alpha)(x, y)\right| \lesssim \begin{cases} 1, & \text{if } \alpha + |m| < -n; \\ 1 + \log \dfrac{1}{[x, y]}, & \text{if } \alpha + |m| = -n; \\ \dfrac{1}{[x, y]^{n+\alpha+|m|}}, & \text{if } \alpha + |m| > -n. \end{cases}$$

It follows from the above lemma that if $K \subset \mathbb{B}$ is compact and m is a multi-index, then

$$|\partial^m R_\alpha(x,y)| \lesssim 1 \quad (x \in K,\ y \in \overline{\mathbb{B}}), \tag{13}$$

where differentiation is performed in the first variable.

The next lemma shows that the above estimate holds in two directions on the diagonal $x = y$. For a proof see [20, Proposition 4 (i)] when $\alpha > -1$ and [7, Lemma 2.9] when $\alpha \in \mathbb{R}$.

Lemma 2.7. *Let $\alpha \in \mathbb{R}$. For all $x \in \mathbb{B}$,*

$$R_\alpha(x,x) \sim \begin{cases} \dfrac{1}{(1-|x|^2)^{\alpha+n}}, & \text{if } \alpha > -n; \\ 1 + \log \dfrac{1}{1-|x|^2}, & \text{if } \alpha = -n; \\ 1, & \text{if } \alpha < -n. \end{cases}$$

The lemma below is taken from [7, Lemma 3.2] and it shows that if x stays close to 0, then $R_\alpha(x,y)$ is uniformly away from 0 for every $y \in \mathbb{B}$. Recall also that $R_\alpha(0,y) = 1$ for every $\alpha \in \mathbb{R}$ and $y \in \mathbb{B}$.

Lemma 2.8. *Let $\alpha \in \mathbb{R}$. There exists $\epsilon > 0$ such that for all $|x| < \epsilon$ and for all $y \in \mathbb{B}$, we have $R_\alpha(x,y) \geq 1/2$.*

For $1 \leq p < \infty$, we have bounded projections from the L^p_α onto the b^p_α.

Definition 2.9. *For $s \in \mathbb{R}$, the harmonic Bergman-Besov projection is*

$$Q_s f(x) = \frac{1}{V_s} T_{ss} = \int_{\mathbb{B}} R_s(x,y) f(y)\, d\nu_s(y),$$

for suitable f.

The following two theorems describes the boundedness of Bergman-Besov projections on b^p_α and b^∞_α spaces, and are Theorem 1.5 of [12] and Theorem 1.6 of [7], respectively.

Theorem 2.10. *Let $1 \leq p < \infty$ and $\alpha, s \in \mathbb{R}$. Then $Q_s : L^p_\alpha \to b^p_\alpha$ is bounded (and onto) if and only if*

$$\alpha + 1 < p(s+1). \tag{14}$$

Given an s satisfying (14) if t satisfies

$$\alpha + pt > -1, \tag{15}$$

then for $f \in b^p_\alpha$, we have

$$Q_s I^t_s f = \frac{V_{s+t}}{V_s} f. \tag{16}$$

Theorem 2.11. *Let $\alpha, s \in \mathbb{R}$. Then $Q_s : L^\infty_\alpha \to b^\infty_\alpha$ is bounded (and onto) if and only if*

$$s > \alpha - 1. \tag{17}$$

Given an s satisfying (17), if t satisfies

$$\alpha + t > 0, \tag{18}$$

then for $f \in b^\infty_\alpha$, we have

$$Q_s I^t_s f = \frac{V_{s+t}}{V_s} f. \tag{19}$$

3. Properties of the Operators

We now formulate the behavior of the operators T_{bc} in many different circumstances. These are adapted from similar results in [16]. First, we insert some obvious inequalities which will be useful in the proofs. If $a_1 < a_2$, $u > 0$, and $v \in \mathbb{R}$, then for $0 \le t < 1$,

$$(1-t^2)^{a_1} \le (1-t^2)^{a_2} \qquad \text{and} \qquad (1-t^2)^u\Big(1+\log(1-t^2)^{-1}\Big)^{-v} \lesssim 1. \tag{20}$$

The second inequality above leads to an estimate that we need many times.

Lemma 3.1. *For $u, v \in \mathbb{R}$,*

$$\int_0^1 (1-t^2)^u\Big(1+\log\frac{1}{1-t^2}\Big)^{-v} dt < \infty \tag{21}$$

if $u > -1$ or $u = -1$ and $v > 1$, and the integral diverges otherwise.

Proof. The integral have only one singularity at $t = 1$. Polynomial growth dominates a logarithmic one for $u \ne -1$. For $u = -1$, we reduce the integral into one studied in calculus after changes of variables and at this time we need $v > 1$ for the convergence of the integral. □

We will use the functions

$$f_{uv}(x) = (1-|x|^2)^u\Big(1+\log\frac{1}{1-|x|^2}\Big)^{-v} \quad (u, v \in \mathbb{R})$$

as test functions to obtain some of the necessary conditions of our theorems from the action of T_{bc} on them. If we apply Lemma 3.1 to the f_{uv}, we get the following result.

Lemma 3.2. *For $1 \le p < \infty$, we have $f_{uv} \in L^p_\alpha$ if and only if $\alpha + pu > -1$, or $\alpha + pu = -1$ and $pv > 1$. For $p = \infty$, we have $f_{uv} \in \mathcal{L}^\infty_\alpha$ if and only if $\alpha + u > 0$, or $u = -\alpha$ and $v \ge 0$.*

Lemma 3.3. *If $b + u > -1$ or if $b + u = -1$ and $v > 1$, then $T_{bc} f_{uv}$ is a finite positive constant. Otherwise, $T_{bc} f_{uv}(x) = \infty$ for $|x| \le \epsilon$, where ϵ is as in Lemma 2.8.*

Proof. If $b + u > -1$ or if $b + u = -1$ and $v > 1$, then integrating in polar coordinates to obtain

$$\begin{aligned} T_{bc} f_{uv}(x) &= \int_{\mathbb{B}} R_c(x,y)\,(1-|y|^2)^{b+u}\Big(1+\log\frac{1}{1-|x|^2}\Big)^{-v} d\nu(y) \\ &= \int_0^1 n t^{n-1}(1-t^2)^{b+u}\Big(1+\log\frac{1}{1-t^2}\Big)^{-v} \int_{\mathbb{S}} R_c(x, t\zeta)\,d\sigma(\zeta)\,dt. \end{aligned}$$

By the mean-value property the integral over $\mathbb{S}$ is $R_c(x, 0)$ which is 1 by (5). Thus,

$$\begin{aligned} T_{bc} f_{uv}(x) &= \int_0^1 n t^{n-1}(1-t^2)^{b+u}\Big(1+\log\frac{1}{1-t^2}\Big)^{-v} R_c(x,0)\,dt. \\ &= \int_0^1 n t^{n-1}(1-t^2)^{b+u}\Big(1+\log\frac{1}{1-t^2}\Big)^{-v} dt. \end{aligned}$$

The last integral is finite by Lemma 3.1, and then clearly $T_{bc} f_{uv}$ is a constant.

For the other values of the parameters,

$$T_{bc} f_{uv}(x) \ge \frac{1}{2}(1-|y|^2)^{b+u}\Big(1+\log\frac{1}{1-|x|^2}\Big)^{-v} d\nu(y) = \infty$$

by Lemma 2.8 for $|x| < \epsilon$ and Lemma 3.1. □

One can easily compute the adjoint of T_{bc}.

Proposition 3.4. *The formal adjoint* $T^*_{bc} : L^{q'}_\beta \to L^{p'}_\alpha$ *of the operator* $T_{bc} : L^p_\alpha \to L^q_\beta$ *for* $1 \le p, q < \infty$ *is* $T^*_{cb} = (1-|x|^2)^{b-\alpha} T_{\beta c}$.

Proof. Let $f \in L^p_\alpha$ and $g \in L^q_\beta$. Then by the definition, the real-valuedness and symmetry in its two variables of $R_c(x,y)$ along with Fubini theorem, we obtain

$$\begin{aligned}
[T_{bc} f, g]_{L^2_\beta} &= \int_{\mathbb{B}} \int_{\mathbb{B}} R_c(x,y) f(x)(1-|x|^2)^b d\nu(x) \overline{g(y)} (1-|y|^2)^\beta d\nu(y) \\
&= \int_{\mathbb{B}} f(x)(1-|x|^2)^{b-\alpha} \overline{\int_{\mathbb{B}} R_c(x,y) g(y)(1-|y|^2)^\beta d\nu(y)} \\
&\times (1-|x|^2)^\alpha d\nu(x) \\
&= \int_{\mathbb{B}} f \, \overline{T^*_{bc}} d\nu_\alpha = [f, T^*_{bc} g]_{L^2_\alpha}.
\end{aligned}$$

Hence,

$$T^*_{bc} g(x) = (1-|x|^2)^{b-\alpha} \int_{\mathbb{B}} R_c(x,y) g(y)(1-|y|^2)^\beta d\nu(y)$$

□

We will use the following simple but very important result. It must have been known by the experts, even though we could not find a reference in the literature.

Lemma 3.5. *Let* $0 < q < \infty$, $\beta \le -1$ *and* $f \in h(\mathbb{B})$. *If* $f \not\equiv 0$, *then*

$$\int_{\mathbb{B}} |f(x)|^q (1-|x|^2)^\beta \, d\nu(x) = \infty.$$

Corollary 3.6. *If* $T_{bc} : L^p_\alpha \to L^q_\beta$ *is bounded and* $f \in L^p_\alpha$, *then* $g = T_{bc} f$ *is harmonic on* $\mathbb{B}$. *If also* $q < \infty$, *then* $\beta > -1$. *Therefore* $T_{bc} : L^p_\alpha \to b^q_\beta$ *when it is bounded with* $\beta > -1$ *and* $q < \infty$. *Moreover, if* $\beta \le -1$ *and* $q < \infty$, *then* $T_{bc} : L^p_\alpha \to L^q_\beta$ *is not bounded. On the other hand, if* $T_{bc} : L^p_\alpha \to L^\infty$ *is bounded and* $f \in L^p_\alpha$, *then* $g = T_{bc} f \in h^\infty$. *Finally, If* $T_{bc} : L^p_\alpha \to \mathcal{L}^\infty_\beta$ *is bounded,* $f \in L^p_\alpha$, *and* $\beta > 0$ *then* $g = T_{bc} f \in b^\infty_\beta$. *Moreover, if* $\beta < 0$, *then* $T_{bc} : L^p_\alpha \to \mathcal{L}^\infty_\beta$ *is not bounded.*

Proof. That g is harmonic follows, for example, by differentiation under the integral sign, from the fact that $R_\alpha(x,y)$ is harmonic in x. That $\beta > -1$ when $q < \infty$ follows from Lemma 3.5. For $\beta < 0$, $h(\mathbb{B}) \cap \mathcal{L}^\infty_\beta$ contains only $g \equiv 0$ by the maximum principle for harmonic functions. □

4. Main Tools

Let (X, μ) and (Y, υ) be σ-finite measure spaces. Let $K(x,y)$ be a non-negative measurable function on $X \times Y$. Let us denote by G the integral operator with kernel K:

$$Gf(y) = \int_X K(x,y) f(x) d\mu(x).$$

Schur test is a sufficiency condition for the boundedness of G from $L^p(X,\mu)$ to $L^q(Y,\upsilon)$.

First we take up the Schur test for the case $1 < p \le q < \infty$. For a proof, see [21, Theorem 2.1] or [25, Theorem 1].

Theorem 4.1. *Suppose $1 < p \le q < \infty$. Let γ and δ be two real numbers with $\gamma + \delta = 1$. If there exists two strictly positive functions ϕ (on X) and ψ (on Y) with positive constants C_1 and C_2 such that*

$$\int_X (K(x,y))^{\gamma p'} (\phi(x))^{p'} d\mu(x) \le C_1 (\psi(y))^{p'},$$

for almost every $y \in Y$ and

$$\int_Y (K(x,y))^{\delta q} (\psi(y))^q d\upsilon(y) \le C_2 (\phi(x))^q,$$

for almost every $x \in X$, then G is bounded from $L^p(X,\mu)$ into $L^q(Y,\upsilon)$ and the norm of G does not exceed $C_1^{1/p'} C_2^{1/q}$.

We also have the following Schur test for the case $1 < q < p < \infty$. For a proof, see [10, Theorem 1] which also attributes it to [1].

Theorem 4.2. *Suppose $1 < q < p < \infty$. If there exists two strictly positive functions ϕ (on X) and ψ (on Y) with positive constant C such that*

$$\int_X K(x,y)\phi(x)^{p'} d\mu(x) \le C(\psi(y))^{q'},$$
$$\int_Y K(x,y)\psi(y)^q d\upsilon(y) \le C(\phi(x))^p,$$

for almost every $y \in Y$ and $x \in X$, respectively and

$$\int\int_{X\times Y} K(x,y)\phi(x)^{p'} \psi(y)^q d\mu \times d\upsilon(x,y) \le C,$$

then G is bounded from $L^p(X,\mu)$ into $L^q(Y,\upsilon)$ and the norm of G does not exceed C.

We also need the following less known Minkowski integral inequality that in effect exchanges the order of integration; for a proof, see [22, Theorem 3.3.5] for example.

Lemma 4.3. *If $1 \le p \le \infty$ and $f(x,y)$ is a measurable function on $X \times Y$, then*

$$\left(\int_Y \left(\int_X |f(x,y)| d\mu(x) \right)^p d\upsilon(y) \right)^{1/p} \le \int_X \left(\int_Y |f(x,y)|^p d\upsilon(y) \right)^{1/p} d\mu(x),$$

with an appropriate interpretation with the L^∞ norm when $p = \infty$.

The next lemma provides an estimate on weighted integrals of powers of $R_\alpha(x,y)$. When $\alpha > -1$ and $w > 0$, it is proved in [20, Proposition 8]. For the whole range $\alpha \in \mathbb{R}$ see [12, Theorem 1.5].

Lemma 4.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $d > -1$. Set $w = p(n+\alpha) - (n+d)$. Then*

$$\int_{\mathbb{B}} |R_\alpha(x,y)|^p (1-|y|^2)^d d\nu(y) \sim \begin{cases} 1, & \text{if } w < 0; \\ 1 + \log \dfrac{1}{1-|x|^2}, & \text{if } w = 0; \\ \dfrac{1}{(1-|x|^2)^w}, & \text{if } w > 0. \end{cases}$$

Notice that the kernel $R_\alpha(x,y)$ is dominated above by $1/[x,y]^{n+\alpha}$ by taking $|m| = 0$ when $\alpha > -n$ in Lemma 2.6. The following integral estimate of these dominating terms will be crucial to the proof our main results. For a proof see [17, Proposition 2.2] or [24, Lemma 4.4].

Lemma 4.5. *Let $d > -1$ and $s \in \mathbb{R}$. Then*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^d}{[x,y]^{n+d+s}}\, d\nu(y) \sim \begin{cases} 1, & \text{if } s < 0; \\ 1 + \log \dfrac{1}{1-|x|^2}, & \text{if } s = 0; \\ \dfrac{1}{(1-|x|^2)^s}, & \text{if } s > 0. \end{cases}$$

We can push D_s^t into some certain integrals. The following lemma is taken from [7, Lemma 2.3].

Lemma 4.6. *Let $b \in \mathbb{R}$ and $f \in L_b^1$. For every $s, t \in \mathbb{R}$ and $x \in \mathbb{B}$,*

$$D_s^t \int_{\mathbb{B}} R_b(x,y) f(y) d\nu_b(y) = \int_{\mathbb{B}} D_s^t R_b(x,y) f(y) d\nu_b(y).$$

In some cases, D_s^t can be written as an integral operator. More precisely we have the following result of [7, Corollary 2.5].

Corollary 4.7. *Let $s > -1$ and $f \in L_s^1 \cap h(\mathbb{B})$. For every $t \in \mathbb{R}$,*

$$D_s^t f(x) = \int_{\mathbb{B}} R_{s+t}(x,y) f(y) d\nu_s(y). \tag{22}$$

The following lemma states that when $f \in b_b^1 (b > -1)$, the operator T_{bc} acts like D_s^t.

Lemma 4.8. *Let $b > -1$, $c \in \mathbb{R}$ and $f \in b_b^1$. Then*

$$\frac{1}{V_b} T_{bc} f(x) = \int_{\mathbb{B}} R_c(x,y) f(y) d\nu_b(y) = D_b^{c-b} f(x).$$

Proof. It is obvious from the definition of $d\nu_b$ and the previous corollary. □

The following result is significant in our necessity proofs.

Lemma 4.9. *If $b + t > -1$, then $T_{bc} I_b^t h = C D_b^{c-b} h$ for $h \in b_b^1$. As consequences, $D_c^{b-c} T_{bc} I_b^t h = Ch$ for $h \in b_b^1$ and $T_{bc} I_b^t D_c^{b-c} h = Ch$ for $h \in b_c^1$.*

Proof. If $b + t > -1$ and $h \in b_b^1$, then $D_b^t h \in b_{b+t}^1 \subset L_{b+t}^1$ by Lemma 2.2 (i). Since

$$T_{bc} I_b^t h(x) = \int_{\mathbb{B}} R_c(x,y) D_b^t h(y) (1-|y|^2)^{b+t} d\nu(y),$$

we have

$$T_{bc} I_b^t h = T_{b+t,c} D_b^t h = C D_{b+t}^{c-b-t} D_b^t h = C D_b^{c-b} h.$$

by Lemma 4.8 and (9). The identities on triple compositions are just consequences of the identities in (8). □

We require the inclusion relations between harmonic Bergman-Besov and weighted Bloch spaces in necessity proofs. We refer to [8] for results on inclusions where also references to earlier work can be found. First, we single out the following simple inclusions:

$$b_\alpha^p \subset b_\beta^p \qquad \text{and} \qquad b_\alpha^\infty \subset b_\beta^\infty \qquad (\alpha \le \beta). \tag{23}$$

We have the following inclusion relations between harmonic Bergman-Besov spaces. For proofs see [8, Theorems 1.1 and 1.2].

Theorem 4.10. *Let $0 < q < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$b^p_\alpha \subset b^q_\beta \quad \textit{if and only if} \quad \frac{\alpha+1}{p} < \frac{\beta+1}{q}.$$

Theorem 4.11. *Let $0 < p \leq q < \infty$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$b^p_\alpha \subset b^q_\beta \quad \textit{if and only if} \quad \frac{\alpha+n}{p} \leq \frac{\beta+n}{q}.$$

We also have the following inclusion relation between a Bergman-Besov space b^p_α and a weighted Bloch space b^∞_β. For a proof see [8, Theorem 1.3].

Theorem 4.12. *Let $0 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. Then*

(i) $b^\infty_\beta \subset b^p_\alpha$ *if and only if* $\beta < \dfrac{\alpha+1}{p}$.

(ii) $b^p_\alpha \subset b^\infty_\beta$ *if and only if* $\beta \geq \dfrac{\alpha+n}{p}$.

Note that all the inclusions above are continuous, strict, and the best possible.

We now mention two more theorems about inclusion relations that we will invoke later. The following theorem gives the inclusion relation between h^∞ and b^∞_α.

Theorem 4.13. *Let $\alpha \in \mathbb{R}$.*

(i) If $\alpha < 0$, then $b^\infty_\alpha \subset h^\infty$.

(ii) If $\alpha \geq 0$, then $h^\infty \subset b^\infty_\alpha$.

Proof. (i): Let $\alpha < 0$. Pick $s, t \in \mathbb{R}$ such that $\alpha + t > 0$ and $s > \alpha - 1$ holds. Assume that $f \in b^\infty_\alpha$. By (19) we have the following integral representation

$$f(x) = \frac{V_s}{V_{s+t}} \int_{\mathbb{B}} R_s(x, y) I^t_s f(y)(1 - |y|^2)^s \, d\nu(y),$$

and therefore

$$|f(x)| \lesssim \int_{\mathbb{B}} |R_s(x, y)||I^t_s f(y)|(1 - |y|^2)^s \, d\nu(y).$$

Using that $\|f\|_{b^\infty_\alpha} = \|I^t_s f\|_{L^\infty_\alpha} = \sup_{x\in\mathbb{B}}(1 - |x|^2)^\alpha |I^t_s f(x)|$, we get $(1 - |x|^2)^\alpha |I^t_s f(x)| \leq \|f\|_{b^\infty_\alpha}$ for all $x \in \mathbb{B}$. Thus

$$|f(x)| \lesssim \|f\|_{b^\infty_\alpha} \int_{\mathbb{B}} |R_s(x, y)|(1 - |y|^2)^{s-\alpha} \, d\nu(y).$$

Since $s - \alpha > -1$ and $n + s - (n + s - \alpha) = \alpha < 0$, by Lemma 4.4 we have $|f(x)| \lesssim \|f\|_{b^\infty_\alpha}$ for all $x \in \mathbb{B}$. We conclude that $f \in h^\infty$.

(ii) Let $f \in h^\infty$. First, we take $\alpha > 0$. So that it is enough to show that $f \in \mathcal{L}^\infty_\alpha$. Since $f \in h^\infty$, there exist an $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{B}$. Together with $(1 - |x|^2)^\alpha \leq 1$, this yields $(1 - |x|^2)^\alpha |f(x)| \leq M$ for all $x \in \mathbb{B}$. Hence we have $f \in \mathcal{L}^\infty_\alpha$ and this implies $f \in b^\infty_\alpha$.

Let now $\alpha = 0$. This time we must show that $\sup_{x\in\mathbb{B}}(1 - |x|^2)|\nabla f(x)| < \infty$. Since $f \in h^\infty$, again there exist an $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{B}$. By Cauchy's estimate (see [2, 2.4]), there exists a positive constant C such that

$$|\nabla u(x)| \leq \frac{CM}{r},$$

for every $x \in \mathbb{B}$. Since $1 + |x| \leq 2$ when $x \in \mathbb{B}$, we obtain

$$(1 - |x|^2)|\nabla u(x)| \leq \frac{2CM(1 - |x|)}{r} \leq 4CM.$$

Hence $f \in b^\infty_0$. $\square$

We also have the following inclusion theorem between h^∞ and b^p_α. For a proof see Section 13.1 of [12] and discussion in there when $1 \le p < \infty$ and [6, Theorem 5.1] when $0 < p < 1$.

Theorem 4.14. *Let $0 < p < \infty$ and $\alpha \in \mathbb{R}$. Then*

$$b^p_\alpha \subset h^\infty \quad \text{if and only if} \quad \alpha < -n, \text{ or } \alpha = -n \quad \text{and} \quad 0 < p \le 1.$$

5. Proofs of Necessity Parts of Theorems 1.1–1.7

In this section, we obtain necessary conditions for the boundedness of the operator T_{bc}, that is, (i) implies (iii) in all of our seven theorems. Before the necessity proofs, we first show the following lemma and corollary after that.

Lemma 5.1. *Let $a, a_1, a_2 \in \mathbb{R}$. Define $\mathcal{R}_a(x, y)$ such that*

$$\mathcal{R}_a(x,y) := \begin{cases} 1, & \text{if } a < -n; \\ 1 + \log \dfrac{1}{[x,y]}, & \text{if } a = -n; \\ \dfrac{1}{[x,y]^{n+a}}, & \text{if } a > -n. \end{cases}$$

If $a_1 < a_2$, then we have $\mathcal{R}_{a_1}(x, y) \lesssim \mathcal{R}_{a_2}(x, y)$ for all $x, y \in \mathbb{B}$.

Proof. There are five cases $a_1 < a_2 < -n$, $-n < a_1 < a_2$, $a_1 < -n < a_2$, $a_1 < a_2 = -n$ and $-n = a_1 < a_2$ all of them can be elementary verified. If $a_1 < a_2 < -n$, it is clear that this estimate holds. If $-n < a_1 < a_2$, we write

$$\frac{1}{[x,y]^{n+a_1}} = \frac{[x,y]^{a_2-a_1}}{[x,y]^{n+a_2}} \lesssim \frac{1}{[x,y]^{n+a_2}}.$$

Now let $a_1 < n < a_2$. Since $[x, y] \le 2$ and $0 < n + a_2$, its obvious that $\frac{1}{[x,y]} \ge \frac{1}{2}$ and thus we have

$$1 \lesssim \frac{1}{[x,y]^{n+a_2}}.$$

If $a_1 < a_2 = -n$, then $1 + \log \frac{1}{[x,y]} \ge 1 + \log \frac{1}{2} \gtrsim 1$. Finally, let $-n = a_1 < a_2$. Note that, $1 + \log \frac{1}{[x,y]}$ and $\frac{1}{[x,y]^{n+a_2}}$ are bounded both above and below when $[x, y]$ away from zero. On the other side, $1 + \log \frac{1}{[x,y]}$ is dominated by $\frac{1}{[x,y]^{n+a_2}}$ when $[x, y]$ near zero, because

$$\lim_{t\to 0} \frac{\log \frac{1}{t}}{(\frac{1}{t})^\delta} = 0,$$

for $\delta = n + a_2 > 0$ and $t = [x, y]$. Hence, if $a_1 < a_2$ then $\mathcal{R}_{a_1}(x, y) \lesssim \mathcal{R}_{a_2}(x, y)$ for all $x, y \in \mathbb{B}$. □

Corollary 5.2. *If $S_{bd} : L^p_\alpha \to L^q_\beta$ is bounded and $c < d$, then $S_{bc} : L^p_\alpha \to L^q_\beta$ is also bounded*

Proof. This is just because $|R_c(x, y)| \lesssim |R_d(x, y)|$ by the previous lemma. □

Firstly, we derive the first inequality in (iii) of each theorem. In this section, we do not need to assume $\beta > -1$ when $q < \infty$ or $\beta \ge 0$ when $q = \infty$ since the boundedness of T_{bc} implies one of them by Corollary 3.6. More precisely, the following theorem gives the first necessary conditon for all of Theorems 1.1–1.7.

Theorem 5.3. *Let $b, c, \alpha, \beta \in \mathbb{R}$ and $1 \le p, q \le \infty$. Suppose that T_{bc} is bounded from L^p_α to L^q_β, then $\alpha + 1 \le p(b + 1)$ for $1 = p \le q \le \infty$, also $\alpha - 1 < b$ for $1 \le q < p = \infty$ and the strict inequality $\alpha + 1 < p(b + 1)$ holds for the remainig cases.*

Proof. The proof can be seperated in three cases depending on the value of p. We first show the case $1 < p < \infty$. Consider f_{uv} with $u = -(1+\alpha)/p$ and $v = 1$ so that $f_{uv} \in L^p_\alpha$ by Lemma 3.2. Then its clear that $T_{bc} f_{uv} \in L^q_\beta$ and Lemma 3.3 implies $b + u > -1$. This yields $(1+\alpha)/p < 1 + b$ with the value of u chosen.

We now show the second case $p = 1$. Consider f_{uv} with $u > -(1+\alpha)$ and $v = 0$ so that $f_{u0} \in L^1_\alpha$ by Lemma 3.2. Then $T_{bc} f_{u0} \in L^q_\beta$ and Lemma 3.3 implies $b + u > -1$. Writing $u = -1 - \alpha + \varepsilon$ with $\varepsilon > 0$, we obtain $\alpha < b + \varepsilon$. This is just $\alpha \leq b$.

The last case is $p = \infty$. Let now f_{uv} with $u = -\alpha$ so that $f_{u0} \in \mathcal{L}^\infty_\alpha$ by Lemma 3.2. Then $T_{bc} f_{u0} \in L^q_\beta$ and Lemma 3.3 implies $b - \alpha > -1$. □

We next derive the second inequality in (iii) of each theorem. As indicated before, we do this by an original method depends on the inclusion relations between Bergman-Besov and weighted Bloch spaces appears in [16]. A key step of this method is Lemma 4.9.

Theorem 5.4. *Let $b, c, \alpha, \beta \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Suppose that T_{bc} is bounded from L^p_α to L^q_β. Then $c \leq b + \frac{n+\beta}{q} - \frac{n+\alpha}{p}$ for $1 \leq p \leq q < \infty$, $c < b + \frac{1+\beta}{q} - \frac{1+\alpha}{p}$ for $1 \leq q < p < \infty$, $c \leq b + \beta - \frac{n+\alpha}{p}$ for $1 \leq p < q = \infty$ and the strict inequality holds for $p \neq 1$ when $\beta = 0$, also $c < b + \frac{1+\beta}{q} - \alpha$ for $1 \leq q < p = \infty$, $c \leq b + \beta - \alpha$ for $p = q = \infty$ and the strict inequality holds when $\beta = 0$.*

Proof. First note that, the boundedness of T_{bc} implies $\beta > -1$ when $q < \infty$ and $\beta \geq 0$ when $q = \infty$ by Corollary 3.6.

Now we handle the all cases in four groups. The first group again consist of the cases $1 \leq p \leq q < \infty$ and $1 \leq q < p < \infty$. Let $h \in b^p_\alpha$. In order to able to use Lemma 4.9, we need to show that $h \in b^1_b$. In the cases $1 < p \leq q < \infty$ and $1 \leq q < p < \infty$, we have $(1+\alpha)/p < 1 + b$ by the first necessary condition, and then Theorem 4.10 gives $h \in b^1_b$. In the case $1 = p \leq q < \infty$, we have $\alpha \leq b$ again by the first necessary condition, and then (23) shows $h \in b^1_b$. Pick t such that $\alpha + pt > -1$. Together with the first necessary condition, it is easy to check that $b + t > -1$. We will consider the composition of the bounded maps

$$b^p_\alpha \xrightarrow{\mathrm{I}^t_\mathrm{b}} L^p_\alpha \xrightarrow{\mathrm{T_{bc}}} b^q_\beta \xrightarrow{\mathrm{D}^{\mathrm{b-c}}_\mathrm{c}} b^q_{\beta+q(b-c)}.$$

Note that since $T_{bc} h$ is harmonic and $\beta > -1$ range of T_{bc} is $b^q_\beta \subset L^q_\beta$. Lemma 4.9 yields that b^p_α is imbedded in $b^q_{\beta+q(b-c)}$ by the inclusion map. But by Theorem 4.11 this is possible only if $(\alpha+n)/p \leq (\beta + q(b-c) + n)/q$ which is equivalent to $c \leq b + \frac{n+\beta}{q} - \frac{n+\alpha}{p}$ in the case $1 \leq p \leq q < \infty$. Similarly, by Theorem 4.10 this is possible only if $(\alpha+1)/p \leq (\beta + q(b-c) + 1)/q$ that is $c < b + \frac{1+\beta}{q} - \frac{1+\alpha}{p}$ in the case $1 \leq q < p < \infty$.

The second group consist of the case $1 \leq p < q = \infty$. Let $H \in b^p_{\alpha+p(c-b)}$. By Lemma 2.2 (i), $D^{b-c}_c H = h \in b^p_\alpha$. Exactly as in the proof of the first group of cases, $h \in b^1_b$. Pick t such that $\alpha + pt > -1$ and this gives $b + t > -1$ with the first necessary condition. We will consider the composition of the bounded maps

$$b^p_{\alpha+p(c-b)} \xrightarrow{\mathrm{D}^{\mathrm{b-c}}_\mathrm{c}} b^p_\alpha \xrightarrow{\mathrm{I}^t_\mathrm{b}} L^p_\alpha \xrightarrow{\mathrm{T_{bc}}} b^\infty_\beta.$$

Note that since $T_{bc} h$ is harmonic, range of T_{bc} is $b^\infty_\beta \subset \mathcal{L}^\infty_\beta$ when $\beta > 0$ and $\mathcal{L}^\infty \cap h(\mathbb{B}) = h^\infty \subset L^\infty$ when $\beta = 0$. Lemma 4.9 yields that $b^p_{\alpha+p(c-b)}$ is imbedded in b^∞_β when $\beta > 0$ and in h^∞ when $\beta = 0$ by the inclusion map. By Theorem 4.12 (ii) this is possible only if $c \leq b + \beta - \frac{n+\alpha}{p}$ when $\beta > 0$. Similarly, by Theorem 4.14 this is possible only if $c < b - \frac{n+\alpha}{p}$ in the case $1 < p < q = \infty$ and it is possible only if $c \leq b - (n+\alpha)$ in the case $1 = p, q = \infty$ when $\beta = 0$.

The third group consist of the case $1 \leq q < p = \infty$. Let $h \in b^\infty_\alpha$. The first necessary condition gives $\alpha - 1 < b$ and then Theorem 4.12(i) yields $h \in b^1_b$. Pick t such that $\alpha + t > 0$ and this gives $b + t > -1$ with the first necessary condition. We will consider the composition of the bounded maps

$$b_\alpha^\infty \xrightarrow{\mathrm{I}_\mathrm{b}^\mathrm{t}} \mathcal{L}_\alpha^\infty \xrightarrow{\mathrm{T}_{\mathrm{bc}}} b_\beta^q \xrightarrow{\mathrm{D}_\mathrm{c}^{\mathrm{b-c}}} b_{\beta+q(b-c)}^q.$$

Note that since $T_{bc}h$ is harmonic and $\beta > -1$, range of T_{bc} is $b_\beta^q \subset L_\beta^q$. Lemma 4.9 yields that b_α^∞ is imbedded in $b_{\beta+q(b-c)}^q$ by the inclusion map. By Theorem 4.12 (i) this is possible only if $c < b + \frac{1+\beta}{q} - \alpha$.

The last group consist of the case $p = q = \infty$. Let $H \in b_{\alpha+(c-b)}^\infty$. By Lemma 2.2 (ii), $D_c^{b-c}H = h \in b_\alpha^\infty$. Exactly as in the proof of the third group of cases, $h \in b_b^1$. Pick t such that $\alpha + t > 0$ and this gives $b + t > -1$ with the first necessary condition. We will consider the composition of the bounded maps

$$b_{\alpha+(c-b)}^\infty \xrightarrow{\mathrm{D}_\mathrm{c}^{\mathrm{b-c}}} b_\alpha^\infty \xrightarrow{\mathrm{I}_\mathrm{b}^\mathrm{t}} \mathcal{L}_\alpha^\infty \xrightarrow{\mathrm{T}_{\mathrm{bc}}} b_\beta^\infty.$$

Note that since $T_{bc}h$ is harmonic, range of T_{bc} is $b_\beta^\infty \subset \mathcal{L}_\beta^\infty$ when $\beta > 0$ and $\mathcal{L}^\infty \cap h(\mathbb{B}) = h^\infty \subset L^\infty$ when $\beta = 0$. Lemma 4.9 yields that $b_{\alpha+(c-b)}^\infty$ is imbedded in b_β^∞ when $\beta > 0$ and in h^∞ when $\beta = 0$ by the inclusion map. By (23) this is possible only if $c \le b + \beta - \alpha$ when $\beta > 0$. Similarly, by Theorem 4.13 this is possible only if $c < b - \alpha$ in when $\beta = 0$. $\square$

Finaly, we must prove that in the case $1 = p \le q \le \infty$, if one of the inequalities in (iii) of Theorems 1.2 and 1.5 is an equality, then the other must be a strict inequality. Our method of proof will be an adaptation of the reasoning used in Theorem 6.3 of [16].

Theorem 5.5. *Let $b, c, \alpha, \beta \in \mathbb{R}$ and $1 \le p, q \le \infty$. Suppose that T_{bc} is bounded from L_α^p to L_β^q. Then equality cannot hold simultaneously in the inequalities of Theorems 1.2 and 1.5.*

Proof. First note that, the boundedness of T_{bc} implies $\beta > -1$ when $\infty < q$ and $\beta \ge 0$ when $q = \infty$ by Corollary 3.6.

If $\alpha = b$ and $c = b+\beta-\alpha$ simultaneously in the case $1 = p = q$, then also $c = \beta > -1$ and $T_{bc}^* = T_{\beta\beta} : L^\infty \to L^\infty$ is bounded. Let

$$f_x(y) = \begin{cases} \frac{|R_\beta(x,y)|}{R_\beta(x,y)}, & \text{if } R_\beta(x,y) \neq 0; \\ 1, & \text{if } R_\beta(x,y) = 0, \end{cases}$$

which is a uniformly bounded family for $x \in \mathbb{B}$. The same is true also of $\{T_{\beta\beta} f_x\}$. But

$$T_{\beta\beta} f_x(x) = \int_{\mathbb{B}} |R_\beta(x,y)|(1-|y|^2)^\beta d\nu(y) \sim 1 + \log \frac{1}{1-|x|^2}$$

by Lemma 4.4, this contradicts to the uniform boundedness.

If $\alpha = b$ and $c = b + (n+\beta)/q - (n+\alpha)$ simultaneously in the case $1 = p < q < \infty$, then also $c = (n+\beta)/q - n$ and $T_{bc}^* = T_{\beta c} : L_\beta^{q'} \to L^\infty$ is bounded with $q' > 1$. By Theorem 4.14, there is an unbounded $g \in b_{-n}^{q'}$. Then $h = D_{\beta-(n+\beta)/q'}^{(n+\beta)/q'} g \in b_\beta^{q'} \subset L_\beta^{q'} \subset L_\beta^1$. By Lemma 4.8 and (9), we obtain

$$T_{bc}^* h = T_{\beta c} h = V_\beta D_\beta^{c-\beta} h = V_\beta D_\beta^{c-\beta} D_{\beta-(n+\beta)/q'}^{(n+\beta)/q'} g = V_\beta g.$$

Nevertheless $g \notin L^\infty$, and this contradicts that $T_{bc}^* : L_\beta^{q'} \to L^\infty$.

If $\alpha = b$ and $c = b + \beta - (\alpha + n)$ simultaneously in the case $1 = p, q = \infty$, then also $c = \beta - n$. For $i = 1, 2, ...,$ let $x_i = (1 - 1/i, 0, ..., 0)$ and E_i the ball of radius $1/2i$ centered at x_i, and define

$$f_i(x) = \frac{V_\alpha \chi_{E_i}(x)}{\nu(E_i)(1-|x|^2)^\alpha}.$$

Clearly $f_i \in L^1_\alpha$ and $\|f_i\|_{L^1_\alpha} = 1$ for every i. Then $\{T_{bc} f_i\} = \{T_{\alpha,\beta-n} f_i\}$ is a uniformly bounded family. By the mean value property,

$$T_{\beta,\beta-n} f_i(y) = \frac{V_\alpha}{\nu(E_i)} \int_{E_i} R_{\beta-n}(x, y) d\nu(x) = V_\alpha R_{\beta-n}(y, x_i).$$

But

$$T_{\beta,\beta-n} f_i(x_i) = V_\alpha R_{\beta-n}(x_i, x_i) \sim \begin{cases} \dfrac{1}{(1-|x_i|^2)^\beta}, & \text{if } \beta > 0; \\ 1 + \log \dfrac{1}{1-|x_i|^2}, & \text{if } \beta = 0; \end{cases}$$

by Lemma 2.7, this contradicts to the uniform boundedness. □

6. Proofs of Sufficiency Parts of Theorems 1.1–1.7

In this section we will present the proofs that the inequalities in (iii) of Theorems 1.1–1.7 imply the boundedness of S_{bc}. By Corollary 5.2, it is enough to prove this only for large values of c. In all theorems except Theorem 1.5, there are values of $c > -n$ satisfying the inequalities in (iii), thus we make this the standing assumption in this section. For Theorem 1.5, we deal with the values of c separately.

We consider each theorem separately since each of the cases has a sufficiently different proof those from the others. Throughout this section, we assume that the two inequalities in (iii) hold.

The following sufficiency proof of Theorem 1.1 follows the same lines as the proofs of [25, Lemma 6].

Proof of sufficiency for Theorem 1.1. First, taking c to have its largest value

$$c = b + \frac{n+\beta}{q} - \frac{n+\alpha}{p} \tag{24}$$

causes no loss of generality by Corollary 5.2. We have $c > -n$ by the condition $\alpha + 1 < p(b+1)$ and $\beta > -1$. We employ the Schur test in Theorem 4.1 with the measures $\mu = \nu_\alpha, \upsilon = \nu_\beta$ and the kernel $K(x, y) = \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}}$ which together give us us the desired boundedness of the operator S_{bc}. Thus we need to find two positive constant γ and δ such that $\gamma + \delta = 1$ and two strictly positive functions $\phi(x) = (1-|x|^2)^A$ and $\psi(y) = (1-|y|^2)^B$ on $\mathbb{B}$ with $A, B \in \mathbb{R}$ to be determined. The two inequalities that need to be satisfied for the Schur test are

$$\int_{\mathbb{B}} \frac{(1-|x|^2)^{(b-\alpha)\gamma p'}}{[x, y]^{(n+c)\gamma p'}} (1-|x|^2)^{Ap'} (1-|x|^2)^\alpha d\nu(x) \lesssim (1-|y|^2)^{Bp'},$$
$$\int_{\mathbb{B}} \frac{(1-|x|^2)^{(b-\alpha)\delta q}}{[x, y]^{(n+c)\delta q}} (1-|y|^2)^{Bq} (1-|y|^2)^\beta d\nu(y) \lesssim (1-|x|^2)^{Aq}.$$

One way to satisfy them is by matching the growth rates of their two sides, that is, the powers of the $1-|\cdot|^2$. This is possible if $A, B < 0$ and

$$\begin{aligned} -Bp' &= (n+c)\gamma p' - n - (b-\alpha)\gamma p' - Ap' - \alpha, \\ -Aq &= (n+c)\delta q - n - Bq - \beta - (b-\alpha)\delta q, \end{aligned} \tag{25}$$

by Lemma 4.5. But we must also make sure that the conditions of Lemma 4.5 for this to happen are met, that is,

$$\begin{aligned} F_1 := (b-\alpha)\gamma p' + Ap' + \alpha &> -1, \\ Bq + \beta &> -1, \\ F_2 := (n+c)\gamma p' - n - (b-\alpha)\gamma p' - Ap' - \alpha &> 0, \\ (n+c)\delta q - n - Bq - \beta &> 0. \end{aligned} \tag{26}$$

Keep in mind that the two equations in (25) are linearly dependent. The variables A, B and δ are determined in the following way. We first choose an B to satisfy the second inequality in (26); so

$$-\frac{1+\beta}{q} < B < 0. \tag{27}$$

This is possible since $\beta > -1$. Next we pick a δ to satisfy the fourth inequality in (26) and naturally let $\gamma = 1 - \delta$; so we take

$$\begin{aligned} \delta &= \frac{1}{n+c}\Big(B + \frac{n+\beta}{q} + \varepsilon\Big), \\ \gamma &= \frac{1}{n+c}\Big(-B + n + b - \frac{n+\alpha}{p} - \varepsilon\Big) \end{aligned} \tag{28}$$

with $\varepsilon > 0$ by (24). Using the chosen values of B, γ and δ, we then solve for A from, say, the second equation in (25), and simplify it using the definition of δ; so

$$\begin{aligned} A &= (b-\alpha)\delta - (n+c)\delta + B + \frac{n+\beta}{q} \\ &= (b-\alpha)\delta - B - \frac{n+\beta}{q} - \varepsilon + B + \frac{n+\beta}{q} = (b-\alpha)\delta - \varepsilon. \end{aligned} \tag{29}$$

Finally, we must check that the remaining first and third inequalities in (26) hold for some $\varepsilon > 0$. Substituting in the value of A from (29), since $\gamma + \delta = 1$,

$$\begin{aligned} F_1 + 1 &= (b-\alpha)\gamma p' + (b-\alpha)\delta p' - \varepsilon p' + \alpha + 1 = (b-\alpha)p' + \alpha + 1 - \varepsilon p' \\ &= p'\Big(b - \alpha + \frac{\alpha+1}{p'}\Big) - \varepsilon p' = p'\Big(b + 1 - \frac{\alpha+1}{p}\Big) - \varepsilon p' \\ &= p'\Big(b + 1 + \frac{\alpha+1}{p} - \varepsilon\Big) > 0 \end{aligned}$$

by the condition $\alpha + 1 < p(b+1)$ provided $\varepsilon < b + 1 - \frac{\alpha+1}{p}$. Substituting in for A and γ from (28) and (29), again since $\gamma + \delta = 1$,

$$\begin{aligned} F_2 &= p'\left((n+c)\gamma - (b-\alpha)\gamma - (b-\alpha)\delta + \varepsilon - \frac{n+\alpha}{p'}\right) \\ &= p'\left(-B + n + b - \frac{n+\alpha}{p} - \varepsilon - b + \alpha + \varepsilon - \frac{n+\alpha}{p'}\right) \\ &= -p'B > 0 \end{aligned}$$

by (27). Hence for

$$0 < \varepsilon < b + 1 - \frac{\alpha+1}{p},$$

Theorem 4.1 using the chosen functions ϕ and ψ with the powers in (29) and (27) applies with the parameters in (28) and proves that S_{bc} is bounded from L^p_α to L^q_β with $1 < p \le q < \infty$ when the inequalities in (iii) hold. □

Proof of sufficiency for Theorem 1.2. First, let $1 = p = q$ and $f \in L^1_\alpha$. Writing the L^1_β norm of $S_{bc}f$ explicitly and applying Fubini theorem, we get that

$$\begin{aligned}
\|S_{bc}f\|_{L^1_\beta} &\lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} |R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x)(1-|y|^2)^\beta d\nu(y) \\
&\lesssim \int_{\mathbb{B}} |f(x)|(1-|x|^2)^b \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{n+c}} d\nu(y) d\nu(x) \\
&= V_\alpha \int_{\mathbb{B}} |f(x)|(1-|x|^2)^{b-\alpha} \int_{\mathbb{B}} \frac{(1-|x|^2)^\beta}{[x,y]^{n+c}} d\nu(y) d\nu_\alpha(x).
\end{aligned}$$

Let $J(x)$ be that part of the integrand of the outer integral multiplying $|f(x)|$. We will show that J is bounded on $\mathbb{B}$ by using Lemma 4.5 since $\beta > -1$ as required.

Consider $(n+c) - n - \beta = c - \beta$. Firstly, if $c - \beta < 0$, then the integral in $J(x)$ is bounded and $J(x)$ is also bounded since $b - \alpha \geq 0$ by the first inequality of (iii). Next, if $c - \beta = 0$, then the integral in $J(x)$ is $1 + \log(1-|x|^2)^{-1}$. But this time $b > \alpha$ since the two inequalities are the same in (iii) and $b = \alpha$ can not hold as stated in Theorem 5.5. Then $J(x)$ is bounded by (20). Lastly, if $c - \beta > 0$, then $J(x) \sim (1-|x|^2)^{b-\alpha-c+\beta}$. But by the second inequality in (iii) we have $b - \alpha - c + \beta \geq 0$ and thus $J(x)$ is bounded once again. Therefore $\|S_{bc}f\|_{L^1_\beta} \lesssim \|f\|_{L^1_\alpha}$ and S_{bc} is bounded from L^1_α to L^1_β.

Next, let $1 = p < q < \infty$ and $f \in b^1_\alpha$. Writing the L^q_β norm of $S_{bc}f$ explicitly and using Lemma 4.3 with the measures ν_α and ν_β, we obtain

$$\begin{aligned}
\|S_{bc}f\|_{L^q_\beta} &= \left(\int_{\mathbb{B}} \left| V_\alpha \int_{\mathbb{B}} R_c(x,y) f(x)(1-|x|^2)^{b-\alpha} d\nu_\alpha(x)\right|^q d\nu_\beta(y)\right)^{1/q} \\
&\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |R_c(x,y)|^q |f(x)|^q (1-|x|^2)^{(b-\alpha)q} d\nu_\beta(y)\right)^{1/q} d\nu_\alpha(x) \\
&\lesssim \int_{\mathbb{B}} |f(x)|(1-|x|^2)^{b-\alpha} \left(\int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{(n+c)q}} d\nu(y)\right)^{1/q} d\nu_\alpha(x).
\end{aligned}$$

Let $J(x)$ be that part of the integrand of the outer integral multiplying $|f(x)|$ for $x \in \mathbb{B}$. We will show that J is bounded on $\mathbb{B}$ by using Lemma 4.5 since $\beta > -1$ as required.

Now we consider $\rho = (n+c)q - n - \beta$. Firstly, if $\rho < 0$, then the integral in $J(x)$ is bounded and $J(x)$ is also bounded since $b - \alpha \geq 0$ by the first inequality of (iii). Next, if $\rho = 0$, then the integral in $J(x)$ is $1 + \log(1-|x|^2)^{-1}$. Then since the two inequalities are the same in (iii), we have $b > \alpha$ again by Theorem 5.5. Therefore $J(x)$ is bounded by (20). Lastly, if $\rho > 0$, then $J(x) \sim (1-|x|^2)^{b-\alpha-\rho/q}$. But by the second inequality in (iii), we have $b - \alpha - \rho/q = b - \alpha - (n+c) + (n+\beta)/q \geq 0$ and thus $J(x)$ is bounded once again. Hence $\|S_{bc}f\|_{L^q_\beta} \lesssim \|f\|_{L^1_\alpha}$ and S_{bc} is bounded from L^1_α to L^q_β with $1 = p < q < \infty$. $\square$

Proof of sufficiency for Theorem 1.3. First, let $1 < q < p < \infty$. The proof of this case starts out as in the proof of sufficiency for Theorem 1.1. Now we employ the Schur test in Theorem 4.2 but with the same test data as sufficiency proof in the case $1 < p < q < \infty$. So we have the $\mu = \nu_\alpha, \upsilon = \nu_\beta$ and the kernel $K(x,y) = \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}}$ which together give us $V_\alpha G = S_{bc}$. Thus we need to find two strictly positive functions $\phi(x) = (1-|x|^2)^A$ and $\psi(y) = (1-|y|^2)^B$ on $\mathbb{B}$ with $A, B \in \mathbb{R}$ to be determined. Two of the three inequalities that need to be satisfied for the Schur test are

$$\begin{aligned}
&\int_{\mathbb{B}} \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}} (1-|x|^2)^{Ap'} (1-|x|^2)^\alpha d\nu(x) \lesssim (1-|y|^2)^{Bq'}, \\
&\int_{\mathbb{B}} \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}} (1-|y|^2)^{Bq} (1-|y|^2)^\beta d\nu(y) \lesssim (1-|x|^2)^{Ap}.
\end{aligned}$$

One way to satisfy them is by matching the growth rates of their two sides, that is, the powers of the $1 - |\cdot|^2$.

This is possible if $A, B < 0$ and

$$\begin{aligned} &-Bq' = c - (b + Ap'), \\ &-Ap = c - (Bq + \beta) - (b - \alpha), \end{aligned} \tag{30}$$

by Lemma 4.5. But we must also make sure that the conditions of Lemma 4.5 for this to happen are met, that is,

$$\begin{aligned} b + Ap' > -1, &\quad Bq + \beta > -1, \\ c - (b + Ap') > 0, &\quad c - (Bq + \beta) > 0. \end{aligned} \tag{31}$$

Substituting for p', q' in terms of p, q, we can write (30) as a system of two linear equations in the two unknowns A, B as

$$\begin{aligned} p(q-1)A - q(p-1)B &= (c-b)(p-1)(q-1), \\ -pA + qB &= c - b + \alpha - \beta. \end{aligned} \tag{32}$$

this system has the following unique solution

$$\begin{aligned} A &= \frac{(p-1)(q(c-b) + \alpha - \beta)}{p(q-p)}, \\ B &= \frac{(q-1)(p(c-b) + \alpha - \beta)}{q(q-p)} \end{aligned} \tag{33}$$

for A, B. The second inequality in (iii) can be written in the form

$$c = b + \frac{1+\beta}{q} - \frac{1+\alpha}{p} - \varepsilon \tag{34}$$

with $\varepsilon > 0$. So by Corollary 5.2, it suffices to prove that S_{bc} is bounded when (34) holds for small enough $\varepsilon > 0$. Substituting this value of c into (33), the solution

$$\begin{aligned} A &= \frac{(p-1)}{p}\left(-\frac{1+\alpha}{p} + \frac{\varepsilon q}{p-q}\right), \\ B &= \frac{(q-1)}{q}\left(-\frac{1+\beta}{q} + \frac{\varepsilon p}{p-q}\right), \end{aligned} \tag{35}$$

Now, It remains to show that this solution satisfies all the necessary conditions for sufficiently small $\varepsilon > 0$. Recall that $\beta > -1$. First, by the inequality $\alpha + 1 < p(b+1)$,

$$c = b + \frac{1+\beta}{q} - \frac{1+\alpha}{p} - \varepsilon > \frac{1+\beta}{q} - 1 - \varepsilon > -(1+\varepsilon) > -n$$

provided $\varepsilon < n - 1$. Next we need to check that the inequalities in (31). By (35) and the inequality $\alpha + 1 < p(b+1)$,

$$b + Ap' = b - \frac{1+\alpha}{p} + \frac{\varepsilon q}{p-q} > -1 + \frac{\varepsilon q}{p-q} > -1. \tag{36}$$

By (35) again,

$$Bq + \beta = (q-1)\left(-\frac{1+\beta}{q} + \frac{\varepsilon p}{p-q}\right) + \beta = -1 + \frac{1+\beta}{q} + \frac{\varepsilon p(q-1)}{p-q} > -1. \tag{37}$$

By (34) and (36),

$$c-(b+Ap')=-\varepsilon+\frac{1+\beta}{q}-\frac{\varepsilon q}{p-q}=\frac{1+\beta}{q}-\frac{\varepsilon p}{p-q}>0 \tag{38}$$

provided $\varepsilon<(\frac{1}{q}-\frac{1}{p})(1+\beta)$. Lastly, by (34), (37), and the inequality $\alpha+1<p(b+1)$,

$$\begin{aligned} c-(Bq+\beta) &= b+\frac{1+\beta}{q}-\frac{1+\alpha}{p}-\varepsilon+1-\frac{1+\beta}{q}-\frac{\varepsilon p(q-1)}{p-q} \\ &= b+1-\frac{1+\alpha}{p}-\frac{\varepsilon q(p-1)}{p-q}>0 \end{aligned} \tag{39}$$

provided $\varepsilon<\frac{p}{p-1}(\frac{1}{q}-\frac{1}{p})(b+1-\frac{1+\alpha}{p})$. Finally, we verify the third condition of Theorem 4.2, that is the finiteness of the double integral

$$\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{(1-|x|^2)^{(b-\alpha)}}{[x,y]^{(n+c)}}(1-|x|^2)^{Ap'}(1-|y|^2)^{Bq}d\nu_\alpha(x)d\nu_\beta(y).$$

We call it $\mathcal{I}$. We first estimate the integral with respect to $d\nu(x)$ by Lemma 4.5 and obtain

$$\mathcal{I}\sim\int_{\mathbb{B}}(1-|y|^2)^{Bq+\beta-c+b+Ap'}d\nu(y)$$

by (38). Moreover, by (37) and (38), the power of the $(1-|y|^2)$ is

$$\begin{aligned} Bq+\beta-(c-(b+Ap')) &= -1+\frac{1+\beta}{q}+\frac{\varepsilon p(q-1)}{p-q}-\frac{1+\beta}{q}+\frac{\varepsilon p}{p-q} \\ &= -1+\frac{\varepsilon pq}{p-q}>-1, \end{aligned}$$

and this makes $\mathcal{I}$ finite. Hence for

$$0<\varepsilon<\min\left\{n-1,\left(\frac{1}{q}-\frac{1}{p}\right)(1+\beta),\frac{p}{p-1}\left(\frac{1}{q}-\frac{1}{p}\right)\left(b+1-\frac{1+\alpha}{p}\right)\right\},$$

Theorem 4.2 using the selected functions ϕ and ψ with the powers in (35) applies and proves that S_{bc} is bounded from L^p_α to L^q_β with $1<q<p<\infty$ when the inequalities in (iii) hold.

Next, let $1=q<p<\infty$. Assume that $f\in L^p_\alpha$. Writing the L^1_β norm of $S_{bc}f=Sf$ explicitly and applying Fubini's theorem, then applying the Hölder inequality, we obtain

$$\begin{aligned} \|Sf\|_{L^1_\beta} &\lesssim \int_{\mathbb{B}}\int_{\mathbb{B}}|R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x)(1-|y|^2)^\beta d\nu(y) \\ &\lesssim \int_{\mathbb{B}}|f(x)|(1-|x|^2)^b\int_{\mathbb{B}}\frac{(1-|y|^2)^\beta}{[x,y]^{n+c}}d\nu(y)d\nu(x) \\ &= \int_{\mathbb{B}}|f(x)|(1-|x|^2)^{\alpha/p}\int_{\mathbb{B}}\frac{(1-|y|^2)^\beta d\nu(y)}{[x,y]^{n+c}}(1-|x|^2)^{b-\alpha/p}d\nu(x) \\ &\lesssim \|f\|_{L^p_\alpha}\left(\int_{\mathbb{B}}\left(\int_{\mathbb{B}}\frac{(1-|y|^2)^\beta d\nu(y)}{[x,y]^{n+c}}\right)^{p'}(1-|x|^2)^{(b-\alpha/p)p'}d\nu(x)\right)^{1/p'} \\ &=: J^{1/p'}\|f\|_{L^p_\alpha}. \end{aligned}$$

We will show that J is finite using Lemma 4.5 since $\beta>-1$ as required.

Firstly, if $c-\beta<0$, $J\sim\int_{\mathbb{B}}(1-|x|^2)^{(b-\alpha/p)p'}d\nu(x)$. By the first inequality in (iii), we have

$$(b-\frac{\alpha}{p})p'=p'\big(b+1-\frac{1+\alpha}{p}-\frac{1}{p'}\big)>p'\big(-\frac{1}{p'}\big)=-1.$$

Thus J is finite by Lemma 3.1. Next, if $c-\beta=0$, then

$$J\sim\int_{\mathbb{B}}\left(1+\log\frac{1}{(1-|x|^2)}\right)^{p'}(1-|x|^2)^{(b-\alpha/p)p'}d\nu(x)<\infty$$

also by Lemma 3.1. Lastly, if $c-\beta>0$, then $J\sim\int_{\mathbb{B}}(1-|x|^2)^{(b-\alpha/p)p'-(c-\beta)p'}$ $d\nu(x)$. But by the second inequality in (iii), we have

$$(b-\frac{\alpha}{p})p'-(c-\beta)p'=p'\big(b-c+1+\beta-\frac{1+\alpha}{p}-\frac{1}{p'}\big)>p'\big(-\frac{1}{p'}\big)=-1$$

and thus J is finite once again. Therefore $\|S_{bc}f\|_{L^1_\beta}\lesssim\|f\|_{\mathcal{L}^p_\alpha}$ and S_{bc} is bounded from L^p_α to L^1_β when $1=q<p<\infty$. □

Proof of sufficiency for Theorem 1.4. Let $f\in L^p_\alpha$. Writing $S_{bc}f(y)$ explicitly and applying Hölder inequality with the measure ν_α yields

$$\begin{aligned}(1-|y|^2)^\beta|S_{bc}f(y)|&\lesssim(1-|y|^2)^\beta\int_{\mathbb{B}}|R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x)\\&\lesssim(1-|y|^2)^\beta\int_{\mathbb{B}}|f(x)|\frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}}d\nu_\alpha(x)\\&\lesssim\|f\|_{L^p_\alpha}(1-|y|^2)^\beta\left(\int_{\mathbb{B}}\frac{(1-|x|^2)^{(b-\alpha)p'+\alpha}}{[x,y]^{(n+c)p'}}d\nu(x)\right)^{1/p'}\\&=:J(y)\|f\|_{L^p_\alpha}.\end{aligned}$$

We will show that J is bounded on $\mathbb{B}$ by using Lemma 4.5. First, notice that

$$(b-\alpha)p'+\alpha+1=p'\left(b-\alpha+\frac{(\alpha+1)(p-1)}{p}\right)=p'\left(1+b-\frac{\alpha+1}{p}\right)>0$$

by the the first inequality of (iii) as required. Consider that

$$\begin{aligned}\rho&=(n+c)p'-n-(b-\alpha)p'-\alpha=p'\left(n+c-b+\alpha-\frac{n+\alpha}{p'}\right)\\&=p'\left(c-b+\frac{n+\alpha}{p}\right).\end{aligned}$$

If $\rho<0$, then the integral in $J(y)$ is bounded and $J(y)$ is also bounded for all $y\in\mathbb{B}$ since $\beta\geq 0$. Note that this is obvious from the second inequality in (iii) when $\beta=0$. Next, if $\rho=0$, then the integral in $J(y)$ is $(1+\log(1-|y|^2)^{-1})^{1/p'}$ and since (iii) reads $\beta>0$, and therefore $J(y)$ is bounded for all $y\in\mathbb{B}$ by (20). Lastly, if $\rho>0$, then $J(y)\sim(1-|y|^2)^{\beta-\rho/p'}$. But by the second inequality in (iii), we have $\beta-\rho/p'=\beta-c+b-\frac{n+\alpha}{p}\geq 0$ and thus $J(y)$ is bounded for all $y\in\mathbb{B}$ once again. Then $(1-|y|^2)^\beta|S_{bc}f(y)|\lesssim\|f\|_{L^p_\alpha}$ for all $y\in\mathbb{B}$ and $\|S_{bc}f\|_{\mathcal{L}^\infty_\beta}\lesssim\|f\|_{L^p_\alpha}$. Thus S_{bc} is bounded from L^p_α to $\mathcal{L}^\infty_\beta$ with $1<p<q=\infty$. □

Proof of sufficiency for Theorem 1.5. Let $f\in L^1_\alpha$. If $\alpha=b$ and $\beta=0$, then $c<-n$ by the second inequality in (iii) and $|R_c(x,y)|$ is bounded by Lemma 2.6. So we have

$$|S_{bc}f(y)|\lesssim\int_{\mathbb{B}}|R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x)\lesssim\|f\|_{L^1_\alpha}\quad(y\in\mathbb{B}).$$

Then $\|S_{bc}f\|_{L^\infty} \lesssim \|f\|_{L^1_\alpha}$.

Otherwise $\alpha \le b$ and $\beta > 0$, and there are values of $c > -n$ satisfying the inequalities in (iii). So in the rest of proof we can assume $c > -n$ by Corollary 5.2. Then we write $S_{bc}f(y)$ explicitly, and obtain

$$\begin{aligned}(1-|y|^2)^\beta |S_{bc}f(y)| &\le (1-|y|^2)^\beta \int_{\mathbb{B}} |R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x)\\ &\lesssim (1-|y|^2)^\beta \int_{\mathbb{B}} |f(x)| \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}} d\nu_\alpha(x)\\ &= \int_{\mathbb{B}} |f(x)|(1-|y|^2)^\beta \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}} d\nu_\alpha(x)\\ &=: \int_{\mathbb{B}} |f(x)| J(x,y) d\nu_\alpha(x).\end{aligned}$$

Since $[x,y] \gtrsim (1-|x|^2)$ and $[x,y] \gtrsim (1-|y|^2)$ for $x, y \in \mathbb{B}$, we have $J(x,y) \lesssim (1-|x|^2)^{b-\alpha+\beta-(n+c)}$ for all such y. Note that the power here is nonnegative by the second inequality in (iii) yielding that $J(x,y)$ is bounded for all $x, y \in \mathbb{B}$. So we get that

$$(1-|x|^2)^\beta |S_{bc}f(y)| \lesssim \int_{\mathbb{B}} |f(x)| d\nu_\alpha(x) = \|f\|_{L^1_\alpha} \quad (y \in \mathbb{B})$$

and $\|S_{bc}f\|_{\mathcal{L}^\infty_\beta} \lesssim \|f\|_{L^1_\alpha}$. Thus S_{bc} is bounded from L^1_α to $\mathcal{L}^\infty_\beta$. □

Proof of sufficiency for Theorem 1.6. First, let $q = 1$. Assume that $f \in \mathcal{L}^\infty_\alpha$. Writing the L^1_β norm of $S_{bc}f$ explicitly and applying Fubini's theorem, taking the $\mathcal{L}^\infty_\alpha$ norm of f out of integral, we obtain

$$\begin{aligned}\|S_{bc}f\|_{L^1_\beta} &\lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} |R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x)(1-|y|^2)^\beta d\nu(y)\\ &\lesssim \int_{\mathbb{B}} |f(x)|(1-|x|^2)^b \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{n+c}} d\nu(y) d\nu(x)\\ &\lesssim \|f\|_{\mathcal{L}^\infty_\alpha} \int_{\mathbb{B}} (1-|x|^2)^{b-\alpha} \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{n+c}} d\nu(y) d\nu(x) =: J \|f\|_{\mathcal{L}^\infty_\alpha}.\end{aligned}$$

We will show that J is finite using Lemma 4.5 since $\beta > -1$ as required.

Firstly, if $c - \beta < 0$, then the inner integral in J is bounded and J is finite since $b - \alpha > -1$ by the first inequality of (iii). Next, if $c - \beta = 0$, then the inner integral is $1 + \log(1/(1-|x|^2))^{-1}$. Then $J(x)$ is finite by $b - \alpha > -1$ and Lemma 3.1. Lastly, if $c - \beta > 0$, then $J \sim \int_{\mathbb{B}} (1-|x|^2)^{b-\alpha-c+\beta} d\nu(x)$. But by the second inequality in (iii) we have $b - \alpha - c + \beta > -1$ and thus J is finite once again. Therefore $\|S_{bc}f\|_{L^1_\beta} \lesssim \|f\|_{\mathcal{L}^\infty_\alpha}$ and S_{bc} is bounded from $\mathcal{L}^\infty_\alpha$ to L^1_β.

Next, let $1 < q < p = \infty$. Assume that $f \in \mathcal{L}^\infty_\alpha$. Writing L^q_β norm of $S_{bc}f = Sf$ explicitly and taking the $\mathcal{L}^\infty_\alpha$ norm of f out of integral, we obtain

$$\begin{aligned}\|Sf\|^q_{L^q_\beta} &= \int_{\mathbb{B}} \left| \int_{\mathbb{B}} |R_c(x,y)||f(x)|(1-|x|^2)^b d\nu(x) \right|^q d\nu_\beta(y)\\ &\lesssim \|f\|^q_{\mathcal{L}^\infty_\alpha} \int_{\mathbb{B}} (1-|y|^2)^\beta \left(\int_{\mathbb{B}} \frac{(1-|x|^2)^{b-\alpha}}{[x,y]^{n+c}} d\nu(x) \right)^q d\nu(y) =: J \|f\|^q_{\mathcal{L}^\infty_\alpha}.\end{aligned}$$

We will show that J is finite using Lemma 4.5 since $\beta - \alpha > -1$ by the first inequality in (iii) as required.

Firstly, if $c - b + \alpha < 0$, then the inner integral in J is bounded and J is finite since $\beta > -1$. Next, if $c - b + \alpha = 0$, then the inner integral is $1 + \log(1/(1-|x|^2))^{-1}$. Then J is finite by Lemma 3.1. Lastly, if

$c - b + \alpha > 0$, then $J \sim \int_{\mathbb{B}} (1 - |y|^2)^{\beta - (c-b+\alpha)q} d\nu(y)$. But we have

$$\beta - (c - b + \alpha)q = \beta - (c - b + \alpha)q + 1 - 1 = q(\frac{\beta + 1}{q} - c + b - \alpha) - 1 > -1$$

by the second inequality in (iii) and thus J is finite once again. Therefore $\|S_{bc}f\|^q_{L^q_\beta} \lesssim \|f\|^q_{\mathcal{L}^\infty_\alpha}$ and S_{bc} is bounded from $\mathcal{L}^\infty_\alpha$ to L^q_β. □

Proof of sufficiency for Theorem 1.7. Let $f \in \mathcal{L}^\infty_\alpha$. Writing $S_{bc}f(y)$ explicitly and taking the $\mathcal{L}^\infty_\alpha$ norm of f out of integral, we obtain

$$\begin{aligned}(1 - |y|^2)^\beta |S_{bc}f(y)| &\lesssim (1 - |y|^2)^\beta \int_{\mathbb{B}} |R_c(x, y)||f(x)|(1 - |x|^2)^b d\nu(x) \\ &\lesssim \|f\|_{L^\infty_\alpha} (1 - |y|^2)^\beta \int_{\mathbb{B}} \frac{(1 - |x|^2)^{b-\alpha}}{[x, y]^{n+c}} d\nu(x) \\ &=: J(y)\|f\|_{L^\infty_\alpha}.\end{aligned}$$

We will show that J is bounded on $\mathbb{B}$ by using Lemma 4.5 since $b - \alpha > -1$ by the first inequality in (iii) as required.

Firstly, if $c - b + \alpha < 0$, then the integral in J is bounded and J is bounded since $\beta > -1$. Next, if $c - b + \alpha = 0$, then the inner integral is $1 + \log(1/(1 - |x|^2))^{-1}$. Then J is finite by (20). Lastly, if $c - b + \alpha > 0$, then $J(y) \sim (1 - |y|^2)^{\beta - c + b - \alpha}$. But we have $\beta - c + b - \alpha \geq 0$ by the second inequality in (iii) and thus J is bounded once again. Therefore $\|S_{bc}f\|_{\mathcal{L}^\infty_\beta} \lesssim \|f\|_{\mathcal{L}^\infty_\alpha}$ and S_{bc} is bounded from $\mathcal{L}^\infty_\alpha$ to $\mathcal{L}^\infty_\beta$. This completes the proof. □

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