Filomat 36:13 (2022), 4277–4291
https://doi.org/10.2298/FIL2213277A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extended Eigenvalues of a Closed Linear Operator

Aymen Ammar[a], Fatima Zohra Boutaf[a], Aref Jeribi[a]

[a]*Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Soukra road km 3.5, B.P. 1171, 3000, Sfax, Tunisia*

Abstract. A complex number λ is an extended eigenvalue of an operator A if there is a nonzero operator B such that $AB = \lambda BA$. In this case, B is said to be an eigenoperator. This research paper is devoted to the investigation of some results of extended eigenvalues for a closed linear operator on a complex Banach space. The obtained results are explored in terms two cases bounded, and closed eigenoperators. In addition, the notion of extended eigenvalues for a 2×2 upper triangular operator matrix is introduced and some of its properties are displayed.

1. Introduction

For a bounded linear operator A on a complex Banach space X, the set $\sigma_{ext}(A)$ of extended eigenvalues for A is defined to be the set of complex numbers λ for which the following Sylvester's operational equation:

$$AB = \lambda BA \tag{1}$$

has a nonzero solution B. Recently, there has been a spate of interest in extended eigenvalue and extended eigenoperator. Much concern has been devoted to this notion dating back to the works of Brown in [4] and Kim, Moore and Pearcy in [10], who set forward a generalization of the well known Lomonosov theorem on the existence of nontrivial hyperinvariant subspace for the compact operators on a Banach space. They demonstrated that if A is compact, then B has a nontrivial hyperinvariant subspace for any number $\lambda \in \mathbb{C}$. Certainly, if $\lambda = 1$ in Eq. (1), then this particular case pertains to Lomonosov's theorem [12] that is the algebra $\{A\}'$, the commutant of A, which possesses a common trivial invariant subspace. It was proved in [11] that if A is a compact operator, there exists spectral algebra B_A which has a nontrivial invariant subspace and contains properly $\{A\}'$ whenever the spectral radius of A is positive, so that the operators commuting with A together with operators satisfying (1) for some $|\lambda| \leq 1$ belong to B_A.

Extended eigenvalues and their corresponding extended eigenoperators whetted the interest and drew the attention of several authors (see, e.g., [1, 5, 9, 13, 16]). In [2], Biswas, Lambert and Petrovic introduced this notion by depicting that the extended eigenvalue of an operator A which has a dense range corresponds to the eigenvalue of closed homomorphism constructed from A. Furthermore, they computed the set of extended eigenvalues of the integral Volterra operator on the space $L^2(0, 1)$. The set of extended eigenoperators of Volterra integration operator was reported by Karaev in [9]. On the other side, it was revealed in

2020 *Mathematics Subject Classification*. Primary 11B05, 47A53 ; Secondary 39B42
Keywords. (Extended eigenvalue; Eigenoperator; Essential spectrum; 2×2 Block matrix)
Received: 13 April 2021; Accepted: 10 May 2021
Communicated by Fuad Kittaneh
Email addresses: ammar_aymen84@yahoo.fr (Aymen Ammar), fatima.boutaf@gmail.com (Fatima Zohra Boutaf), Aref.Jeribi@fss.rnu.tn (Aref Jeribi)

[17] that there are compact quasinilpotent operators, for which the set of extended eigenvalues is the one point set $\{1\}$.

In [3], Biswas and Petrovic applied the Rosenblum theorem [14] that is stated as follows:

$$\text{if } \sigma(A) \cap \sigma(E) = \emptyset, \text{ then } B = 0 \text{ is the only solution of } AB - BE = 0,$$

with the special case where $E = \lambda A$, leading to derive the following important inclusion:

$$\sigma_{ext}(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\},$$

where $\sigma(A)$ is the spectrum of A. In the same paper, it was proved that this inclusion is an equality on the finite dimensional spaces.

The intrinsic objective of this work is to generalize some obtained results of extended eigenvalues of a bounded linear operator in Banach space to the closed case. Furthermore, this paper is dedicated to investigate some results of extended eigenvalues of a 2×2 upper triangular operator matrix.

This paper is organized as follows: In section 2, we display some notations and preliminaries which will be needed in the sequel. In section 3, we exhibit some properties of extended eigenvalues of a closed linear operator for both cases, when the corresponding eigenoperator is a bounded linear operator and when the corresponding eigenoperator is a closed linear operator. The main result of this section is Theorem 3.7 which characterizes the set of extended eigenvalues of an invertible closed linear operator by means of its Schechter essential spectrum. In section 4, we introduce the notion of extended eigenvalues of a 2×2 upper triangular operator matrix. The basic goal of this section is to discuss the relationship between the set of extended eigenvalue of a 2×2 upper triangular operator matrix and that of its diagonal entries. We close the last section by setting forward some results of extended eigenvalue of a 2×2 block operator matrix.

2. Preliminaries

In this paper, the symbol X stands for a complex Banach space. We denote by $C(X)$ the set of all densely defined closed linear operators on X, and by $\mathcal{L}(X)$ the set of all bounded linear operators on X. For a closed linear operator A we write $\mathcal{D}(A)$, $R(A)$ and $N(A)$ to denote the domain, the range and the kernel of A, respectively. The resolvent set, the spectrum and the point spectrum of A are, respectively, defined as

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is injective and } (\lambda I - A)^{-1} \in \mathcal{L}(X)\},$$
$$\sigma(A) = \mathbb{C} \setminus \rho(A),$$
$$\sigma_p(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is not injective}\}.$$

For $A \in C(X)$, we shall use the symbol A^* to denote the adjoint of A.

Definition 2.1. *Let $A \in C(X)$. The Schechter essential spectrum is defined by*

$$\sigma_s(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K),$$

where $\mathcal{K}(X)$ stands for the ideal of all compact operators on X. ◇

Lemma 2.2. ([15, Theorem 7.28]) *Let $A \in C(X)$ and $K \in \mathcal{K}(X)$. Therefore,*

$$\sigma_s(A) = \sigma_s(A + K).$$ ◇

Proposition 2.3. [8, Proposition 2.2.5] *Let $A \in C(X)$. Then, we have $\sigma_s(A)$ which is closed.* ◇

Definition 2.4. *A set D in the complex plane is a Cauchy domain if the following conditions are satisfied:*

(i) *D is bounded and open,*

(ii) *D has a finite number of components, the closures of any two of which are disjoint, and*

(iii) *the boundary, ∂D, of D is composed of a finite positive number of closed rectifiable Jordan curves.* ◇

Theorem 2.5. *[18] Let F be a closed and G be a bounded open subset of the complex plane such that $F \subset G$. Then, there exists a Cauchy domain D such that $F \subset D$ and $\bar{D} \subset G$.* ◊

Lemma 2.6. [8, Theorem 7.3.2] *Let $A \in C(X)$. If $0 \in \rho(A)$, then we have*

$$\lambda \in \sigma_s(A) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_s(A^{-1}).$$ ◊

Definition 2.7. *Two operators $U \in \mathcal{L}(X)$ and $V \in \mathcal{L}(X)$ are similar if there exists an invertible operator S such that $SU = VS$.* ◊

Theorem 2.8. *[6] Let H_1, H_2 be two Hilbert spaces. Let $A \in \mathcal{L}(H_1)$ and $B \in \mathcal{L}(H_2)$ be given operators. There exists $C \in \mathcal{L}(H_2, H_1)$ such that $R(M_c)$ is not dense in $H_1 \times H_2$ if, and only if, one of the following conditions is satisfied:*

(*i*) *$R(B)$ is not dense in H_1.*

(*ii*) *$R(A)$ is not dense in H_2.* ◊

3. Extended eigenvalues of a closed linear operator

The basic objective of this section is a twofold one. First, we will present a description of the set of extended eigenvalues of a closed linear operator, whenever the corresponding eigenoperator is a bounded linear operator. In the same way, we characterize the set of extended eigenvalues of an invertible closed linear operator whose inverse is bounded. Second, we will provide some results for the extended point spectrum of a closed linear operator when the corresponding eigenoperator is a closed linear operator. We conclude this section by recording some results if the generalized inverse of a closed linear operator is the corresponding closed eigenoperator.

3.1. Bounded eigenoperator

Definition 3.1. *Let A be a closed linear operator on X. A complex number λ is an extended eigenvalue of A if there is a nonzero bounded linear operator B such that*

$$\begin{cases} ABx = \lambda BAx, \\ \text{for all } x \in \mathcal{D}(A). \end{cases} \tag{2}$$

The operator B is called eigenoperator corresponding to λ. The set of extended eigenvalues and the set of eigenoperators corresponding to λ are represented, respectively, by $\sigma_{ext}(A)$ and $E_{ext}(A, \lambda)$. ◊

Remark 3.2. *From relation* (2), *it follows that $R(B) \subset \mathcal{D}(A)$.* ◊

Proposition 3.3. *Let A be a closed linear operator on X.*

(*i*) *We have for all $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$*

$$B \in E_{ext}(A, 1) \text{ if, and only if, } B \in E_{ext}(\alpha A + \beta B, 1).$$

(*ii*) *Let $B_i \in E_{ext}(A, \lambda)$, $i \in \{1, ..., n\}$. If $\prod_{i=1,...,n} B_i \neq 0$, then $\prod_{i=1,...,n} B_i \in E_{ext}(A, \lambda^n)$ for any $n = 1, 2, 3, ...$*

(*iii*) *Assume that $0 \in \rho(A)$, we have*

$$0 \neq \lambda \in \sigma_{ext}(A) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ext}(A^{-1}).$$

(*iv*) *If $\alpha \neq 0$, then*

$$\sigma_{ext}(\alpha A) = \sigma_{ext}(A).$$ ◊

Proof. (i) Suppose that $B \in E_{ext}(A, 1)$, then

$$ABx = BAx \text{ for all } x,\ Bx \in \mathcal{D}(A). \tag{3}$$

Since $\alpha A + \beta B$ is a closed linear operator on X and $\mathcal{D}(\alpha A + \beta B) = \mathcal{D}(A)$, for all $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$. Then, grounded on relation (3), we infer that

$$\begin{aligned}(\alpha A + \beta B)Bx &= \alpha ABx + \beta B^2 x\\ &= \alpha BAx + \beta B^2 x\\ &= B(\alpha A + \beta B)x, \text{ for all } x,\ Bx \in \mathcal{D}(\alpha A + \beta B).\end{aligned}$$

This implies that $B \in E_{ext}(\alpha A + \beta B, 1)$. The proof of the converse may be checked in a same way.
(ii) Let $B_i \in E_{ext}(A, \lambda)$ for all $i \in \{1, ..., n\}$. Hence, we have for all $i \in \{1, ..., n\}$

$$AB_i x = \lambda B_i A x, \text{ for all } x,\ B_i x \in \mathcal{D}(A).$$

Since $\prod_{i=1,...,n} B_i \neq 0$ and $R(B_1 ... B_n) \subset R(B_1)$, we obtain

$$AB_1 ... B_n x = \lambda^n B_1 ... B_n A x, \text{ for all } x,\ B_1 ... B_n x \in \mathcal{D}(A).$$

In other words,

$$\prod_{i=1,...,n} B_i \in E_{ext}(A, \lambda^n).$$

(iii) Let $0 \neq \lambda \in \sigma_{ext}(A)$. Then, there exists $B \in \mathcal{L}(X) \setminus \{0\}$ such that

$$ABx = \lambda BAx, \text{ for all } x \in \mathcal{D}(A).$$

Since $0 \in \rho(A)$, we have

$$BA^{-1}y = \lambda A^{-1}By, \text{ for all } y \in X.$$

Multiplying by $\frac{1}{\lambda}$, we get

$$A^{-1}By = \frac{1}{\lambda}BA^{-1}y, \text{ for all } y \in X.$$

It follows that

$$\frac{1}{\lambda} \in \sigma_{ext}(A^{-1}).$$

Conversely, let $\lambda \neq 0$ such that $\frac{1}{\lambda} \in \sigma_{ext}(A^{-1})$. Then, there exists $B \in \mathcal{L}(X) \setminus \{0\}$ such that

$$A^{-1}By = \frac{1}{\lambda}BA^{-1}y, \text{ for all } y \in X.$$

Since for all $y \in X$, we have $y = Ax$, for all $x \in \mathcal{D}(A)$. Then,

$$BAx = \frac{1}{\lambda}ABx, \text{ for all } x \in \mathcal{D}(A).$$

Multiplying by λ, we obtain

$$ABx = \lambda BAx, \text{ for all } x \in \mathcal{D}(A).$$

In other words, $\lambda \in \sigma_{ext}(A)$.
(iv) Suppose that $\alpha \neq 0$ and let $\lambda \in \sigma_{ext}(\alpha A)$. Then, there exists $B \in \mathcal{L}(X) \setminus \{0\}$ such that

$$(\alpha A)Bx = \lambda B(\alpha A)x, \text{ for all } x \in \mathcal{D}(\alpha A).$$

Since $\mathcal{D}(\alpha A) = \mathcal{D}(A)$, we get

$$\alpha ABx = \alpha\lambda BAx, \quad \text{for all } x \in \mathcal{D}(A).$$

Multiplying by $\dfrac{1}{\alpha}$, we obtain

$$\lambda \in \sigma_{ext}(A).$$

The inverse inclusion may be proved in a similar way. □

Remark 3.4. *The spectral mapping theorem does not hold for extended spectrum. In fact, let A be the identity operator on X and P(.) be the nonzero complex polynomial defined by* $P(\lambda) = \lambda(\lambda - 2)$. *It is easy to check that* $\sigma_{ext}(A) = \{1\}$. *Then, we have*

$$\begin{aligned} P(\sigma_{ext}(A)) &= P(\{1\}) \\ &= \{-1\}. \end{aligned}$$

However,

$$\sigma_{ext}(P(A)) = \sigma_{ext}(-I).$$

Based on Proposition 4.3 (v), we obtain

$$\sigma_{ext}(P(A)) = \{1\}.$$

In other words, there is no inclusion among $P(\sigma_{ext}(A))$ *and* $\sigma_{ext}(P(A))$. ◊

Theorem 3.5. *Let* $A \in C(X)$. *Then,*

$$\left\{\frac{\alpha}{\bar{\beta}} : \ \alpha \in \sigma_p(A) \text{ and } \beta \in \sigma_p(A^*)\right\} \subset \sigma_{ext}(A). \quad ◊$$

Proof. Let $\lambda \in \left\{\frac{\alpha}{\bar{\beta}} : \ \alpha \in \sigma_p(A) \text{ and } \beta \in \sigma_p(A^*)\right\}$. Then, there exist $\alpha \in \sigma_p(A)$ and $\beta \in \sigma_p(A^*)$ such that $\lambda = \dfrac{\alpha}{\bar{\beta}}$. Furthermore, there exist $x \in \mathcal{D}(A) \setminus \{0\}$ and $y \in \mathcal{D}(A^*) \setminus \{0\}$ such that $Ax = \alpha x$ and $A^*y = \beta y$. Let's define the operator $B = x \otimes y$ by

$$(x \otimes y)z = (z, y)x, \quad \text{for all } z \in X,$$

where $(.,.)$ denotes the duality mapping of $y \in \mathcal{D}(A^*)$ and $z \in X$. As $\mathcal{D}(A)$ proves to be a subspace, then $Bz \in \mathcal{D}(A)$ for all $z \in X$. Now, we attempt to check that B is an eigenoperator for A associated with λ. On the one hand, we have for all $z \in X$ and $Bz \in \mathcal{D}(A)$

$$\begin{aligned} (AB)z &= A(Bz) \\ &= A(z, y)x \\ &= (z, y)Ax \\ &= (z, y)\alpha x \\ &= \alpha Bz. \end{aligned}$$

On the other hand, we have for all $z \in \mathcal{D}(A)$ and $Az \in X$

$$\begin{aligned} (BA)z &= B(Az) \\ &= (Az, y)x \\ &= (z, A^*y)x \\ &= (z, \beta y)x \\ &= \bar{\beta}(z, y)x \\ &= \bar{\beta}Bz. \end{aligned}$$

It follows that

$$ABz = \frac{\alpha}{\overline{\beta}}BAz, \quad \text{for all } z, \; Bz \in \mathcal{D}(A).$$

As a result, $\lambda \in \sigma_{ext}(A)$. □

As a direct consequence of Theorem 3.5, we infer the following result:

Corollary 3.6. *Let X be a complex Banach space and $A \in C(X)$.*
(i) If $1 \in \sigma_p(A^*)$, then $\sigma_p(A) \subset \sigma_{ext}(A)$.
(ii) If A and A^* have nontrivial kernels, then $\sigma_{ext}(A) = \mathbb{C}$.
(iii) Let $\lambda \in \mathbb{R}$. If $\lambda \in \sigma_p(A) \cap \sigma_p(A^*)$, then $\sigma_{ext}(\lambda I - A) = \mathbb{C}$. ◊

Proof. (i) Let $1 \in \sigma_p(A^*)$. Since $\overline{1} = 1$, it follows that $\sigma_p(A) \subset \sigma_{ext}(A)$.
(ii) Supposing that A and A^* have nontrivial kernels. Then, there exist $x \in \mathcal{D}(A) \setminus \{0\}$ and $y \in \mathcal{D}(A^*) \setminus \{0\}$ such that $Ax = A^*y = 0$. Hence, the operator $B = x \otimes y$ holds for all $\lambda \in \mathbb{C}$

$$ABz = \lambda BAz = 0, \quad \text{for all } z, \; Bz \in \mathcal{D}(A).$$

Therefore, $\sigma_{ext}(A) = \mathbb{C}$.
(iii) Let $\lambda \in \mathbb{R}$ such that $\lambda \in \sigma_p(A) \cap \sigma_p(A^*)$. It follows that $0 \in \sigma_p(\lambda I - A) \cap \sigma_p((\bar{\lambda} I - A)^*)$. As λ is a real number, $0 \in \sigma_p(\lambda I - A) \cap \sigma_p((\lambda I - A)^*)$. Departing from ($ii$), we infer that $\sigma_{ext}(\lambda I - A) = \mathbb{C}$. □

Theorem 3.7. *Let $A \in C(X)$ such that $0 \in \rho(A)$. Then,*

$$\sigma_{ext}(A) \subset \Big\{\lambda \in \mathbb{C} \setminus \{0\} : \sigma_s(A) \cap \sigma_s(\lambda A) \neq \emptyset\Big\}. \qquad ◊$$

In order to prove Theorem 3.7, we shall invest the following result:

Lemma 3.8. *Let $E, F \in C(X)$ and $B \in \mathcal{L}(X)$ such that $0 \in \rho(E) \cap \rho(F)$. If $\sigma_s(E) \cap \sigma_s(F) = \emptyset$, then $B = 0$ is the only solution of the equation*

$$BF - EB = 0. \qquad ◊$$

Proof. First, we demonstrate that

$$\sigma_s(E) \cap \sigma_s(F) = \emptyset \;\text{ if, and only if, }\; \sigma_s(E^{-1}) \cap \sigma_s(F^{-1}) = \emptyset.$$

To prove the "if "part, let $\sigma_s(E^{-1}) \cap \sigma_s(F^{-1}) = \emptyset$, suppose that $\sigma_s(E) \cap \sigma_s(F) \neq \emptyset$, and select $\lambda \in \sigma_s(E) \cap \sigma_s(F)$. By using Lemma 2.6, we have $\frac{1}{\lambda} \in \sigma_s(E^{-1}) \cap \sigma_s(F^{-1})$. This contradicts our assumption. The proof of the "only if "part can be evaluated in the same way.
Second, we consider the following equation

$$E^{-1}B - BF^{-1} = Q,$$

with $Q \in \mathcal{L}(X)$. It is well known that both $\sigma(E^{-1})$ and $\sigma(F^{-1})$ are compact nonempty subsets of complex plane $\mathbb{C}$ and as $\sigma_s(E^{-1})$ and $\sigma_s(E^{-1})$ are closed based on Proposition 2.3; we infer that $\sigma_s(E^{-1})$ and $\sigma_s(F^{-1})$ are compact subsets. The fact that $\sigma_s(E^{-1}) \cap \sigma_s(F^{-1}) = \emptyset$ and by applying Theorem 2.5, we deduce the existence of a Cauchy domain D such that $\sigma_s(E^{-1}) \subset D$ and $\sigma_s(F^{-1}) \cap \bar{D} = \emptyset$. Now, suppose that B is a solution of the considered equation and let $\omega \in \partial D$. Then,

$$\omega \notin \sigma_s(E^{-1})$$

and

$$\omega \notin \sigma_s(F^{-1}).$$

Hence, there exist $K_0, K_1 \in K(X)$ such that $\omega \notin \sigma(F^{-1} + K_0)$ and $\omega \notin \sigma(E^{-1} + K_1)$. Since $K_0, K_1 \in K(X)$ and using Lemma 2.2, then we have

$$\sigma_s(F^{-1} - K_0) = \sigma_s(F^{-1})$$

and

$$\sigma_s(E^{-1} - K_1) = \sigma_s(E^{-1}).$$

That's why $\sigma(F^{-1} + K_0)$ and $\sigma(E^{-1} + K_1)$ can be replaced, respectively, by $\sigma(F^{-1})$ and $\sigma(E^{-1})$. From this perspective, $\omega \notin \sigma(F^{-1})$ and $\omega \notin \sigma(E^{-1})$. Hence, we get

$$\omega \in \rho(E^{-1}) \cap \rho(F^{-1}).$$

Thus,

$$B(\omega I - F^{-1}) - (\omega I - E^{-1})B = Q.$$

Moreover,

$$(\omega I - E^{-1})^{-1}B - B(\omega I - F^{-1})^{-1} = (\omega I - E^{-1})^{-1}Q(\omega I - F^{-1})^{-1}.$$

Using Dunford's functional calculus (refer back to [7]), and integrating both sides over ∂D, we obtain

$$\frac{1}{2\pi i}\int_{\partial D} (\omega I - E^{-1})^{-1} B d\omega = B,$$

$$\frac{1}{2\pi i}\int_{\partial D} B(\omega I - F^{-1})^{-1} d\omega = 0,$$

and

$$B = \frac{1}{2\pi i}\int_{\partial D} (\omega I - E^{-1})^{-1}Q(\omega I - F^{-1})^{-1} d\omega. \tag{4}$$

Based upon Eq. (4), if we put $Q = 0$, then we get $B = 0$ which is the unique solution of the following equation

$$E^{-1}B - BF^{-1} = 0.$$

As a result, $B = 0$ is the only solution of

$$BF - EB = 0.$$ □

Now, we are ready to proof Theorem 3.7.

Proof. Using the contraposition of Lemma 3.8, where it is sufficient to replace E by A and F by λA. In such a way that for $\lambda \neq 0$, there exists $B \in \mathcal{L}(X) \setminus \{0\}$ satisfying

$$ABx = \lambda BAx, \quad \text{for all } x, Bx \in \mathcal{D}(A).$$

Therefore, we can deduce that

$$\sigma_s(A) \cap \sigma_s(\lambda A) \neq \emptyset.$$ □

Corollary 3.9. *If $\sigma_s(A) = \{\alpha\}$ with $\alpha \neq 0$, then $\sigma_{ext}(A) = \{1\}$.* ◊

Proof. First, it should be pointed out here that having $0 \in \rho(A)$ entails the fact that $1 \in \sigma_{ext}(A^{-1})$ (the operator B in relation (2) can be taken as the identity operator on X). Referring to Proposition 3.3 (*iv*), we obtain $1 \in \sigma_{ext}(A)$.
Now, it remains to prove the inverse inclusion. For this reason, let's suppose that $\sigma_s(A) = \{\alpha\}$ with $\alpha \neq 0$. We have $\sigma_s(A) \cap \sigma_s(\lambda A) \neq \emptyset$, which implies that $\alpha \in \sigma_s(\lambda A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(\lambda A + K)$. Hence, $\dfrac{\alpha}{\lambda} \in \bigcap_{\frac{1}{\lambda}K \in \mathcal{K}(X)} \sigma(A + \frac{1}{\lambda}K)$.
Thus, $\lambda = 1$. As a matter of fact, $\sigma_{ext}(A) \subset \{1\}$. □

3.2. Closed eigenoperator

In this subsection, we will focus upon relation

$$\begin{cases} ABx = \lambda BAx, \\ for\ all\ x \in \mathcal{D}(A) \cap \mathcal{D}(B), \end{cases} \tag{5}$$

instead of relation (2), with B is a nonzero closed linear operator on a Banach space X. In this case, the set of extended eigenvalues and the set of eigenoperators corresponding to λ are represented, respectively, by $\tilde{\sigma}_{ext}(A)$ and $\tilde{E}_{ext}(A, \lambda)$.

Remark 3.10. (*i*) *From relation* (5)*, it follows that* $R(B) \subset \mathcal{D}(A)$ *and* $R(A) \subset \mathcal{D}(B)$.

(*ii*) *Let A be a closed linear operator on a Banach space X. If* $R(A) \subset \mathcal{D}(A)$*, then* $1 \in \tilde{\sigma}_{ext}(A)$*. As an illustration, let's examine the following example:*

Example 3.11. *Let* $X = C[0,1]$ *be the space of continuous functions on* $[0,1]$*. Let A be the closed linear operator on X defined by*

$$Af = f', \ for\ f, f' \in \mathcal{D}(A),$$

where $\mathcal{D}(A) = C^1[0,1]$ *(the set of all continuously differentiable functions). It follows that its square* A^2 *can be expressed as*

$$A^2 f = f'', \ f \in \mathcal{D}(A^2),$$

where $\mathcal{D}(A^2) = C^2[0,1]$ *(the set of all differentiable functions whose derivative is in* $C^1[0,1]$*). We deduce that* $1 \in \tilde{\sigma}_{ext}(A)$.

(*iii*) *Let* A, B *be nonzero closed linear operators and* $\lambda \neq 0$,

$$B \in \tilde{E}_{ext}(A, \lambda) \ \text{ if, and only if, } \ A \in \tilde{E}_{ext}(B, \tfrac{1}{\lambda}). \qquad \diamond$$

Definition 3.12. *Let A and B be two closed linear operators on X. B is called a generalized inverse of A if* $R(B) \subset \mathcal{D}(A)$ *and* $R(A) \subset \mathcal{D}(B)$ *such that*

$$Au = ABAu, \ \text{ for all } \ u \in \mathcal{D}(A),$$

$$Bv = BABv, \ \text{ for all } \ v \in \mathcal{D}(B). \qquad \diamond$$

Proposition 3.13. *Let A be a nonzero closed linear operator and let B be the generalized inverse of A.*

(*i*) $B \in \tilde{E}_{ext}(A, \lambda)$ *if, and only if,* $BAB \in \tilde{E}_{ext}(A, \lambda)$.

(*ii*) *Let* $\lambda \neq 0$*. Then,* $B \in \tilde{E}_{ext}(A, \lambda)$ *if, and only if,* $ABA \in \tilde{E}_{ext}(B, \dfrac{1}{\lambda})$. $\diamond$

Proof. (*i*) Let $B \in \tilde{E}_{ext}(A, \lambda)$. Then,

$$ABx = \lambda BAx, \ \text{ for all } \ x \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

Consequently,

$$ABABx = \lambda BABAx, \ \text{ for all } \ x \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

Thus,

$$A(BAB)x = \lambda(BAB)Ax, \ \text{ for all } \ x \in \mathcal{D}(A) \cap \mathcal{D}(BAB).$$

It follows that

$$BAB \in \tilde{E}_{ext}(A, \lambda).$$

Conversely, supposing that $BAB \in \tilde{E}_{ext}(A, \lambda)$, provides

$$ABABx = \lambda BABAx, \ \text{ for all } \ x \in \mathcal{D}(A) \cap \mathcal{D}(BAB).$$

Hence,

$$A(BAB)x = \lambda(BAB)Ax, \quad \text{for all } \ x \in \mathcal{D}(A) \cap \mathcal{D}(BAB).$$

So, we get

$$ABx = \lambda BAx \ \text{ for all } \ x \in \mathcal{D}(A) \cap \mathcal{D}(B).$$

As a result,

$$B \in \tilde{E}_{ext}(A, \lambda).$$

The proof of (*ii*) is analogous to the previous one. □

4. Extended eigenvalues of a 2 × 2 upper triangular operator matrix

In this section, we shall determine some properties of extended eigenvalues of a 2×2 upper triangular operator matrix. Let X_1 and X_2 be two Banach spaces and consider the 2×2 upper triangular operator matrices defined on $X_1 \times X_2$ by

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{6}$$

and

$$M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{7}$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and $C \in \mathcal{L}(X_2, X_1)$.

Definition 4.1. *A complex number λ is an extended eigenvalue of M_C if there is a nonzero*

$$Y = \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}, \tag{8}$$

where $Y_1 \in \mathcal{L}(X_1)$, $Y_3 \in \mathcal{L}(X_2, X_1)$ and $Y_2 \in \mathcal{L}(X_2)$ such that

$$M_C Y = \lambda Y M_C. \tag{9}$$

The set of extended eigenvalues is represented by $\sigma_{ext}(M_C)$. ◇

Remark 4.2. (*i*) *$\sigma_{ext}(M_C) \neq \emptyset$. Indeed, one may take $Y = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$, where the I_1 and I_2 are, respectively, identity operators on X_1 and X_2, so that $1 \in \sigma_{ext}(M_C)$.*
(*ii*) *If $A = B = 0$, then $\sigma_{ext}(M_C) = \mathbb{C}$. In fact, we have for all $Y_3 \in \mathcal{L}(X_2, X_1) \setminus \{0\}$*

$$\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix},$$

for any $\lambda \in \mathbb{C}$.
(*iii*) *$\sigma_{ext}(M_0) = \sigma_{ext}(A) \cup \sigma_{ext}(B) \cup \{\lambda \in \mathbb{C} :$ there exists $0 \neq Y_3 \in \mathcal{L}(X_2, X_1)$ such that $AY_3 = \lambda Y_3 B\}$.*
(*iv*) *If Y_1, Y_2 and Y_3 are non zeros, then*

$$\sigma_{ext}(M_C) = \sigma_{ext}(A) \cap \sigma_{ext}(B) \cap \{\lambda \in \mathbb{C} : AY_3 + CY_2 = \lambda Y_1 C + \lambda Y_3 B\}.$$ ◇

The following theorem sets a relation between the extended spectrum of a 2×2 upper triangular operator matrix and the extended spectrum of its diagonal entries.

Theorem 4.3. *Let M_C be the 2×2 upper triangular operator matrices defined in Eq.* (6) *and consider the 2×2 upper triangular operator matrices, Y, defined in Eq.* (8)*. Then, we have the following results:*
(*i*) *Let $CY_2 = \lambda Y_1 C$, for any $\lambda \in \mathbb{C}$.*

$$\text{If } \lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B), \text{ then } \lambda \in \sigma_{ext}(M_C). \tag{10}$$

(*ii*) *If $Y_1 \neq 0$ or $Y_2 \neq 0$, then*

$$\sigma_{ext}(M_C) \subseteq \sigma_{ext}(A) \cup \sigma_{ext}(B). \tag{11}$$

◊

Proof. (*i*) Suppose that $CY_2 = \lambda Y_1 C$, for any $\lambda \in \mathbb{C}$. Let's consider the following cases:
First case, if $\lambda \in \sigma_{ext}(A)$, then there exists $Y_1 \in \mathcal{L}(X_1) \setminus \{0\}$ such that

$$AY_1 = \lambda Y_1 A.$$

In this case, Y in Eq. (8) can be chosen as $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}$, in such a way that $M_C Y = \lambda Y M_C$. This implies that $\lambda \in \sigma_{ext}(M_C)$.
Second case, if $\lambda \in \sigma_{ext}(B)$, then there exists $Y_2 \in \mathcal{L}(X_2) \setminus \{0\}$ such that

$$BY_2 = \lambda Y_2 B.$$

In this case, one can take Y in Eq. (8) as $Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_2 \end{pmatrix}$, so that $M_C Y = \lambda Y M_C$. As a result, $\lambda \in \sigma_{ext}(M_C)$.
(*ii*) First, let's notice that $\sigma_{ext}(A) \cup \sigma_{ext}(B) \neq \emptyset$. In fact, by choosing $Y_1 = I_1$ ($Y_2 = I_2$), we get, respectively, $1 \in \sigma_{ext}(A)$ ($1 \in \sigma_{ext}(B)$). It follows that $1 \in \sigma_{ext}(A) \cup \sigma_{ext}(B)$. Now, let $\lambda \in \sigma_{ext}(M_C)$ and consider the following cases:
First case, if $Y_1 \neq 0$, then Eq. (9) implies, in particular, the existence of $Y_1 \in \mathcal{L}(X_1) \setminus \{0\}$ such that

$$AY_1 = \lambda Y_1 A.$$

Therefore,

$$\lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B).$$

Second case, if $Y_2 \neq 0$, the use of Eq. (9) leads, in particular, to the existence of $Y_2 \in \mathcal{L}(X_2) \setminus \{0\}$ such that

$$BY_2 = \lambda Y_2 B.$$

Consequently,

$$\lambda \in \sigma_{ext}(A) \cup \sigma_{ext}(B).$$ □

Remark 4.4. (*i*) *The converse of implication* (10) *in Theorem 4.3 is not always true. Indeed, let $A = V$ be the Volterra integral operator on $X_1 = L^2(0,1)$, $B = I$ be the identity operator on $X_2 = L^2(0,1)$ and let $C = 0$ be the zero operator on $L^2(0,1)$. Clearly, we notice that $CY_2 = \lambda Y_1 C$, for any $\lambda \in \mathbb{C}$. It is shown in [2] that $\sigma_{ext}(V) =]0,\infty[$. Furthermore, it is easy to observe that $\sigma_{ext}(I) = \{1\}$, so we get*

$$\sigma_{ext}(V) \cup \sigma_{ext}(I) =]0,\infty[.$$

On the other side, we have

$$\sigma_{ext}\left(\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\right) = \Big\{\lambda \in \mathbb{C} : \text{ there exists } \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ such that}$$

$$\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix} = \lambda \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\},$$

which implies that

$$\begin{aligned}
\sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big) = &\Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_1 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & Y_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_1, Y_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} = \lambda \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_1, Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_3 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} Y_1 & Y_3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_2, Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & Y_3 \\ 0 & Y_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 & Y_3 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_1, Y_2, Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } \\
&\qquad \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix} = \lambda \begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big) = &\Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_1 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_1 = \lambda Y_1 V\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } Y_2 = \lambda Y_2\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_3 = \lambda Y_3\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_1, Y_2 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_1 = \lambda Y_1 V \text{ and } Y_2 = \lambda Y_2\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_1, Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_1 = \lambda Y_1 V \text{ and } VY_3 = \lambda Y_3\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_2, Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_3 = \lambda Y_3 \text{ and } Y_2 = \lambda Y_2\Big\} \\
&\bigcup \Big\{\lambda \in \mathbb{C} : \text{ there exist } Y_1, Y_2, Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_1 = \lambda Y_1 V, Y_2 = \lambda Y_2, \text{ and } VY_3 = \\
&\qquad \lambda Y_3\Big\}.
\end{aligned}$$

We have

$$\Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_3 \in \mathcal{L}(L^2(0,1)) \setminus \{0\} \text{ such that } VY_3 = \lambda Y_3\Big\} = \sigma(V).$$

Since $\sigma(V) = \{0\}$, then we can deduce that

$$\begin{aligned}
\sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big) &= \;]0,\infty[\cup \{1\} \cup \{0\} \cup \{1\} \cup \emptyset \cup \emptyset \cup \emptyset \\
&= [0,\infty[.
\end{aligned}$$

As a result, there exists $\lambda = 0 \in \sigma_{ext}\Big(\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}\Big)$. *However,* $\lambda = 0 \notin \sigma_{ext}(V) \cup \sigma_{ext}(I)$.

(*ii*) *If* $Y_1 = Y_2 = 0$, *then there is no inclusion relation among* $\sigma_{ext}(A) \cup \sigma_{ext}(B)$ *and* $\sigma_{ext}(M_C)$. *In fact, let H be a Hilbert space,* $A \in \mathcal{L}(H)$ *such that* $\sigma(A) = \{\lambda\}$ *with* $\lambda \neq 0$, $B = I$ *be the identity operator on H and* $C \in \mathcal{L}(H)$. *Then, it follows that* $\sigma_{ext}(A) = \{1\}$ *(see [3]). Furthermore, it is easy to notice that* $\sigma_{ext}(I) = \{1\}$. *Hence, we obtain*

$$\sigma_{ext}(A) \cup \sigma_{ext}(I) = \{1\}.$$

On the other side, we have

$$\begin{aligned}\sigma_{ext}\Big(\begin{pmatrix} A & C \\ 0 & I \end{pmatrix}\Big) &= \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_3 \in \mathcal{L}(H)\backslash\{0\} \text{ such that } \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & C \\ 0 & I \end{pmatrix}\Big\} \\ &= \Big\{\lambda \in \mathbb{C} : \text{ there exists } Y_3 \in \mathcal{L}(H) \setminus \{0\} \text{ such that } (A - \lambda I)Y_3 = 0\Big\} \\ &= \{\lambda\}.\end{aligned}$$

That would be clear when $\lambda \neq 1$. ◊

Corollary 4.5. *If* $A = B$ *and* $CY_2 = \lambda Y_1 C$, *for any* $\lambda \in \mathbb{C}$. *Then,*

$$\lambda \in \sigma_{ext}(M_C) \text{ if, and only if, } \lambda \in \sigma_{ext}(A).$$ ◊

Proof. Suppose that $A = B$ and $CY_2 = \lambda Y_1 C$, for any $\lambda \in \mathbb{C}$. Let $\lambda \in \sigma_{ext}(A)$. Based on Theorem 4.3 (*i*), we conclude that $\lambda \in \sigma_{ext}(M_C)$. Conversely, let $\lambda \in \sigma_{ext}(M_C)$. Using Theorem 4.3 (*ii*), it is sufficient to prove $\lambda \in \sigma_{ext}(A)$ when Y of Eq. (8) is equal to $\begin{pmatrix} 0 & Y_3 \\ 0 & 0 \end{pmatrix}$. In this case, Eq. (9) implies the existence of $Y_3 \in \mathcal{L}(X_1) \setminus \{0\}$ such that

$$AY_3 = \lambda Y_3 A.$$

As a result, we obtain $\lambda \in \sigma_{ext}(A)$. □

In the following theorem, we will extend results obtained in Theorem 4.3 from bounded 2×2 upper triangular block operator matrices to invertible closed ones.

Theorem 4.6. *Let* X_1 *and* X_2 *two Banach spaces, we consider an unbounded* 2×2 *upper triangular block operator matrices defined on* $\mathcal{D}(M_C) = \mathcal{D}(A) \times \mathcal{D}(B) \subset X_1 \times X_2$ *by*

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{12}$$

where A and B are, respectively, two closed linear operators on X_1 *and* X_2 *and* $C \in \mathcal{L}(X_2, X_1)$ *such that* $0 \in \rho(A) \cap \rho(B)$. *Consider the* 2×2 *upper triangular block operator matrices, Y, defined in Eq.* (8). *Then, we have the following results:*

(*i*) *Let* $CY_2 = \lambda Y_1 C$, *for any* $\lambda \in \mathbb{C}$. *Then,*

$$\text{if } \lambda \in \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\}, \text{ then } \lambda \in \sigma_{ext}(M_C) \setminus \{0\}. \tag{13}$$

(*ii*) *If* $Y_1 \neq 0$ *or* $Y_2 \neq 0$, *then*

$$\sigma_{ext}(M_C) \setminus \{0\} \subseteq \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\}. \tag{14}$$

◊

Proof. We denote that if A and B are closed linear operators and C is a bounded linear operator, then M_C with its domain $\mathcal{D}(A) \times \mathcal{D}(B)$ is closed as it is the sum of a closed and a bounded operator. Using the fact that $0 \in \rho(A) \cap \rho(B)$, we infer that $0 \in \rho(M_C)$ such that

$$M_C^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}.$$

(i) Again, relying on the fact that $0 \in \rho(A) \cap \rho(B)$ together with Proposition 3.3 (iii) leads to

$$\lambda \in \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\} \text{ if, and only if, } \frac{1}{\lambda} \in \{\sigma_{ext}(A^{-1}) \cup \sigma_{ext}(B^{-1})\} \setminus \{0\}.$$

Resting upon Theorem 4.3 (i), we get

$$\frac{1}{\lambda} \in \sigma_{ext}(M_C^{-1}) \setminus \{0\}.$$

Again, according to Proposition 3.3 (iii), we have

$$\lambda \in \sigma_{ext}(M_C) \setminus \{0\}.$$

(ii) The fact that $0 \in \rho(M_C)$ together with Proposition 3.3 (iii) leads to

$$\lambda \in \sigma_{ext}(M_C) \setminus \{0\} \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ext}(M_C^{-1}) \setminus \{0\}.$$

Investing Theorem 4.3 (ii), allows us to deduce that

$$\frac{1}{\lambda} \in \{\sigma_{ext}(A^{-1}) \cup \sigma_{ext}(B^{-1})\} \setminus \{0\}.$$

Applying again Proposition 3.3 (iii), we infer that

$$\lambda \in \{\sigma_{ext}(A) \cup \sigma_{ext}(B)\} \setminus \{0\}.$$ □

Lemma 4.7. *If $\sigma_s(A) \cap \sigma_s(B) = \emptyset$, then for every $C \in \mathcal{L}(X_2, X_1)$ the operator M_C is similar to M_0.* ◊

Proof. Let $\sigma_s(A) \cap \sigma_s(B) = \emptyset$. Following the same reasoning as in the proof of Lemma 3.8, we get for every $C \in \mathcal{L}(X_2, X_1)$, the equation $A\psi - \psi B = C$ has a solution ψ. Since

$$M_C = \begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix} M_0 \begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix},$$

where $\begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix}$ is the inverse of $\begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix}$. We get M_C which is similar to M_0. □

Theorem 4.8. *If $\sigma_s(A) \cap \sigma_s(B) = \emptyset$, then for every $C \in \mathcal{L}(X_2, X_1)$*

$$\sigma_{ext}(M_C) = \sigma_{ext}(M_0).$$ ◊

Proof. Let $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$ and suppose that $\sigma_s(A) \cap \sigma_s(B) = \emptyset$. By using Lemma 4.7, it follows that for every $C \in \mathcal{L}(X_2, X_1)$ the operator M_C is similar to M_0 such that

$$M_C = \begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix} M_0 \begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix},$$

where $\begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix}$ is the inverse of $\begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix}$. Now, let's suppose that $\lambda \in \sigma_{ext}(M_C)$, then there exists a nonzero operator such that

$$M_C Y = \lambda Y M_C.$$

It follows that,

$$\begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix} M_0 \begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix} Y = \lambda Y \begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix} M_0 \begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix}.$$

Therefore, we get

$$M_0\begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix} = \lambda\begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix}M_0.$$

In other words, there exists a nonzero operator

$$\begin{aligned} Z &= \begin{pmatrix} I & -\psi \\ 0 & I \end{pmatrix}\begin{pmatrix} Y_1 & Y_3 \\ 0 & Y_2 \end{pmatrix}\begin{pmatrix} I & \psi \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} Y_1 & Y_1\psi + Y_3 - \psi Y_2 \\ 0 & Y_2 \end{pmatrix}, \end{aligned}$$

such that

$$M_0 Z = \lambda Z\, M_0.$$

So, $\lambda \in \sigma_{ext}(M_0)$. The proof of the inverse inclusion follows the same way. □

Using the same method of Biswas and Petrovic [3], for which it was established that the extended spectrum is invariant under a quasisimilarity, we can set forward the following Lemma:

Lemma 4.9. *Let $A, B \in \mathcal{L}(X)$ such that A has a dense range and B is injective. Then, we have*
(*i*) $\sigma_{ext}(AB) \subset \sigma_{ext}(BA)$.
(*ii*) *If, further A is injective and B has a dense range, then* $\sigma_{ext}(AB) = \sigma_{ext}(BA)$. ◊

Proof. (*i*) Suppose that $\lambda \in \sigma_{ext}(AB)$, then there exists a nonzero operator Z satisfying

$$ABZ = \lambda ZAB. \tag{15}$$

Multiplying Eq. (15) by A on the left and by B on the right, we obtain

$$BABZA = \lambda BZABA. \tag{16}$$

In Eq. (16), we have $Z \neq 0$. Since B is injective, then $BZ \neq 0$. The fact that A has a dense range, we infer that $BZA \neq 0$, which assures that $\lambda \in \sigma_{ext}(BA)$, with $BZA \in E(BA, \lambda)$.
(*ii*) The inverse inclusion follows by symmetry. □

Theorem 4.10. *Let $A \in \mathcal{L}(H_1)$ and $B \in \mathcal{L}(H_2)$ be given injective operators such that $R(A)$ dense in H_1 and $R(B)$ dense in H_2. We have for all $C \in \mathcal{L}(H_2, H_1)$*

$$\sigma_{ext}\Big(\begin{pmatrix} A & AC \\ 0 & B \end{pmatrix}\Big) = \sigma_{ext}\Big(\begin{pmatrix} A & CB \\ 0 & B \end{pmatrix}\Big).$$ ◊

Proof. We have the following formula

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

We set $R = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$ and $S = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & I \end{pmatrix}$. The fact that A, B and I are injective, allows us to prove easily that both R and S are injective. Furthermore, A, B and I have dense ranges. Applying Theorem 2.8, we infer that both R and S have dense ranges. Now, using Lemma 4.9 leads to

$$\sigma_{ext}(M_C) = \sigma_{ext}(SR) = \sigma_{ext}\Big(\begin{pmatrix} A & CB \\ 0 & B \end{pmatrix}\Big). \tag{17}$$

Next, if we take $R = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & B \end{pmatrix}$ and $S = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$, then in a similar way, we get

$$\sigma_{ext}(M_C) = \sigma_{ext}(SR) = \sigma_{ext}\Big(\begin{pmatrix} A & AC \\ 0 & B \end{pmatrix}\Big). \tag{18}$$

Based upon Eqs. (17) and (18), we obtain for every $C \in \mathcal{L}(X_2, X_1)$

$$\sigma_{ext}\Big(\begin{pmatrix} A & AC \\ 0 & B \end{pmatrix}\Big) = \sigma_{ext}\Big(\begin{pmatrix} A & CB \\ 0 & B \end{pmatrix}\Big). \qquad \square$$

References

[1] H. Alkanjo, On extended eigenvalues and extended eigenvectors of truncated shift. Concr. Oper. 1 (2013), 19–27.
[2] A. Biswas, A. Lambert, S. Petrovic, Extended eigenvalues and the Volterra operator. Glasg. Math. J. 44 (2002), no. 3, 521-534.
[3] A. Biswas, S. Petrovic, On extended eigenvalues of operators. Integral Equations Operator Theory 55 (2006), no. 2, 233-248.
[4] S. Brown, Connections between an operator and a compact operator that yield hyperinvariant subspaces. J. Operator Theory 1 (1979), no. 1, 117–121.
[5] G. Cassier, H. Alkanjo, Extended spectrum, extended eigenspaces and normal operators. J. Math. Anal. Appl. 418 (2014), no. 1, 305–316.
[6] D. S. Cvetković-Ilić, The point, residual and continuous spectrum of an upper triangular operator matrix. Linear Algebra Appl. 459 (2014), 357–367.
[7] N. Dunford, J. Schwartz, Linear Operators. I. General Theory. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7 Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London 1958 xiv+858 pp.
[8] A. Jeribi, Spectral theory and applications of linear operators and block operator matrices. Springer-Verlag, New York, 2015.
[9] M. T. Karaev, On extended eigenvalues and extended eigenvectors of some operator classes. Proc. Amer. Math. Soc. 134 (2006), no. 8, 2383-2392.
[10] H. W. Kim, R. Moore, C. M. Pearcy, A variation of Lomonosov's theorem. J. Operator Theory 2 (1979), no. 1, 131-140.
[11] A. Lambert, S. Petrovic, Beyond hyperinvariance for compact operators. J. Funct. Anal. 219 (2005), no. 1, 93-108.
[12] V. Lomonosov, Invariant subspaces for the family of operators commuting with completely continuous operators, Funct. Anal. Appl. 7 (1974), no. 1, 213-214.
[13] E. Otkun Çevik, Z. I. Ismailov, Spectrum of the direct sum of operators. Electron. J. Differential Equations (2012), No. 210, 8 pp.
[14] M. Rosenblum, On the operator equation $BX - XA = Q$. Duke Math. J. 23x, (1956) 263-269.
[15] M. Schechter, Principles of functional analysis. Second edition. Graduate Studies in Mathematics, 36. American Mathematical Society, Providence, RI, 2002.
[16] M. Sertbaş, F. Yilmaz, On the extended spectrum of some quasinormal operators. Turkish J. Math. 41 (2017), no. 6, 1477–1481.
[17] S. Shkarin, Compact operators without extended eigenvalues. J. Math. Anal. Appl. 332 (2007), no. 1, 455-462.
[18] A. E. Taylor, Spectral theory of closed distributive operators. Acta Math. 84 (1951), 189–224.