Filomat 36:12 (2022), 4241–4249
https://doi.org/10.2298/FIL2212241R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Variants of Normality in Relative Topological Spaces

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Abstract. With each topological property $\mathcal{P}$ one can associate a relative version of it formulated in terms of the location of Y in X in such a natural way that when Y coincides with X, then this relative property coincides with $\mathcal{P}$. Arhangel'skii and Genedi introduced this concept of relative topological properties in 1989. The concept of mild normality or κ-normality was introduced independently by Singal and Singal in 1973 and Ščepin in 1972. A few years earlier in 1969, Singal and Arya studied the concept of almost normality. V. Zaǐcev in 1968 introduced the concept of quasi normal spaces while π-normality was studied by Kalantan in 2008. In this paper we study these variants of normality in a relative sense.

1. Introduction and Preliminaries

The main motivation behind the study of relative topological properties is actually the location problem of how a certain space Y is located in a larger space X. In different situations topologists have studied various relative topological properties. In [6] Grothendieck studied relative countable compactness. Relative topological dimensions was studied by Tkachuk [28] and Chigogidze [5]. A few others who have contributed initially are Rančin [22], Dow, Varmeer [12], Kočinac [16] etc. A systematic study of relative topological properties was begun by A. V. Arhangel'skii and H. M. M. Genedi in [1]. According to them each topological property $\mathcal{P}$ can be associated with a relative version of it formulated in terms of the location of Y in X in such a natural way that when Y coincides with X then this relative property coincides with $\mathcal{P}$. In [2] Arhangel'skii presented a survey on some relative topological spaces.

Normality is one of the most important topological properties and various versions of normality have been studied in the past such as κ-normality [23] or mild normality [24], almost normality [25], π-normality [18], quasi normality [30], seminormality, Δ-normality [7] etc. Arhangel'skii in [1] defined normality in a relative sense. In [1] Arhangel'skii introduced the concept of densely normal spaces which is a relative topological property and proved its relationship with κ-normality. In [3] Arhangel'skii's raised a question asking whether every κ-normal regular space is a densely normal space and in [15] as an answer to this question Just and Tartir have given an example of a Tychonoff κ-normal space which is not densely normal. Das and Bhat in [9] introduced another class of spaces which lies between densely normal spaces and κ-normal spaces. In the recent past various relative topological properties were investigated (see [11, 13, 14, 20]).

2020 *Mathematics Subject Classification*. Primary 54D15
Keywords. Normal, strongly normal, κ-normal, almost normal, π-normal, quasi normal, almost regular
Received: 13 April 2021; Revised: 27 March 2022; Accepted: 01 April 2022
Communicated by Ljubiša D.R. Kočinac
Research supported by Department of Science and Technology (DST), Government of India through INSPIRE fellowship (IF160701)
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Among the above mentioned versions of normality in the present paper we study κ-normality, almost normality, quasi normality and π-normality in a relative sense, prove some of their properties and their relationship with one another and arrive at the conclusion that all other versions of relative normality discussed in this paper lie between relative strong normality and relative κ-normality. In other words relative κ-normality is the most generalized version of relative normality amongst the discussed variants. Further, it is observed that in the class of β-normal spaces (or seminormal spaces) variants of relative normality studied in this paper coincide with each other.

Let X be a topological space and $A \subset X$. Throughout this paper the closure of a set A will be denoted by $\overline{A}$ and the interior by A^o.

A set $U \subset X$ is said to be regularly open [17] if $U = \overline{U}^o$. The complement of a regularly open set is called a regularly closed set. The intersection (union) of two regularly closed (regularly open) sets need not be regularly closed (regularly open). Therefore π-open sets and π-closed sets are defined as finite unions of regular open sets and finite intersections of regular closed sets respectively.

Definition 1.1. A space X is said to be:

1. ([23], [24]) κ-normal or mildly normal if for every pair of disjoint regularly closed sets A and B of X there exist disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$.
2. ([25]) almost normal if for every pair of disjoint sets A and B one of which is regularly closed and other is closed, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.
3. ([30]) quasi-normal for every pair of disjoint π-closed subsets A and B of X there exist two open disjoint subsets U and V of X such that $A \subset U$ and $B \subset V$.
4. ([18]) π-normal if for every pair of disjoint closed subsets A and B of X, one of which is π-closed, there exist two open disjoint subsets U and V of X such that $A \subset U$ and $B \subset V$.
5. ([26]) almost regular if for every regularly closed set A of X and a point $x \notin A$, there exist disjoint open sets U and V such that $A \subset U$ and $x \in V$.

Definition 1.2. Let $Y \subset X$. Y is said to be:

1. ([1]) relatively T_1 in X if for every $y \in Y$, $\{y\}$ is closed in X.
2. ([1]) normal in X or relatively normal in X, if for each pair A, B of disjoint closed subsets of X, there are disjoint open subsets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
3. ([1]) strongly normal in X or relatively strongly normal in X, if for each pair A, B of disjoint closed sets in Y, there are disjoint open subsets U and V in X such that $A \subset U$ and $B \subset V$.

2. Variants of Normality in a Relative Sense

Definition 2.1. Let X be a topological space. Then $Y \subset X$ is said to be:

1. *relative κ-normal* in X if for every pair of disjoint regularly closed sets A and B of X, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
2. *relative almost normal* in X if for any two disjoint closed subsets A and B of X one of which is regularly closed, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
3. *relative quasi normal* in X if for any two disjoint π-closed subsets A and B of X, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $B \cap Y \subset V$.
4. *relative π-normal* in X if for any two disjoint subsets A and B of X one of which is π-closed and other is closed, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

It is clear from the definitions that if X is κ-normal (almost normal, quasi normal, π-normal), then $Y \subset X$ is relative κ-normal (relative almost normal, relative quasi normal, relative π-normal respectively) in X. The following example shows that none of the converses are true.

Example 2.2. Let X be the set of integers. Define a topology τ on X, where every odd integer is open and a set U is open if for every even integer $p \in U$, the successor and the predecessor of p also belong to U. Let Y be the set of all odd integers. Then Y is relative κ-normal, relative almost normal, relative quasi normal as well as relative π-normal in X. But X is none of the absolute versions of these properties because $A = \{2, 3, 4\}$ and $B = \{6, 7, 8\}$ are disjoint regularly closed sets in X which are π-closed as well and there do not exist disjoint open sets in X separating them.

Note. If Y is κ-normal (almost normal, quasi normal, π-normal) in itself that is with respect to the subspace topology, then Y need not be relative κ-normal (relative almost normal, quasi normal, π-normal) in X respectively. See the following example.

Example 2.3. Let X be the set of integers with the topology defined in Example 2.2. Let Y be the set of all even integers. Then Y has discrete topology. So Y is a normal space and hence Y is κ-normal as well as quasi normal and π-normal in itself but Y is none of the relative versions of any of these variants in X because $A = \{2, 3, 4\}$ and $B = \{6, 7, 8\}$ are disjoint regularly closed sets which are π-closed as well in X and there do not exist disjoint open sets in X separating $A \cap Y = \{2, 4\}$ and $B \cap Y = \{6, 8\}$.

Example 2.4. Let $X = \{1, 2, 3, 4\}$ and $\tau_X = \{\{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3\}, X, \phi\}$. Let $Y = \{1, 3, 4\}$. It is clear that Y is almost normal in itself, i.e. with respect to the subspace topology but it is not relative almost normal in X because $\{3, 4\}$ is regularly closed in X and $\{1\}$ is closed in X such that $\{3, 4\} \cap Y$ and $\{1\} \cap Y$ cannot be separated by disjoint open sets in X.

Theorem 2.5. *Let $Y \subset X$. If Y is relative κ-normal in X, then for every regularly closed set A and every regularly open set B of X such that $A \subset B$, there exists a regularly open set O of X such that $A \cap Y \subset O \subset \overline{O} \subset B \cup (X \setminus Y)$.*

Proof. Let A be a regularly closed set and B be a regularly open set in X such that $A \subset B$. Then A and $X \setminus B$ are disjoint regularly closed subsets of X. Since Y is relative κ-normal in X, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $(X \setminus B) \cap Y \subset V$. Thus, $(X \setminus V) \subset B \cup (X \setminus Y)$. So, $A \cap Y \subset U \subset (X \setminus V) \subset B \cup (X \setminus Y)$. Since $X \setminus V$ is a closed set containing U and $\overline{U}$ is a smallest closed set containing U, $A \cap Y \subset U \subset \overline{U} \subset B \cup (X \setminus Y)$. Take $O = \overline{U}^{\circ}$. Then O is a regularly open set such that $A \cap Y \subset O \subset \overline{O} \subset B \cup (X \setminus Y)$. □

Theorem 2.6. *Let $Y \subset X$. If Y is relative almost normal in X, then:*

1. *for every closed subset A and every regularly open subset B of X such that $A \subset B$, there exists an open set U of X such that $A \cap Y \subset U \subset \overline{U} \subset B \cup (X \setminus Y)$.*
2. *for every regularly closed subset A and every open subset B of X such that $A \subset B$, there exists an open set U of X such that $A \cap Y \subset U \subset \overline{U} \subset B \cup (X \setminus Y)$.*

Proof. 1. Let A be a closed subset and B be a regularly open subset of X such that $A \subset B$. Then $X \setminus B$ is regularly closed subset of X and $A \cap (X \setminus B) = \emptyset$. Since Y is almost normal in X, there exist disjoint open sets U and V of X such that $A \cap Y \subset U$ and $(X \setminus B) \cap Y \subset V$. Thus, $(X \setminus V) \subset B \cup (X \setminus Y)$. So, $A \cap Y \subset U \subset (X \setminus V) \subset B \cup (X \setminus Y)$. Since $X \setminus V$ is a closed set containing U and $\overline{U}$ is a smallest closed set containing U, $A \cap Y \subset U \subset \overline{U} \subset B \cup (X \setminus Y)$.

To prove 2., let A be a regularly closed subset of X and B be an open subset of X such that $A \subset B$. Then $X \setminus A$ is regularly open set containing closed set $X \setminus B$. By (1), there is an open set U of X such that $(X \setminus B) \cap Y \subset U \subset \overline{U} \subset (X \setminus A) \cup (X \setminus Y)$. Thus $A \cap Y \subset X \setminus \overline{U} \subset X \setminus U \subset B \cup X \setminus Y$. Let $X \setminus \overline{U} = V$. Then V is open in X and $A \cap Y \subset V \subset \overline{V} \subset B \cup (X \setminus Y)$. □

Theorem 2.7. *Let $Y \subset X$. If Y is relative quasi normal in X, then for every π-closed subset A and every π-open subset B of X such that $A \subset B$, there exists a regularly open set O such that $A \cap Y \subset O \subset \overline{O} \subset B \cup (X \setminus Y)$.*

Proof. Let A be a π-closed set and B be a π-open set of X such that $A \subset B$. Then A and $X \setminus B$ are disjoint π-closed subsets of X. Since Y is relative quasi normal in X, there exist disjoint open sets U and V in X such that $A \cap Y \subset U$ and $(X \setminus B) \cap Y \subset V$. Thus, $(X \setminus V) \subset B \cup (X \setminus Y)$. So, $A \cap Y \subset U \subset (X \setminus V) \subset B \cup (X \setminus Y)$. Since $X \setminus V$ is a closed set containing U and $\overline{U}$ is the smallest closed set containing U, $A \cap Y \subset U \subset \overline{U} \subset B \cup (X \setminus Y)$. Take $O = \overline{U}^{\circ}$. Then O is a regularly open set such that $A \cap Y \subset O \subset \overline{O} \subset B \cup (X \setminus Y)$. □

Theorem 2.8. *If Y is relative π-normal in X, then:*

1. *for every closed subset A and every π-open subset B of X such that $A \subset B$, there exists an open set V such that $A \cap Y \subset V \subset \overline{V} \subset B \cup (X \setminus Y)$.*
2. *for every π-closed subset A and every open subset B of X such that $A \subset B$, there exists an open set U of X such that $A \cap Y \subset U \subset \overline{U} \subset B \cup (X \setminus Y)$.*

Proof. The proof is similar to the proof of Theorem 2.6. □

Definition 2.9. ([2]) A subset A of X is said to be concentrated on Y if A is contained in the closure in X of the trace $A \cap Y$ of the set A on Y. A space X is normal on Y if every two disjoint closed subsets of X concentrated on Y can be separated by disjoint open neighborhoods in X.

Definition 2.10. ([2]) A space X is called densely normal if there exists a dense subspace Y of X such that X is normal on Y.

Definition 2.11. ([9]) A subset A of X is said to be strongly concentrated on Y if $A \subset \overline{(A \cap Y)^{\circ}}$. Let Y be a subspace of X. Then X is said to be weakly normal on Y if for every disjoint closed subsets A and B of X strongly concentrated on Y, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

Definition 2.12. ([9]) A space is said to be weakly densely normal if there exists a proper dense subspace Y of X such that X is weakly normal on Y.

Theorem 2.13. ([2]) *Every densely normal space is κ-normal.*

Theorem 2.14. ([9]) *Every weakly densely normal space is κ-normal.*

From above theorems the following result is obvious.

Theorem 2.15. *If X is densely normal (or weakly densely normal), then every subset of X is relative κ-normal.*

Recall that a space X is called extremally disconnected if it is T_1 and the closure of any open set in X is open. Any π-open (π-closed) subset of an extremally disconnected space is an open domain (closed domain). Any extremally disconnected space is π-normal space [18]. Also a space X is called weakly extremally disconnected [19] if the closure of any open set is open. In a weakly extremally disconnected space any regularly closed set is clopen. Every weakly extremally disconnected space is almost normal [19]. The proofs of the following theorems follows from the above discussed facts.

Theorem 2.16. *Every subset of an extremally disconnected space is relative π-normal.*

Theorem 2.17. *Every subset of a weakly extremally disconnected space is relative almost normal.*

Theorem 2.18. *$Y \subset X$ is relative κ-normal in X if for every pair of disjoint regularly closed sets A and B in X, there exists a continuous function f on X into closed interval $[0, 1]$ such that $f(A \cap Y) = \{0\}$ and $f(A \cap Y) = \{1\}$.*

Proof. Let A and B be two disjoint regularly closed subsets of X and $f : X \to [0, 1]$ be a continuous function such that $f(A \cap Y) = \{0\}$ and $f(B \cap Y) = \{1\}$. Since f is continuous on X, $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets in X containing $A \cap Y$ and $B \cap Y$ respectively. Hence Y is relative κ-normal in X. □

Theorem 2.19. *$Y \subset X$ is relative almost normal in X if for every pair of disjoint closed sets A and B of X one of which is regularly closed in X, there exists a continuous function f on X into closed interval $[0,1]$ such that $f(A \cap Y) = \{0\}$ and $f(A \cap Y) = \{1\}$.*

Theorem 2.20. *$Y \subset X$ is relative quasi normal in X if for every pair of disjoint π-closed sets A and B in X, there exists a continuous function f on X into closed interval $[0,1]$ such that $f(A \cap Y) = \{0\}$ and $f(A \cap Y) = \{1\}$.*

Theorem 2.21. *$Y \subset X$ is relative π-normal in X if for every pair of disjoint closed sets A and B of X one of which is π-closed in X, there exists a continuous function f on X into closed interval $[0,1]$ such that $f(A \cap Y) = \{0\}$ and $f(A \cap Y) = \{1\}$.*

Remark 2.22. To provide characterization of relative κ-normality, relative almost normality, relative quasi normality and relative π-normality, it is natural to ask whether converses of Theorem 2.18 - 2.21 holds or not.

The example below establishes that a continuous image of a relative κ-normal (relative almost normal) space need not be relative κ-normal (relative almost normal). Thus it is natural to ask Questions 2.24 and 2.25.

Example 2.23. Let X_1 be the set of integers with the discrete topology and Y be the set of all odd integers. Then clearly Y is relative κ-normal as well as almost normal in X_1. Let X_2 be the set of integers. Define a topology on X_2 by taking every odd integer to be open and a set $U \subset X_2$ is open if for every even integer $p \in U$, the predecessor and the successor of p are also in U. Define $f : X_1 \to X_2$ as $f(x) = x + 1$. Clearly, f is continuous, one-one but not open. Also $f(Y)$ is the set of all even integers. Since X_2 has odd-even topology, $A = \{4, 5, 6\}$ and $B = \{8, 9, 10\}$ are regularly closed in X_2. But $A \cap f(Y) = \{4, 6\}$ and $B \cap f(Y) = \{8, 10\}$, which cannot be separated by disjoint open sets in X_2. Hence $f(Y)$ is neither relative κ-normal nor almost normal in X_2.

Question 2.24. *Under what conditions is an image of a relative κ-normal (relative almost normal) space is relative κ-normal (relative almost normal)?*

Question 2.25. *Under what conditions is an image of a relative π-normal (relative quasi normal) space is relative π-normal (relative quasi normal)?*

Definition 2.26. ([21]) $Y \subset X$ is said to be relative almost regular in X if for every regularly closed set A in X and a point $y \in Y$ such that $y \notin A$, there exist disjoint open sets U and V such that $A \cap Y \subset U$ and $y \in V$.

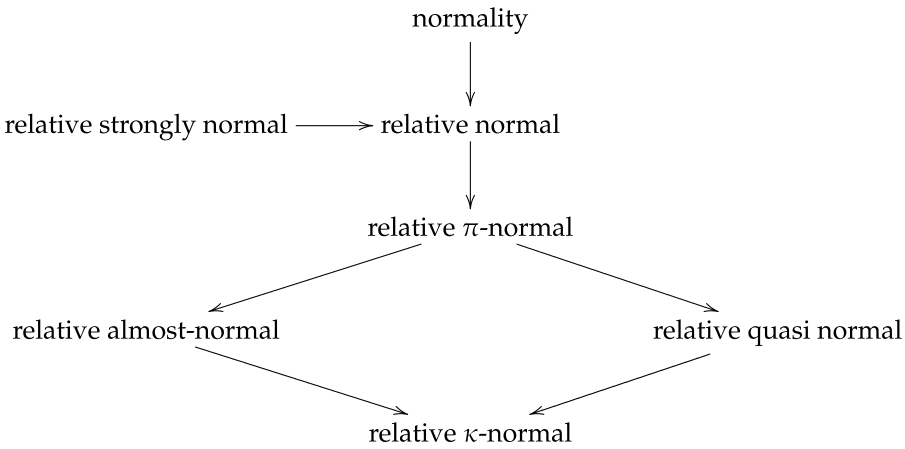
Theorem 2.27. ([21]) *If Y is relative almost normal and relative T_1 in X, then Y is relative almost regular.*

In general relative almost normality does not necessarily imply relative almost regularity. See the following example.

Example 2.28. Let $X = \{p, q, r\}$ and $\tau = \{\{p\}, \{q\}, \{p, q\}, X, \phi\}$. Let $Y = \{p, r\}$. Here, the set $\{q, r\}$ is regularly closed in X and $p \in Y$ such that $p \notin \{q, r\}$. But p and $\{q, r\} \cap Y = \{r\}$ cannot be separated by two disjoint open sets in X. Hence Y is not relative almost regular in X. But Y is relative almost normal in X as there is no pair of disjoint closed sets in X.

3. Interrelations

The interrelations shown in the following diagram follows immediately from the definitions.



4. Some Counter Examples

Example 4.1. A finite relative π-normal space which is not relative normal.

Let $X = \{a, b, c\}$ and $\tau = \{\{a\}, \{a, b\}, \{a, c\}, X, \phi\}$. Let $Y = \{b, c\}$. Then Y is relative π-normal in X because there is no proper, non-empty regularly closed set in X. But Y is not relative normal in X because $\{b\}$ and $\{c\}$ are closed in X such that $\{b\} \cap Y$ and $\{c\} \cap Y$ cannot be separated by disjoint open sets in X.

Example 4.2. An infinite relative π-normal space which is not relative normal.

Let $X= \mathbb{R}$ with cofinite topology and $Y= \mathbb{Z}$. Then Y is relative π-normal in X because any regularly closed set in X is either empty or all of X. But Y is not relative normal in X.

Example 4.3. A relative κ-normal space which is not relative almost normal.

Let $X = \{a, b, c, d\}$ and $\tau = \{\{a, b\}, \{b\}, \{b, c\}, \{c\}, \{b, c, d\}, \{a, b, c\}, X, \phi\}$. Let $Y = \{a, c, d\}$. The non-trivial regularly closed sets in X are $\{c, d\}$ and $\{a, b, d\}$ which are not disjoint. Hence Y is relative κ-normal in X. But Y is not relative almost normal in X because $\{c, d\}$ is regularly closed in X and $\{a\}$ is closed in X such that $\{c, d\} \cap Y$ and $\{a\} \cap Y$ cannot be separated by disjoint open sets in X.

Example 4.4. A relative quasi normal space which is not π-normal.

Consider Example 4.3. The non-trivial π-closed sets in X are $\{c, d\}, \{a, b, d\}$ and $\{d\}$ which are not disjoint. Hence Y is relative quasi normal in X. But Y is not relative π-normal in X as π-closed set $\{d\}$ and closed set $\{a\}$ cannot be separated by disjoint open sets in X.

Example 4.5. A relative quasi normal space which is not relative almost normal.

Consider Example 4.3. From Example 4.4 Y is relative quasi normal. But from Example 4.3 Y is not relative almost normal.

Question 4.6. *Does there exists a relative almost normal space which is not relative quasi normal?*

5. Spaces where all these Variants are Equivalent

Definition 5.1. ([4]) A space X is said to be β-normal if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $\overline{U} \cap \overline{V} = \phi$.

Theorem 5.2. *Let X be a β-normal space. Then following statements are equivalent:*

(i) Y is relative normal in X.

(ii) Y is relative π-normal in X.

(iii) Y is relative almost normal in X.

(iv) Y is relative quasi normal in X.

(v) Y is relative κ-normal in X.

Proof. The implications (i) ⇒ (ii) ⇒ (iii)⇒ (v) and (i)⇒ (ii)⇒ (iv) ⇒ (v) are obvious from the interrelations.

To prove (v) ⇒ (i), let A and B be two closed sets in X. Since X is β-normal, there exist open subsets U and V of X such that $\overline{U} \cap \overline{V} = \phi$, $\overline{U \cap A} = A$, and $\overline{V \cap B} = B$. So, $\overline{U}$ and $\overline{V}$ are disjoint regularly closed sets such that $A \subset \overline{U}$ and $B \subset \overline{V}$. Which implies $A \cap Y \subset \overline{U} \cap Y$ and $B \cap Y \subset \overline{V} \cap Y$. Since Y is relative κ-normal in X, there exist disjoint open subsets U_1 and V_2 of X such that $A \cap Y \subset \overline{U} \cap Y \subset U_1$ and $B \cap Y \subset \overline{V} \cap Y \subset V_1$. Hence Y is relative normal in X. □

Definition 5.3. ([29]) A space is said to be seminormal if for every closed set F and each open set U containing F, there exists a regular open set V such that $F \subset V \subset U$.

Theorem 5.4. *Let X be a seminormal space. Then following statements are equivalent:*
(i) Y is relative normal in X.
(ii) Y is relative π-normal in X.
(iii) Y is relative almost normal in X.
(iv) Y is relative quasi normal in X.
(v) Y is relative κ-normal in X.

Proof. The implications (i) ⇒ (ii) ⇒ (iii)⇒ (v) and (i)⇒ (ii)⇒ (iv) ⇒ (v) are obvious.

To prove (v) ⇒ (i), let A and B be two disjoint closed sets in X. Then $X \setminus B$ is an open set containing A. Since X is seminormal, there exists a regularly open set U in X such that $A \subset U \subset X \setminus B$. Now $X \setminus U$ is a regularly closed set contained in the open set $X \setminus A$. Again by seminormality of X, there exists a regularly open set V in X such that $X \setminus U \subset V \subset X \setminus A$. Here, $X \setminus V$ and $X \setminus U$ are disjoint regularly closet sets in X such that $A \subset X \setminus V$ and $B \subset X \setminus U$. Thus, $A \cap Y \subset (X \setminus V) \cap Y$ and $B \cap Y \subset (X \setminus U) \cap Y$. Since Y is relative κ-normal in X, there exist disjoint open sets P and Q in X such that $A \cap Y \subset (X \setminus V) \cap Y \subset P$ and $B \cap Y \subset (X \setminus U) \cap Y \subset Q$. Hence Y is relative normal in X. □

Definition 5.5. ([8]) A space is said to be weakly seminormal if for every closed set F and each open set U containing F, there exists a π-open set V such that $F \subset V \subset U$.

Every seminormal space is weakly seminormal but the converse need not be true [8].

Theorem 5.6. *Let X be a weakly seminormal space. Then following statements are equivalent:*
(i) Y is relative normal in X.
(ii) Y is relative π-normal in X.
(iii) Y is relative quasi normal in X.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) are obvious from the interrelations.

To prove (iii) ⇒ (i), let A and B be two disjoint closed sets in X. Then $X \setminus B$ is an open set containing A. Since X is weakly seminormal, there exists a π-open set U in X such that $A \subset U \subset X \setminus B$. Now $X \setminus U$ is a π-closed set contained in the open set $X \setminus A$. Again by weak seminormality of X, there exists a π-open set V in X such that $X \setminus U \subset V \subset X \setminus A$. Here, $X \setminus V$ and $X \setminus U$ are disjoint π-closet sets in X such that $A \subset X \setminus V$ and $B \subset X \setminus U$. Thus, $A \cap Y \subset (X \setminus V) \cap Y$ and $B \cap Y \subset (X \setminus U) \cap Y$. Since Y is relative quasi normal in X, there exist disjoint open sets P and Q in X such that $A \cap Y \subset (X \setminus V) \cap Y \subset P$ and $B \cap Y \subset (X \setminus U) \cap Y \subset Q$. Hence Y is relative normal in X. □

Remark. Instead of X if we take Y as a β-normal (or seminormal) subspace of X, the results in Theorem 5.2 and Theorem 5.4 will not be the same. Let us consider the Example 4.3 in which X is not β-normal because for $\{a\}$ and $\{d\}$ disjoint closed sets in X, there does not exist disjoint open sets in X satisfying the conditions of β-normality. Also X is not seminormal because $\{d\}$ is a closed set in X and $\{b, c, d\}$ is an open set in X containing $\{d\}$ and there does not exist any regularly open set V in X such that $\{d\} \subset V \subset \{b, c, d\}$. But we can see that Y is a β-normal as well as seminormal subspace of X. Also Y is relative κ-normal in X but Y is not relative normal in X.

Definition 5.7. ([10]) A space is said to be almost β-normal if for any two disjoint closed subsets A and B of X, one of which is regularly closed, there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $\overline{U} \cap \overline{V} = \phi$.

Theorem 5.8. *Let X be an almost β-normal space. Then Y is relative almost normal in X if and only if Y is relative κ-normal in X.*

Proof. The direct implications are obvious from the interrelations.

Conversely, let X be an almost β-normal space and Y is relative κ-normal in X. Let A and B be two disjoint closed sets in X of which A is regularly closed. Since X is almost β-normal, there exist disjoint open sets U and V in X such that $\overline{U} \cap \overline{V} = \phi$, $\overline{A \cap U} = A$ and $\overline{B \cap V} = B$. Thus $A \subset \overline{U}$ and $B \subset \overline{V}$. Here $\overline{U}$ and $\overline{V}$ are disjoint regularly closed sets in X. Since Y is relative κ-normal in X, there exist disjoint open sets W_1 and W_2 in X such that $\overline{U} \cap Y \subset W_1$ and $\overline{V} \cap Y \subset W_2$ which implies $A \cap Y \subset \overline{U} \cap Y \subset W_1$ and $B \cap Y \subset \overline{V} \cap Y \subset W_2$. Hence Y is relative almost normal in X. □

Definition 5.9. ([11]) $Y \subset X$ is said to be *relative β-normal* in X or β-normal in X if for any two disjoint closed subsets A and B of X, there exist open subsets U and V of X such that $(A \cap Y) \cap U$ is dense in $A \cap Y$ and $(B \cap Y) \cap V$ is dense in $B \cap Y$ and $\overline{U} \cap \overline{V} = \phi$.

Theorem 5.10. ([11]) *Let Y be a relative β-normal space in X. Then following statements are equivalent:*

(a) Y is relative normal in X.
(b) Y is relative π-normal in X.
(c) Y is relative almost normal in X.
(d) Y is relative quasi normal in X.
(e) Y is relative κ-normal in X.

Definition 5.11. ([11]) $Y \subset X$ is said to be strong β-normal in X or *relative strong β-normal* in X if for any two disjoint closed subsets A and B of Y, there exist open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $\overline{U} \cap \overline{V} = \phi$.

Theorem 5.12. ([11]) *Let Y be a relative strong β-normal space in X. Then following statements are equivalent:*

(a) Y is relative strong normal in X.
(b) Y is relative normal in X.
(c) Y is relative π-normal in X.
(d) Y is relative almost normal in X.
(e) Y is relative quasi normal in X.
(f) Y is relative κ-normal in X.

Acknowledgement

The authors are thankful to the referee for his valuable comments and suggestions.

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